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Periodic modules of large periods for extra-special p-groups

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1. Introduction

Let p be an odd prime and let k be a field of characteristic p. We consider periodic modules over the group algebra of the extra special p-group of order p^3 and of exponent p:

$$M(p)=\langle a, b \mid a^p=b^p=[a, b]^p=1,\ [a, [a, b]]=[b, [a, b]]=1\rangle.$$

Periodic modules of period $2p$ had been known only for $M(p)$ by Carlson [6] (1979). In [17] Okuyama and the author gave new examples of such periodic modules for a metacyclic p-group

$$M_m(p)=\langle a, b \mid a^{p^{m-1}}=b^p=1,\ a^b=a^{1+p^{m-2}}\rangle,\ m\geq 3$$

as the kernels of cocycles representing certatin homogeneous elements of the cohomology algebra $H^*(M_m(p), k)$. We note that the maximum possibility of periods of periodic modules over both of the above p-groups is $2p$ by Carlson [7]. In this paper we shall show another examples of periodic $kM(p)$-modules of period $2p$, which are the kernels of cocycles representing particular homogeneous elements of the cohomology algebra $H^*(M(p), k)$.

Before stating our periodic modules we must prepare some notations. Let G be an arbitrary finite group. The cohomology group $H^r(G, k)$ is naturally isomorphic to the set of kG-homomorphisms of the rth syzygy $\Omega^r(k)$ to k. Following Carlson, we denote by $\hat{\varsigma}$ the kG-homomorphism of $\Omega^r(k)$ to k corresponding to an element ς in $H^r(G, k)$ and let L_ς denote the kernel of the homomorphism $\hat{\varsigma}$. The first cohomology group $H^1(G, k)$ is isomorphic with $\mathrm{Hom}(G, k)$. We identify elements in $H^1(G, k)$ with those of $\mathrm{Hom}(G, k)$ via this isomorphism. For a normal subgroup H of G of index p, the image of an element in $H^1(G, k)$ which has H as the kernel under the Bockstein homomorphism of $H^1(G, k)$ to $H^2(G, k)$ is called a Bockstein element (or a Bockstein for short) corresponding to the subgroup H.

Now let us define our periodic $kM(p)$-module. Let $c=[a, b]$. First of all we note that a Bocksitein corresponding to a maximal subgroup is not zero, because its restriction to another maximal subgroup is a Bockstein corresponding to the subgroup $\langle c\rangle$. Let λ and μ be the elements in $H^1(G, k)$ such that

$$(a)\lambda=0, \quad (b)\lambda=1$$

and

$$(a)\mu=1, \quad (b)\mu=0$$

and let α and β in $H^2(M(p), k)$ be the Bocksteins of the element λ and μ, respectively. Then α and β are Bocksteins corresponding to the maximal subgroups $\langle a, c\rangle$ and $\langle b, c\rangle$, respectively. For a polynomial

$$f(X)=s_0+s_1X+\cdots+s_{n-1}X^{n-1}+X^n$$

in $k[X]$ such that

$$f(i)\neq 0 \text{ for } i=0, 1, \dots, p-1,$$

we define a homogeneous element χ of degree $2n$ as follows:

$$\chi=s_0\beta^n+s_1\alpha\beta^{n-1}+\cdots+s_{n-1}\alpha^{n-1}\beta+\alpha^n.$$

With this notation

THEOREM. *If $n\equiv 1$ mod p, then the module L_χ is an indecomposable periodic $kM(p)$-module of period $2p$.*

The periodicity and the indecomposability of the module L_χ can be shown by using the theory of the module varieties associated with modules, which have been developed by Quillen, Alperin-Evens, Carlson, and Benson, etc.

Let B denote the maximal subgroup $\langle b, c\rangle$ of $M(p)$. Let η be the element in $H^1(B, k)$ such that $(b)\eta=0$ and $(c)\eta=1$, and let ξ in $H^2(B, k)$ be its Bockstein. Let τ be the image of ξ under the norm map from $H^2(B, k)$ to $H^{2p}(M(p), k)$. In Section 2 we will show the following:

LEMMA 2.1. *The tensor product $L_\chi\otimes L_\tau$ is a projective $kM(p)$-module. In particular the module L_χ is periodic.*

LEMMA 2.2. *The module L_χ is indecomposable.*

In order to determine the period of the module L_χ we need more information on the cohomology algebra $H^*(M(p), k)$. In Section 3 we will show a dimension formula:

PROPOSITION 3.1. *The element τ is not a zero-divisor in $H^*(M(p), k)$. Moreover the dimension of the cokernel of the homomorphism induced by multiplication by τ is $2(p+1)$ at each cohomology group $H^r(M(p), k)$. In particular one has*

$$\dim H^{r+2p}(M(p), k) = \dim H^r(M(p), k) + 2(p+1) \text{ for } r \geqq 0.$$

This will be established by investigating the cohomology exact sequences associated with the extension which corresponds to the norm τ. The extension we deal with is the mod p version of that of $\mathbf{Z}M(p)$-lattices in the section 6 in Lewis [15], where some homomorphisms of the integral cohomology groups associated with the extension was considered. Also shown in Section 3 is

LEMMA 3.3. *The second cohomology group $H^2(M(p), k)$ is four dimensional.*

Using these facts we will verify that the period of the module L_χ is in fact $2p$ if $n \equiv 1 \bmod p$ in Section 4.

If the underlying field k is not the prime field, then one can take a χ in $H^2(M(p), k)$. Therefore Lemma 2.1 and Proposition 3.1 together with Lemma 3.1 in Okuyama-Sasaki [17] imply that the cohomology algebra $H^*(M(p), k)$ is generated by χ, τ, and $\sum_{i=0}^{2p} H^i(M(p), k)$. We believe that this is useful to determine the cohomology algebra.

All modules considered are finitely generated right modules. Maps are written on the right with the convention of writing composites.

We fix some more notations. Let G be a finite group. The restriction of a kG-module M to a subgroup H of G is denoted $M_{|H}$. For ϕ a kG-homomorphism of kG-modules we denote by $\phi_{|H}$ the restriction of ϕ to H. If γ is an element in $H^r(G, k)$, then $\gamma_{|H}$ is the restriction of γ to H. And for δ an element in $H^r(H, k)$ we denote by $\delta^{\otimes G}$ the image of δ under the norm map from $H^r(H, k)$ to $H^{r|G:H|}(G, k)$. The restriction map from $H^r(G, M)$ to $H^r(H, M)$ is denoted by res^G_H. If there is no fear of confusion we omit the superscript G. The corestriction map from $H^r(H, M)$ to $H^r(G, M)$ is denoted by cor^G_H. We omit the subscript H if there is no fear of confusion.

Henceforth let G denote the p-group $M(p)$ unless otherwise stated.

2. The periodicity and the indecomposability

In this section we shall prove that the module L_χ is a periodic indecomposable $kM(p)$-module. We use the theory of the cohomology varieties associated with modules. For the convenience of the reader we write down the definition and some results we need here.

For G a finite group let $H(G)=\sum_{i\geq 0} H^{2i}(G, k)$. If M is a kG-module, then we let

$$r_G(M)=\text{rad ann}_{H(G)}H^*(G, M\otimes S),$$

where S is the direct sum of the simple kG-modules one for each isomorphism classes. The cohomology variety $X_G(M)$ associated with the module M is defined to be the prime spectrum which consists of the prime ideals containing $r_G(M)$:

$$X_G(M)=\text{Spec}(H(G),\ r_G(M)).$$

The module M is projective if and only if the variety $X_G(M)$ consists of only the irrelevant maximal ideal $H^+(G)=\sum_{i\geq 1} H^{2i}(G, k)$. The fundamental theorem is the following:

THEOREM (Alperin-Evens). *With the same notation as above it follows that*

$$r_G(M)=\bigcap \text{res}_E^{-1} r_E(M_{|E}),$$

where E runs over all elementary abelian p-subgroups of G.

Chouinard's theorem follows from the above theorem. The following lemma is obtained by mainly J. Carlson, See [8], [9], and [10], or Benson [3].

LEMMA. (1) *Let M and N be kG-modules. Then*

$$X_G(M\otimes N)=X_G(M)\cap X_G(N).$$

(2) *For γ an element in $H^r(G, k)$, one has*

$$X_G(L_\gamma)=X_G(\gamma).$$

Namely $r_G(L_\gamma)=\text{rad}(\gamma)$. The module L_γ is indecomposable if and only if $r_G(L\gamma)$ is a prime ideal.

Now let us proceed to our argument.

LEMMA 2.1. *The tensor product $L_\chi\otimes L_\tau$ is a projective $kM(p)$-module. In particular the module L_χ is periodic.*

PROOF. It is sufficient to show that the restriction of $L_\chi\otimes L_\tau$ to every maximal elementary abelian subgroup of G is projective. One has for H

a subgroup of G

$$(L_\chi\otimes L_\tau)_{|H}\simeq L_{(\chi_{|H})}\otimes L_{(\tau_{|H})}\oplus(\text{projective}).$$

Hence we see that

$$X_H(L_{(\chi_{|H})}\otimes L_{(\tau_{|H})})=X_H(\chi_{|H},\ \tau_{|H}).$$

Recall that

$$\chi=s_0\beta^n+s_1\alpha\beta^{n-1}+\cdots+s_{n-1}\alpha^{n-1}\beta+\alpha^n.$$

First we consider about restriction to the subgroups $\langle ab^i,\ c\rangle$s. For i, $0\leq i\leq p-1$, let $A_i=\langle ab^i,\ c\rangle$ and ξ_i be a Bockstein in $H^2(A_i,\ k)$ corresponding to the subgroup $\langle ab^i\rangle$. And put $\varsigma_i=\beta_{|A_i}$. Then ς_i is a Bockstein in $H^2(A_i,\ k)$ corresponding to the subgroup $\langle c\rangle$. Since $(\alpha-i\beta)_{|A_i}=0$, we have $\alpha_{|A_i}=i\varsigma_i$, so that

$$\begin{aligned}\chi_{|A_i}&=s_0\varsigma_i^n+s_1 i\varsigma_i^n+\cdots+s_{n-1}i^{n-1}\varsigma_i^n+i^n\varsigma_i^n\\&=f(i)\varsigma_i^n.\end{aligned}$$

On the other hand by Mackey formula for the norm map we get

$$\begin{aligned}\tau_{|A_i}&=\xi^{\otimes G}{}_{|A_i}\\&=(\xi_{|B\cap A_i})^{\otimes A_i}\\&=(\xi_{|\langle c\rangle})^{\otimes A_i}.\end{aligned}$$

Since the Bockstein ξ in $H^2(B,\ k)$ corresponds to the subgroup $\langle b\rangle$, its restriction to the subgroup $\langle c\rangle$ is not zero. Therefore we have, by Lemma 3. 1 in Okuyama-Sasaki [16],

$$(\xi_{|\langle c\rangle})^{\otimes A_i}=\xi_i^p-\varsigma_i^{p-1}\xi_i$$

and so

$$\tau_{|A_i}=\xi_i^p-\varsigma_i^{p-1}\xi_i.$$

Consequently we get

$$\begin{aligned}X_{A_i}(L_\chi\otimes L_\tau)&=X_{A_i}(\varsigma_i,\ \xi_i)\\&=\{0\}.\end{aligned}$$

Namely the restriction of the tensor product $L_\chi\otimes L_\tau$ to the subgroup A_i is projective, $i=0,\ 1,\dots,\ p-1$.

Next we deal with the subgroup B. Let $\varsigma=\alpha_{|B}$. Then ς is a Bockstein in $H^2(B,\ k)$ corresponding to the subgroup $\langle c\rangle$. Since $\beta_{|B}=0$, we have

$$\chi_{|B}=(\alpha_{|B})^n = \varsigma^n.$$

On the other hand by Mackey formula we get

$$\begin{aligned}\tau_{|B} &= \xi^{\otimes G}{}_{|B} \\ &= \prod_{i=0}^{p-1} \xi^{a^i} \\ &= \prod_{i=0}^{p-1} (\xi + i\varsigma) \\ &= \xi^p - \varsigma^{p-1}\xi,\end{aligned}$$

because $\xi^a=\varsigma+\xi$. Similarly to the former case, we see that the tensor product $L_\chi \otimes L_\tau$ is projective over kB.

Thus the tensor product $L_\chi \otimes L_\tau$ is a projective module. By the argument in the proof of Theorem 8.7 in Carlson [10] the module L_χ is periodic. This completes the proof of the lemma.

LEMMA 2.2. *The module L_χ is indecomposable.*

PROOF. By Lemma (2) it is enough to prove that the radical of the principal ideal (χ) in $H(G)$ is a homogeneous prime ideal. We use the same notations as in the proof of the previous lemma. Since the radical of the principal ideal (ς) in $H(B)$ is a homogeneous prime ideal, its inverse image $\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma))$ is a homogeneous prime ideal in $H(G)$. We shall show that

$$\mathrm{rad}(\chi)=\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma)),$$

which proves the lemma.

Since $\chi_{|B}=\varsigma^n$, it follows that

$$\mathrm{rad}(\chi)\subset\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma)).$$

Next for an element σ in $\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma))$ we can choose a number m such that

$$(\sigma_{|B})^m=\alpha_{|B}h(\alpha, \tau)_{|B}$$

for some polynomial $h(X, Y)$ in $k[X, Y]$, because $\sigma_{|B}$ is G-invariant and a non-nilpotent G-invariant element in $H(B)$ is a polynomial in $\alpha_{|B}$ and $\tau_{|B}$. By Lemma in Quillen-Venkov [19] the square $(\sigma^m-\alpha h(\alpha, \tau))^2$ is contained in the ideal (β). Thus the element σ is contained in the radical of the ideal (α, β). Namely we have

$$\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma))\subset\mathrm{rad}(\alpha, \beta).$$

Finally by a theorem of Serre [21] it follows that

$$(\prod_{i=0}^{p-1}(\alpha-i\beta))\beta=0.$$

Hence a minimal prime divisor $\mathfrak{p}$ of (χ) contains either one of the $(\alpha-i\beta)$s, $0\leq i\leq p-1$, or β. If β is contained in $\mathfrak{p}$, then so is the element α, beoanse $\chi\equiv 0 \bmod \mathfrak{p}$. If $\alpha-i\beta$ is contained in $\mathfrak{p}$, then we have

$$\begin{aligned}0&\equiv s_0\beta^n+s_1 i\beta^n+\cdots+s_{n-1}i^{n-1}\beta^n+i^n\beta^n\\&=f(i)\beta^n \bmod \mathfrak{p}.\end{aligned}$$

Both α and β are therefore contained in the prime ideal $\mathfrak{p}$ also in this case. Thus the elements α and β are contained in the radical of the ideal (χ). Namely we get

$$\mathrm{rad}(\alpha,\beta)\subset\mathrm{rad}(\chi).$$

Consequently we obtain the equality

$$\mathrm{rad}(\chi)=\mathrm{res}_B^{-1}(\mathrm{rad}(\varsigma)),$$

as desired.

REMARK. Serre's theorem was also proved by Okuyama-Sasaki [16].

3. A dimension formula

The Bockstein element ξ in $H^2(B, k)$ corresponding to the subgroup $\langle b\rangle$ corresponds to the following extension of kB-modules:

$$0\longrightarrow k\xrightarrow{\iota}k^B_{\langle b\rangle}\xrightarrow{\rho}k^B_{\langle b\rangle}\xrightarrow{\varepsilon}k\longrightarrow 0,$$

where

$$\begin{aligned}\iota &: k\longrightarrow k^B_{\langle b\rangle}\,;\ 1\longmapsto 1\otimes(c-1)^{p-1}\\ \rho &: k^B_{\langle b\rangle}\longrightarrow k^B_{\langle b\rangle}\,;\ 1\otimes 1\longmapsto 1\otimes(c-1)\end{aligned}$$

and

$$\varepsilon : k^B_{\langle b\rangle}\longrightarrow k\,;\ 1\otimes 1\longmapsto 1.$$

Regarding $k\xrightarrow{\iota}k^B_{\langle b\rangle}\xrightarrow{\rho}k^B_{\langle b\rangle}$ as a complex of kB-modules, we form its tensor induction to G (see Evens [13]):

$$E_{2p}\longrightarrow E_{2p-1}\longrightarrow E_{2p-2}\longrightarrow\cdots\longrightarrow E_j\longrightarrow\cdots\longrightarrow E_1\longrightarrow E_0.$$

Then the extension

$$0 \longrightarrow k \xrightarrow{\partial_{2p}} E_{2p-1} \xrightarrow{\partial_{2p-1}} E_{2p-2} \longrightarrow \cdots \xrightarrow{\partial_{j+1}} E_j \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \xrightarrow{\partial_0} k \longrightarrow 0$$

corresponds to the norm $\tau = \xi^{\otimes G}$ and each term satisfies the following:

$$E_0 \simeq \bigoplus_{i=0}^{p-1} k^G_{\langle ab^i \rangle} \oplus \text{ (projective)}$$
$$E_p \simeq \bigoplus_{i=0}^{p-1} k^G_{\langle ab^i \rangle} \oplus \text{ (projective)}$$
$$E_{2p-2} \simeq k^G_{\langle b \rangle} \oplus \text{ (projective)}$$
$$E_{2p-1} \simeq k^G_{\langle b \rangle}$$

and

other E_js are projective.

This can be verified as in Section 6 of Lewis [15], so that we omit the proof. Our aim in this section is to prove the following:

PROPOSITION 3.1. *The element* τ *is not a zero-divisor in* $H^*(M(p), k)$. *Moreover the dimension of the cokernel of the homomorphism induced by multiplication by* τ *is* $2(p+1)$ *at each cohomology group* $H^r(M(p), k)$. *In particular one has*

$$\dim H^{r+2p}(M(p), k) = \dim H^r(M(p), k) + 2(p+1) \quad \text{for } r \geq 0.$$

To prove this proposition we need the following lemma.

LEMMA 3.2. *Let* $E = \langle x, y \rangle$ *be an elementary abelian* p*-group of order* p^2, p *an odd prime. Then the homomorphism*

$$\bigoplus_{i=0}^{p-1} \mathrm{res}_{\langle xy^i \rangle} : H^r(E, k) \longrightarrow \bigoplus_{i=0}^{p-1} H^r(\langle xy^i \rangle, k)$$

is epimorphic when $r \geq 2p-3$.

PROOF. Let us denote by RES the above homomorphism. Let λ_x and λ_y be the elements in $H^1(E, k)$ such that

$$(x)\lambda_x = 0, \ (y)\lambda_x = 1 \text{ and } (x)\lambda_y = 1, \ (y)\lambda_y = 0,$$

and let β_x and β_y be the Bocksteins of λ_x and λ_y, respectively. Then one has

$$H^*(E, k) = k[\beta_x, \beta_y] \otimes \Lambda(\lambda_x, \lambda_y).$$

Let η_i be the element in $H^1(\langle xy^i \rangle, k)$ such that

$$(xy^i)\eta_i = 1$$

and let ς_i in $H^2(\langle xy^i \rangle, k)$ be its Bockstein, $i = 0, 1, \cdots, p-1$. Then one has

$$H^*(\langle xy^i\rangle, k)=k[\varsigma_i]\otimes\Lambda(\eta_i).$$

These elements satisfy the following relations:

$$\lambda_{x|\langle xy^i\rangle}=i\eta_i,\ \ \lambda_{y|\langle xy^i\rangle}=\eta_i$$

and

$$\beta_{x|\langle xy^i\rangle}=i\varsigma_i,\ \ \beta_{y|\langle xy^i\rangle}=\varsigma_i$$

When r is even we set $r=2s$. We can take a set $\{\beta_y^s, \beta_x\beta_y^{s-1}, \cdots, \beta_x^s, \beta_y^{s-1}\lambda_x\lambda_y, \beta_x\beta_y^{s-2}\lambda_x\lambda_y, \cdots, \beta_x^{s-1}\lambda_x\lambda_y\}$ as a basis of $H^{2s}(E, k)$. And we take the basis $\{\varsigma_i^s\}$ of $H^{2s}(\langle xy^i\rangle, k)$. With respect to these bases the homomorphism RES is represented by the following matrix:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & \cdots & i & \cdots & p-1 \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^j & \cdots & (p-1)^j \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^s & \cdots & (p-1)^s \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ & & & \cdots\cdots\cdots & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

The rank of this matrix is p if $s\geq p-1$ and $s+1$ otherwise.

When r is odd, we set $r=2s+1$. With respect to the bases $\{\beta_y^s\lambda_x, \beta_x\beta_y^{s-1}\lambda_x, \cdots, \beta_x^s\lambda_x, \beta_y^s\lambda_y, \beta_x\beta_y^{s-1}\lambda_y, \cdots, \beta_x^s\lambda_y\}$ of $H^{2s+1}(E, k)$ and $\{\varsigma_i^s\eta_i\}$ of $H^{2s+1}(\langle xy^i\rangle, k)$ the homomorphism RES is represented by the following matrix:

$$\begin{bmatrix} 0 & 1 & \cdots & i & \cdots & p-1 \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^{j+1} & \cdots & (p-1)^{j+1} \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^{s+1} & \cdots & (p-1)^{s+1} \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^j & \cdots & (p-1)^j \\ & & & \cdots\cdots\cdots & & \\ 0 & 1 & \cdots & i^s & \cdots & (p-1)^s \end{bmatrix}$$

The rank of this matrix is p if $s\geq p-2$ and $s+2$ otherwise. This completes the proof of the lemma.

PROOF OF PROPOSITION 3.1. Let $K_{i+1}=\mathrm{Ker}\ \partial_i$, $i=0, 1, \cdots, 2p-2$. The extension is decomposed into $2p$ short exact sequences:

$$0\longrightarrow K_1\xrightarrow{\kappa_0}E_0\xrightarrow{\partial_0}k\longrightarrow 0$$
$$0\longrightarrow K_{i+1}\xrightarrow{\kappa_i}E_i\xrightarrow{\partial_i}K_i\longrightarrow 0,\ 1\leq_i\leq 2p-2$$
$$0\longrightarrow k\xrightarrow{\kappa_{2p-1}}E_{2p-1}\xrightarrow{\partial_{2p-1}}K_{2p-1}\longrightarrow 0.$$

Associated with these short exact sequences there are $2p$ connecting homomorphisms:

$$\omega_0: H^r(G, k)\longrightarrow H^{r+1}(G, K_1)$$
$$\omega_i: H^{r+i}(G, K_i)\longrightarrow H^{r+i+1}(G, K_{i+1}),\ 1\leq i\leq 2\mathrm{p}-2$$
$$\omega_{2p-1}: H^{r+2p-1}(G, K_{2p-1})\longrightarrow H^{r+2p}(G, k).$$

The composition map of these connecting homomorphisms is exactly the homomorphism induced by multiplication by the element τ. Unless $j=0$, p, $2p-2$, and $2p-1$, the connecting homomorphism ω_j is isomorphic, for the module E_j is projective. In what follows we shall show that the others are all monomorphic so that the dimension of the cokernel of the homomorphism $(\cdot\tau)$ is the sum of those of the connecting homomorphisms. When we deal with one connecting homomorphism ω_j we will omit the index j from the notations κ_j and ∂_j. We shall use the theory of relative projective covers. See Knörr [14] for relative projective covers.

STEP 1. *The connecting homomorphism* ω_0 *is monomorphic and the dimension of its cokernel is* p.

PROOF. First we show that the induced homomorphism ∂_{0*} is the zero homomorphism in the exact cohomology sequence

$$\cdots\longrightarrow H^r(G, K_1)\xrightarrow{\kappa_{0*}}H^r(G, E_0)$$
$$\xrightarrow{\partial_{0*}}H^r(G, k)\xrightarrow{\omega_0}H^{r+1}(G, K_1)\longrightarrow\cdots.$$

Recall that

$$E_0\simeq\bigoplus_{i=0}^{p-1}k_{\langle ab^i\rangle}^G\oplus\ (\text{projective}).$$

Let $\nu_i: k_{\langle ab^i\rangle}^G\longrightarrow E_0$ be the injection with respect to the above decomposition and let $\delta_i=\nu_i\partial: k_{\langle ab^i\rangle}^G\longrightarrow k$, $0\leq i\leq p-1$. Then it follows that

$$\partial_* = \sum_{i=0}^{p-1} \delta_{i*}.$$

By our construction of the module E_0 and the homomorphism ∂_0, one can verify that each homomorphism δ_i is not the zero homomorphism. In particular the module E_0 is the direct sum of the relative $\{\langle ab^i\rangle | i=0, \cdots, p-1\}$-projective cover of the trivial module k and a projective module. Hence we have the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{i=0}^{p-1} H^r(G, k^G_{\langle ab^i\rangle}) & \xrightarrow{\sum_{i=0}^{p-1} \delta_{i*}} & H^r(G, k) \\
\text{Eckmann-Shapiro } \wr & & \uparrow \sum_{i=0}^{p-1} \mathrm{cor}^G_{\langle ab^i\rangle} \\
\bigoplus_{i=0}^{p-1} H^r(\langle ab^i\rangle, k) & \underset{\bigoplus_{i=0}^{p-1} \varepsilon_{i*}}{\overset{\sim}{\longrightarrow}} & \bigoplus_{i=0}^{p-1} H^r(\langle ab^i\rangle, k)
\end{array}$$

where $\varepsilon_i : k \longrightarrow k$ is the homomorphism corresponding to δ_i under the isomorphism of $\mathrm{Hom}_{kG}(k^G_{\langle ab^i\rangle}, k)$ to $\mathrm{Hom}_{k\langle ab^i\rangle}(k, k)$. But since the subgroup A_i is abelian it follows that

$$\begin{aligned}
\mathrm{cor}^G_{\langle ab^i\rangle} &= \mathrm{cor}^{A_i}_{\langle ab^i\rangle} \mathrm{cor}^G_{A_i} \\
&= 0,
\end{aligned}$$

and so we get

$$\partial_* = 0.$$

Thus we obtain the exact sequence

$$0 \longrightarrow H^r(G, k) \xrightarrow{\omega_0} H^{r+1}(G, K_1) \xrightarrow{\kappa_*} H^{r+1}(G, E_0) \longrightarrow 0,$$

therefore

$$\begin{aligned}
\dim \mathrm{Coker}\ \omega_0 &= \dim H^{r+1}(G, E_0) \\
&= \dim \bigoplus_{i=0}^{p-1} H^{r+1}(\langle ab^i\rangle, k) \\
&= p,
\end{aligned}$$

as desired.

STEP 2. *The connecting homomorphism ω_p is monomorphic and the dimension of its cokernel is p.*

PROOF. Recall that

$$E_p \simeq \bigoplus_{i=0}^{p-1} k^G_{\langle ab^i\rangle} \oplus P,$$

where P is a projective kG-module. Similarly to Step 1 it is enough to show that the induced homomorphism ∂_* is the zero homomorphism in the exact cohomology sequence

$$\cdots \longrightarrow H^{r+p}(G, K_{p+1}) \xrightarrow{\kappa_*} H^{r+p}(G, E_p) \xrightarrow{\partial_*} H^{r+p}(G, K_p)$$
$$\xrightarrow{\omega_p} H^{r+p+1}(G, K_{p+1}) \longrightarrow \cdots.$$

Let ϕ_i be the projection of E_p to $k^G_{\langle ab^i\rangle}$ and ν_i be the injection of $k^G_{\langle ab^i\rangle}$ to E_p, respectively, with respect to the direct decomposition above, $0 \leq i \leq p-1$. Then the homomorphism $\kappa\phi_i : K_{p+1} \longrightarrow k^G_{\langle ab^i\rangle}$ is not the zero homomorphism. For otherwise the induced module $k^G_{\langle ab^i\rangle}$ is a direct summand of the module K_p so that the module K_1 has a $\langle ab^i\rangle$-projective direct summand, since the modules $E_{p-1}, \cdots, E_1$ are all projective. But the module K_1 has no such direct summand, since the module E_0 is the direct sum of $\{\langle ab^i\rangle | i=0, \cdots, p-1\}$-projective cover of the trivial module k and a projective module. We set $\delta_i = \nu_i \partial : k^G_{\langle ab^i\rangle} \longrightarrow K_p$. As in the step 1, it is sufficient to show that $\delta_{i*}=0$ for $i=0, \cdots, p-1$. Let

$$\pi_i : k^G_{\langle ab^i\rangle} \longrightarrow k^{A_i}_{\langle ab^i\rangle} \text{ and } \theta_i : k^{A_i}_{\langle ab^i\rangle} \longrightarrow k^G_{\langle ab^i\rangle}$$

be the projection and the injection with respect to the kA_i-decomposition

$$k^G_{\langle ab^i\rangle | A_i} = k^{A_i}_{\langle ab^i\rangle} \oplus \sum_{1 \neq t \in A_i \backslash G} k^{A_i}_{\langle ab^i\rangle} \otimes t.$$

Then the homomorphism

$$\theta_i^* : \mathrm{Hom}_{kG}(k^G_{\langle ab^i\rangle}, K_p) \longrightarrow \mathrm{Hom}_{kA_i}(k^{A_i}_{\langle ab^i\rangle}, K_p)$$

is isomorphic. Let $\psi_i = \theta_i(\delta_{i|A_i})$. We obtain the following commutative diagram:

$$\begin{array}{ccccc}
 & & H^{r+p}(G, k^G_{\langle ab^i\rangle}) & \xrightarrow{\delta_{i*}} & H^{r+p}(A_i, k^G_{\langle ab^i\rangle}) \\
 & \mathrm{res}_{A_i} \swarrow & \wr\!\!\mid \; E-S & & \Big\uparrow \mathrm{cor}^G \\
H^{r+p}(A_i, k^G_{\langle ab^i\rangle}) & \xrightarrow{\pi_{i*}} & H^{r+p}(A_i, k^{A_i}_{\langle ab^i\rangle}) & \xrightarrow{\psi_{i*}} & H^{r+p}(A_i, K_p) \\
 & & \theta_{i*} \Big\uparrow & \nearrow & (\delta_{i|A_i})_* \\
 & & H^{r+p}(A_i, k^G_{\langle ab^i\rangle}) & &
\end{array}$$

where " E−S " means " Eckmann-Shapiro ". Namely we have

$$\begin{aligned}\delta_{i*} &= \mathrm{res}_{A_i}\pi_i\psi_{i*}\mathrm{cor}^G \\ &= \mathrm{res}_{A_i}\pi_{i*}\theta_{i*}(\delta_{i|A_i})_*\mathrm{cor}^G.\end{aligned}$$

Thus it is enough to prove that

$$(\delta_{i|A_i})_*=0.$$

Now since $E_{2p-1}, E_{2p-2}, \cdots, E_{p+1}$ are projective over kA_i, we see that

$$K_{p+1|A_i} \simeq \Omega^{-(p-1)}(k_{A_i}) \oplus (\text{projective})$$

and

$$k^G_{\langle ab^i\rangle|A_i} \simeq \bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle}.$$

These together with the fact that the homomorphism $\kappa\phi_i$ is not the zero homomorphism for each $i=0, \cdots, p-1$ give the following commutative diagram:

$$\begin{array}{ccc}
H^{r+p}(A_i, K_{p+1}) & \xrightarrow{(\kappa\phi_{i|A_i})_*} & H^{r+p}(A_i, \bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle}) \\
\wr\| & & \wr\| \ \text{dimension shifting} \\
H^{r+2p-1}(A_i, k) & \xrightarrow{\eta_*} & H^{r+2p-1}(A_i, \bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle}) \\
\bigoplus_{j=0}^{p-1}\mathrm{res}_{\langle ab^ic^j\rangle} \downarrow & & \downarrow \ \text{Eckmann-Shapiro} \\
\bigoplus_{j=0}^{p-1} H^{r+2p-1}(\langle ab^ic^j\rangle, k) & \underset{\nu_*}{\xrightarrow{\sim}} & \bigoplus_{j=0}^{p-1} H^{r+2p-1}(\langle ab^ic^j\rangle, k)
\end{array}$$

where the homomorphism $\eta: k \longrightarrow \bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle}$ is a kA_i-homomorphism which is determined by the homomorphism $\kappa\phi_{i|A_i}$ of K_{p+1} to $\bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle}$ through projective resolutions, and the homomorphism ν is the homomorphism which corresponds to η under the isomorphism of $\mathrm{Hom}_{kA_i}(k, \bigoplus_{j=0}^{p-1} k^{A_i}_{\langle ab^ic^j\rangle})$ to $\bigoplus_{j=0}^{p-1} Hom_{k\langle ab^ic^j\rangle}(k, k)$. The homomorphism $(\kappa\phi_{i|A_i})_*$ is epimorphic, because so is the homomorphism $\bigoplus_{j=0}^{p-1}\mathrm{res}_{\langle ab^ic^j\rangle}$ by Lemma 3.1. Namely $H^{r+p}(A_i, k^G_{\langle ab^i\rangle})(\nu_{i|A_i})_*$ is contained in the kernel of $(\partial_{|A_i})_*$. Consequently we have

$$(\delta_{i|A_i})_*=0,$$

as desired.

STEP 3. *The connecting homomorphism* ω_{2p-2} *is monomorphic and the dimension of its cokernel is* 1.

PROOF. Recall that

$$E_{2p-2} \simeq k^G_{\langle b\rangle} \oplus P,$$

where P is a projective kG-module. It is enough to show that the induced homomorphism ∂_* is the zero homomorphism in the exact cohomology sequence

$$\cdots \xrightarrow{\partial_*} H^{r+2p-2}(G, K_{2p-1}) \xrightarrow{\kappa_*} H^{r+2p-2}(G, E_{2p-2})$$
$$\xrightarrow{\partial_*} H^{r+2p-2}(G, K_{2p-1}) \xrightarrow{\omega_{2p-2}} H^{r+2p-1}(G, K_{2p-1}) \longrightarrow \cdots.$$

Let ν be the injection of $k^G_{\langle b\rangle}$ to E_{2p-2} with respect to the decomposition above. We set $\delta = \nu\partial : k^G_{\langle b\rangle} \longrightarrow K_{2p-2}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^{r+2p-2}(G, k^G_{\langle b\rangle}) & \xrightarrow{\delta_*} & H^{r+2p-2}(G, K_{2p-2}) \\ \text{Eckmann-Shapiro} \ \wr\!\!| & & \uparrow \mathrm{cor}^G \\ H^{r+2p-2}(\langle b\rangle, k) & \xrightarrow{\psi_*} & H^{r+2p-2}(\langle b\rangle, K_{2p-2}) \end{array}$$

where $\psi : k \longrightarrow K_{2p-2}$ is the kB-homomorphism whihc corresponds to the homomorphism δ under the isomorphism of $\mathrm{Hom}_{kG}(k^G_{\langle b\rangle}, K_{2p-2})$ to $\mathrm{Hom}_{k\langle b\rangle}(k, K_{2p-2})$. Since $K_{2p-2|B}$ is isomorphic with $\Omega^{2(p-1)}(k_B) \oplus$ (projective), we obtain

$$\begin{array}{ccc} H^{r+2p-2}(B, K_{2p-2}) & \simeq & H^r(B, k) \\ \mathrm{cor}^B \uparrow & & \uparrow \mathrm{cor}^B = 0 \\ H^{r+2p-2}(\langle b\rangle, K_{2p-2}) & \simeq & H^r(\langle b\rangle, k) \end{array}$$

Thus we have

$$\partial_* = 0.$$

STEP 4. *The connecting homomorphism* ω_{2p-1} *is monomorphic and the dimension of its cokernel is* 1.

PROOF. Recall that

$$E_{2p-1} \simeq k^G_{\langle b\rangle}.$$

By our construction one has that the homomorphism $\varkappa: k\longrightarrow E_{2p-1}$ is not the zero homomorphism. It is enough to show that the induced homomorphism ∂_* is the zero homomorphism in the exact cohomology sequence

$$\cdots\longrightarrow H^{r+2p-1}(G, k)\xrightarrow{\varkappa_*}H^{r+2p-1}(G, E_{2p-1})\xrightarrow{\partial_*}H^{r+2p-1}(G, K_{2p-1})$$
$$\xrightarrow{\omega_{2p-1}}H^{r+2p}(G, k)\longrightarrow\cdots.$$

Let $\pi: k^G_{\langle b\rangle}\longrightarrow k^B_{\langle b\rangle}$ and $\theta: k^B_{\langle b\rangle}\longrightarrow k^G_{\langle b\rangle}$ be the projection and the injection, respectively, with respect to the kB-decomposition

$$k^G_{\langle b\rangle}=k^B_{\langle b\rangle}\oplus\sum_{1\neq t\in B\backslash G}k^B_{\langle b\rangle}\otimes t.$$

Then we have

$$\partial_*=\mathrm{res}_B\pi_*\theta_*(\partial_{|B})_*\mathrm{cor}^G,$$

as in the proof of Step 2. Hence it is sufficient to verify that

$$(\partial_{|B})_*=0.$$

We have the commutative diagram:

$$\begin{array}{ccccc} H^{r+2p-1}(B, k) & \xrightarrow{(\varkappa_{|B})_*} & H^{r+2p-1}(G, k^G_{\langle b\rangle}) & \xrightarrow{(\partial_{|B})_*} & H^{r+2p-1}(B, K_{2p-1}) \\ \Big\downarrow \bigoplus_{j=0}^{p-1}\mathrm{res}_{\langle bc^j\rangle} & & & \wr\!\mid \text{ Eckmann-Shapiro} & \\ \bigoplus_{j=0}^{p-1}H^{r+2p-1}(\langle bc^j\rangle, k) & \underset{\nu_*}{\xrightarrow{\sim}} & \bigoplus_{j=0}^{p-1}H^{r+2p-1}(\langle bc^j\rangle, k) & & \end{array}$$

where ν is the homomorphism which corresponds to the kB-homomorphism $\varkappa_{|B}: k\longrightarrow\bigoplus_{j=0}^{p-1}k^B_{\langle bc^j\rangle}$ under the isomorphism of $\mathrm{Hom}_{kB}(k, \bigoplus_{j=0}^{p-1}k^B_{\langle bc^j\rangle})$ to $\bigoplus_{j=0}^{p-1}\mathrm{Hom}_{k\langle bc^j\rangle}(k, k)$. By Lemma 3.1 the homomorphism $\bigoplus_{j=0}^{p-1}\mathrm{res}_{\langle bc^j\rangle}$ is epimorphic, and so we have $(\partial_{|B})_*=0$, as desired.

Thus we have established Proposition 3.1.

The following is also needed in Section 4.

LEMMA 3.3. *The second cohomology group $H^2(M(p), k)$ is four dimensional.*

PROOF. This can be verified by the results in the section six in Lewis [15]. But we shall show this lemma by determining a minimal set of generators of the second syzygy $\Omega^2(k)$. For x an element in G let t_x denote the element $(x-1)^{p-1}$ in kG. Clearly the first syzygy $\Omega^1(k)$ is minimally generated by the elements $a-1$ and $b-1$ in kG. Let $\partial: kG\oplus kG \longrightarrow \Omega^1(k)$ be the essential epimorphism defined by

$$(x, y) \longmapsto (a-1)x+(b-1)y \text{ for } (x, y) \text{ in } kG\oplus kG.$$

Let us define four elements w_1, w_2, w_3, and w_4 in $kG\oplus kG$ as follows:

$$w_1=(t_a, 0),\ w_2=(0, t_b)$$
$$w_3=((ac-1)-b(a-1),\ (ac-1)(a-1))$$

and

$$w_4=((bc^{-1}-1)(b-1),\ (bc^{-1}-1)-a(b-1)).$$

It is easily checked that these elements are contained in the kernel $\Omega^2(k)$, by using the equality $ab=bac$. We shall show that the set $\{w_1, w_2, w_3, w_4\}$ is a minimal generating set of $\Omega^2(k)$. Since

$$w_1t_c=(t_at_c, 0)$$
$$w_2t_c=(0, t_bt_c)$$
$$w_3t_c=(-(a-1)(b-1)t_c,\ (a-1)^2t_c)$$

and

$$w_4t_c=((b-1)^2t_c,\ -(a-1)(b-1)t_c),$$

the elements $w_1(b-1)^i$, $0\leq i\leq p-1$, $w_2(a-1)^i$, $0\leq i\leq p-1$, $w_3(a-1)^i(b-1)^j$, $0\leq i\leq p-3$, $0\leq j\leq p-2$, and $w_4(b-1)^j$, $0\leq j\leq p-3$ generate over the group ring $k\langle c\rangle$ a projective $k\langle c\rangle$-module P which is isomorphic with the direct sum of p^2 copies of $k\langle c\rangle$, And one can verify that

$$w_3(a-1)^{p-2}t_b(c-1)^{p-2}=(2a(a-1)^{p-2}t_bt_c,\ t_at_bt_c).$$

This element is $\langle c\rangle$-invariant and is linearly independent to the $\langle c\rangle$-invariant subspace of the projective $k\langle c\rangle$-module P. Thus the kG-submodule generated by the elements w_1, w_2, w_3, and w_4 of $kG\oplus kG$ has dimension greater than or equal to p^3+1. Noting that the dimension of $\Omega^2(k)$ is p^3+1, we see that $\{w_1, w_2, w_3, w_4\}$ is a generating set of $\Omega^2(k)$. It is easily seen that no element of $\{w_1, w_2, w_3, w_4\}$ is contained in the submodule generated by the rest of the set. For example if w_1 is contained in the submodule generated by w_2, w_3, and w_4, then $w_1=w_2x+w_3y+w_4z$ for some x, y, and z in kG. Applying the element t_c to both sides, we have

$$(t_at_c, 0)=(-(a-1)(b-1)t_cy+(b-1)^2t_cz,$$
$$t_bt_cx+(a-1)^2t_cy-(a-1)(b-1)t_cz),$$

a contradiction. The proof is finished.

4. The period of the periodic module

The following lemma is our final goal.

LEMMA 4.1. *If $n \equiv 1$ mod p, then the period of the module L_χ is $2p$.*

PROOF. Lemma 2.1 implies that the module L_χ is periodic. Let L denote the module for short. Restricting the module L to the subgroup B, we see that the period of the module L is 2 or $2p$ by Lemma 4.4 in Benson-Carlson [4]. We must show that the period is not 2. We note that the homomorphism induced by the element χ

$$\chi : H^1(G, k) \longrightarrow H^{1+2n}(G, k)$$

is a monomorphism. In fact for an element $s\lambda + t\mu$ in $H^1(G, k)$, where s, $t \in k$, we have, letting A denote the maximal subgroup $\langle a, c\rangle$, that

$$((s\lambda + t\mu)\chi)_{|A} = t\mu_{|A} f(0)\beta^n{}_{|A}$$

and

$$((s\lambda + t\mu)\chi)_{|B} = s\lambda_{|B}\alpha_{|B}{}^n.$$

Hence if $(s\lambda + t\mu)\chi = 0$, then both s and t are zero. Therefore, since in the exact cohomology sequence

$$0 \longrightarrow \mathrm{Hom}_{kG}(k, k) \longrightarrow \mathrm{Hom}_{kG}(\Omega^{2n}(k), k) \longrightarrow \mathrm{Hom}_{kG}(L, k)$$
$$\xrightarrow{\omega} \mathrm{Ext}^1_{kG}(k, k) \xrightarrow{\hat{\chi}*} \mathrm{Ext}^1_{kG}(\Omega^{2n}(k), k) \longrightarrow \cdots$$

associated with the exact sequence

$$0 \longrightarrow L \longrightarrow \Omega^{2n}(k) \xrightarrow{\hat{\chi}} k \longrightarrow 0$$

it holds that

$$\begin{aligned} \mathrm{Im}\ \omega &\simeq \mathrm{Ker}\ (\cdot\chi) \\ &= 0, \end{aligned}$$

we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{kG}(k, k) \longrightarrow \mathrm{Hom}_{kG}(\Omega^{2n}(k), k) \longrightarrow \mathrm{Hom}_{kG}(L, k) \longrightarrow 0.$$

This implies that

$$\dim \mathrm{Hom}_{kG}(L, k) = \dim H^{2n}(G, k) - 1.$$

On the other hand since the socle of the module L is isomrphic with that of $\Omega^{2n}(k)$, we see that

$$\begin{aligned}\dim \operatorname{Soc}(L) &= \dim \operatorname{Soc}(\Omega^{2n}(k)) \\ &= \dim H^{2n-1}(G, k).\end{aligned}$$

Now suppose that $n \equiv 1 \bmod p$. Then we can put $n = mp+1$. By the dimension formula Proposition 3. 1 and Lemma 3. 3 we have

$$\dim H^{2n}(G, k) = 4 + 2m(p+1)$$

and

$$\dim H^{2n-1}(G, k) = 2 + 2m(p+1).$$

Thus there exist the following exact sequences:

$$0 \longrightarrow \Omega(L) \longrightarrow kG^{3+2m(p+1)} \longrightarrow L \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow kG^{2+2m(p+1)} \longrightarrow \Omega^{-1} \longrightarrow 0.$$

Consequently $\Omega(L)$ and $\Omega^{-1}(L)$ are not isomorphic. This proves the lemma.

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