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The F. and M. Riesz theorem on certain transformation groups, II

by

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§ 1. Introduction.

The classical F. and M. Riesz theorem was extended, by Helson-Lowdenslager and deLeeuw-Glicksberg, to compact abelian groups with ordered duals. As an extension of the result of deLeeuw and Glicksberg, Forelli extended the F. and M. Riesz theorem to a (topological) transformation group in which the reals $\boldsymbol{R}$ acts on a locally compact Hausdorff space.

On the other hand, the author ([14]) obtained several results, corresponding to Forelli's theorems, on a (topological) transformation group in which a compact abelian group acts on a locally compact Hausdorff space under certain conditions. In fact, the author obtained the following in [14].

THEOREM 1.1 (cf. [14, Theorem 1.1]). *Let (G, X) be a transformation group in which G is a compact abelian and X is a locally compact Hausdorff space. Suppose (G, X) satisfies conditions (C. I) and (C. II) (see* [14]). *Let P be a semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M(X)$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Suppose* $\mathrm{sp}(\mu)\subset P$. *Then both* $\mathrm{sp}(\mu_a)$ *and* $\mathrm{sp}(\mu_s)$ *are contained in P. If, in addition, $P\cap(-P)=\{0\}$ and $\pi(|\mu|)\ll\pi(\sigma)$, then* $\mathrm{sp}(\mu_s)\subset P\backslash\{0\}$, *where $\pi: X\to X/G$ is the canonical map.*

THEOREM 1.2 (cf. [14, Theorem 1.2]). *Let (G, X) be as in Theorem 1.1. Let E be a subset of $\hat{G}$ satisfying the following :*

(*) *For any nonzero measure $\lambda\in M_E(G)$, $|\lambda|$ and m_G are mutually absolutely continuous.*

Let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset E$. *Then μ is quasi-invariant.*

THEOREM 1.3 (cf. [14, Theorem 1.3]). *Let (G, X) be as in Theorem 1.1. Let E be a Riesz set in $\hat{G}$. Let μ be a measure in $M(X)$ with*

$\mathrm{sp}(\mu)\subset E$. *Then*

$$\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0,$$

where δ_g *denotes the point mass at* g.

THEOREM 1.4 (cf. [14, Theorem 1.4]). *Let* (G, X) *be as in Theorem 1.1. Let* σ *be a positive Radon measure on* X *that is quasi-invariant, and let* E *be a Riesz set in* $\hat{G}$. *Let* μ *be a measure in* $M(X)$ *with* $\mathrm{sp}(\mu)\subset E$. *Then both* $\mathrm{sp}(\mu_a)$ *and* $\mathrm{sp}(\mu_s)$ *are contained in* $\mathrm{sp}(\mu)$, *where* $\mu=\mu_a+\mu_s$ *is the Lebesgue decomposition of* μ *with respect to* σ.

If (G, X) is a transformation group in which a compact abelian group G acts freely on a locally compact Hausdorff space X or a transformation group in which G is a compact abelian group and X is a locally compact metric space, then (G, X) satisfies conditions (C.I) and (C.II) (cf. [14, Theorem 6.4 and Remark 6.1]). In this paper, we shall prove that Theorems 1.1-1.4 hold for a general (topological) transformation group (G, X) in which G is a compact abelian group and X is a locally compact Hausdorff space. In section 2, we state our results (Theorems 2.1-2.4). In section 3, we give proofs of Theorems 2.1 and 2.2, and we prove Theorems 2.3 and 2.4 in section 4.

§ 2. Notations and results.

Let (G, X) be a (topological) transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Suppose that the action of G on X is given by $(g, x)\to g\cdot x$, where $g\in G$ and $x\in X$.

Let $C_0(X)$ and $C_c(X)$ be the Banach space of continuous functions on X which vanish at infinity and the space of continuous functions on X with compact supports respectively. We note that, if $C_0(X)$ is separable, then X is metrizable (cf. [3, Theorem V.5.1, p. 426]). Let $M(X)$ be the Banach space of complex-valued bounded regular Borel measures on X with the total variation norm. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu\in M(X)$ and $f\in L^1(|\mu|)$, we often write $\mu(f)=\int_X f(x)d\mu(x)$. Let X' be another locally compact Hausdorff space, and let $S: X\to X'$ be a continuous map. For $\mu\in M(X)$, let $S(\mu)\in M(X')$ be the continuous image of μ under S. A (Borel) measure σ on X is called quasi-invariant if $|\sigma|(F)=0$ implies $|\sigma|(g\cdot F)=0$ for all $g\in G$.

Let $\hat{G}$ be the dual group of G. $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra respectively. For $\mu\in M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ. Let m_G be the Haar measure of G. Let $M_a(G)$ be the set of measures in $M(G)$ which are absolutely continuous with respect to m_G. Then by the Radon-Nikodym theorem we can identify $M_a(G)$ with $L^1(G)$. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. A subset E of $\hat{G}$ is called a Riesz set if $M_E(G)\subset L^1(G)$. For a closed subgroup H of G, $H^{\perp}$ denotes the annihilator of H.

For $\lambda\in M(G)$ and $\mu\in M(X)$, we define $\lambda*\mu\in M(X)$ by

$$(2.1)\qquad \lambda*\mu(f)=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu(x)=\int_G\int_X f(g\cdot x)d\mu(x)d\lambda(g)$$

for $f\in C_0(X)$. We note that (2.1) holds for all bounded Baire functions f on X.

REMARK 2.1. Professor Saeki pointed out that (2.1) holds for all bounded Borel functions f on X.

Let $J(\mu)$ be the collection of all $f\in L^1(G)$ with $f*\mu=0$.

DEFINITION 2.1. For $\mu\in M(X)$, we define the spectrum $\mathrm{sp}(\mu)$ of μ by $\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0)$.

We note that $\gamma\in\mathrm{sp}(\mu)$ if and only if $\gamma*\mu\neq 0$ (cf. [14, Remark 1.1 (II. 1)]).

DEFINITION 2.2. We say that $\mu\in M(X)$ translates G-continuously if $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$, where δ_g is the point mass at $g\in G$.

Let $M_{aG}(X)$ be an L-subspace of $M(X)$ defined by

$$M_{aG}(X)=\left\{\mu\in M(X): \begin{array}{l}\mu\ll\rho*\nu \text{ for some } \rho\in L^1(G)\cap M^+(G)\\ \text{and } \nu\in M^+(X)\end{array}\right\}.$$

Put $M_{aG}(X)^{\perp}=\{\nu\in M(X):\nu\perp\mu \text{ for all } \mu\in M_{aG}(X)\}$. Then $M_{aG}(X)^{\perp}$ is also an L-subspace of $M(X)$, and $M(X)=M_{aG}(X)\oplus M_{aG}(X)^{\perp}$. By [14, Proposition 5.1], we note that $\mu\in M_{aG}(X)$ if and only if μ translates G-continuously. Now we state our theorems.

THEOREM 2.1. *Let (G, X) be a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Then Theorem* 1.1 *holds for (G, X).*

THEOREM 2.2. *Let (G, X) be as in Theorem* 2.1. *Then Theorem* 1.2 *holds for (G, X).*

THEOREM 2.3. *Let (G, X) be as in Theorem 2.1. Then Theorem 1.3 holds for (G, X).*

THEOREM 2.4. *Let (G, X) be as in Theorem 2.1. Then Theorem 1.4 holds for (G, X).*

Before closing this section, we give several lemmas. Let (G, X) be a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Suppose there exists an equivalence relation " $\sim$ " on X such that $X/\sim$ is a locally compact Hausdorff space with respect to the quotient topology and $x\sim y$ implies $g\cdot x\sim g\cdot y$ for every $g\in G$. Let $\tau: X\rightarrow X/\sim$ be the canonical map. Define an action of G on $X/\sim$ by $g\cdot\tau(x)=\tau(g\cdot x)$ for $g\in G$ and $x\in X$. We assume that $(G, X/\sim)$ becomes a transformation group by this action. Let $\pi: X\rightarrow X/G$ and $\tilde{\pi}: X/\sim\rightarrow(X/\sim)/G$ be the canonical maps respectively. Then the following lemmas hold.

LEMMA 2.1. *For $\lambda\in M(G)$ and $\mu\in M(X)$, we have*

$$\tau(\lambda*\mu)=\lambda*\tau(\mu).$$

In particular, if $\sigma\in M^{+}(X)$ is quasi-invariant, then $\tau(\sigma)$ is also quasi-invariant.

PROOF. For $f\in C_c(X/\sim)$, we have

$$\begin{aligned}\lambda*\tau(\mu)(f)&=\int_G\int_{X/\sim}f(g\cdot\tilde{x})d\tau(\mu)(\tilde{x})d\lambda(g)\\&=\int_G\int_X f(g\cdot\tau(x))d\mu(x)d\lambda(g)\\&=\int_G\int_X f(\tau(g\cdot x))d\mu(x)d\lambda(g)\\&=\tau(\lambda*\mu)(f).\end{aligned}$$

Hence we have $\tau(\lambda*\mu)=\lambda*\tau(\mu)$. The latter half follows from the fact that $\delta_g*\tau(\sigma)=\tau(\delta_g*\sigma)\ll\tau(\sigma)$ for all $g\in G$. This completes the proof.

LEMMA 2.2. *Let μ be a measure in $M(X)$. Then*

$$\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu).$$

PROOF. By Lemma 2.1, we have $J(\mu)\subset J(\tau(\mu))$. Hence $\mathrm{sp}(\tau(\mu))=\bigcap_{f\in J(\tau(\mu))}\hat{f}^{-1}(0)\subset\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0)=\mathrm{sp}(\mu)$, and the proof is complete.

LEMMA 2.3. *Let μ and ω be measures in $M^{+}(X)$ such that $\pi(\mu)\ll\pi(\omega)$. Then $\tilde{\pi}(\tau(\mu))\ll\tilde{\pi}(\tau(\omega))$.*

PROOF. Let F be a closed set in $(X/\sim)/G$ with $\tilde{\pi}(\tau(\omega))(F)=0$. Then $\omega((\tilde{\pi}\circ\tau)^{-1}(F))=0$. We note that

$$(1)\qquad \pi^{-1}(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))=(\tilde{\pi}\circ\tau)^{-1}(F).$$

In fact, it is sufficient to show that $\pi^{-1}(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))\subset(\tilde{\pi}\circ\tau)^{-1}(F)$ because the reverse inclusion relation is trivial. For any $x\in\pi^{-1}(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))$, $\pi(x)\in\pi((\tilde{\pi}\circ\tau)^{-1}(F))$. Then there exist $y\in(\tilde{\pi}\circ\tau)^{-1}(F)$ and $g\in G$ such that $g\cdot x=y$. Hence

$$\begin{aligned}(\tilde{\pi}\circ\tau)(x)&=\tilde{\pi}(\tau(x))=\tilde{\pi}(g\cdot\tau(x))\\&=\tilde{\pi}(\tau(g\cdot x))=\tilde{\pi}(\tau(y))\\&\in F.\end{aligned}$$

Hence $x\in(\tilde{\pi}\circ\tau)^{-1}(F)$, and (1) holds. By (1), $\pi((\tilde{\pi}\circ\tau)^{-1}(F))$ is a closed set in X/G and

$$\begin{aligned}\pi(\omega)(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))&=\omega(\pi^{-1}(\pi((\tilde{\pi}\circ\tau)^{-1}(F))))\\&=\omega((\tilde{\pi}\circ\tau)^{-1}(F))\\&=0.\end{aligned}$$

Hence, by the hypothesis and (1), we have

$$\begin{aligned}0&=\pi(\mu)(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))=\mu(\pi^{-1}(\pi((\tilde{\pi}\circ\tau)^{-1}(F)))\\&=\mu((\tilde{\pi}\circ\tau)^{-1}(F))=\tilde{\pi}(\tau(\mu))(F).\end{aligned}$$

By regularity, we get $\tilde{\pi}(\tau(\mu))\ll\tilde{\pi}(\tau(\omega))$, and the proof is complete.

§ 3. Proofs of Theorems 2.1 and 2.2.

In this section we prove Theorems 2.1 and 2.2. The following lemma is useful in proving our theorems.

LEMMA 3.1. *Let (G, X) be a transformation group in which G is a compact abelian group and X is a σ-compact, locally compact Hausdorff space. Let μ_1 be a nonzero measure in $M(X)$, and let μ_2 and σ_2 be mutually singular measures in $M^+(X)$. Then there exists an equivalence relation " $\sim$ " on X with the following properties :*

(i) *$X/\sim$ is a (σ-compact) metrizable locally compact Hausdorff space with respect to the quotient topology ;*

(ii) *$(G, X/\sim)$ becomes a transformation group by the action*

(3.1) $g\cdot\tau(x)=\tau(g\cdot x)$ *for* $g\in G$ *and* $x\in X$ *;*

(iii) $\tau(\mu_1)\neq 0$ *;*

(iv) $\tau(\mu_2)\perp\tau(\sigma_2)$,

where $\tau: X/\sim$ *is the canonical map.*

PROOF. Since X is σ-compact, there exists an increasing sequence of compact sets X_n such that $X_n \subset \overset{\circ}{X}_{n+1}$ $(n=1, 2, 3, \cdots)$ and $X=\bigcup_{n=1}^{\infty} X_n$, where $\overset{\circ}{X}_n$ denotes the interior of X_n. Then, by Urysohn's lemma, there exists a function $h_n \in C_c(X_n)$ such that $h_n=1$ on X_n, $h_n=0$ on ${X_{n+1}}^c$ and $0 \leq h_n \leq 1$ on X. Since $\mu_1 \neq 0$, there exists $f_0 \in C_c(X)$ such that $\|f_0\|_\infty \leq 1$ and

(1) $\qquad \mu_1(f_0) \neq 0.$

Since $\mu_2 \perp \sigma_2$, there exists a sequence $\{f_n\}$ of functions in $C_c(X)$ such that $\|f_n\|_\infty \leq 1$ and

(2) $\qquad \sup_{n \geq 1} |(\mu_2 - \sigma_2)(f_n)| = \|\mu_2\| + \|\sigma_2\|.$

We define an equivalence relation " $\sim$ " on X by declaring $x \sim y$ if and only if

(3) $\qquad f_n(g \cdot x) = f_n(g \cdot y)$, $h_k(g \cdot x) = h_k(g \cdot y)$ for all $n \geq 0$, $k \geq 1$ and $g \in G$.

Then we have

(4) $\qquad x \sim y \Longleftrightarrow g \cdot x \sim g \cdot y$ for all $g \in G$.

Let $\tau: X \to X/\sim$ be the canonical map. For $x \in X$, $\tilde{x}$ denotes the equivalence class which contains x. We shall show that this equivalence relation satisfies (i)—(iv). For a subset S of $\boldsymbol{C}$, we note that

(5) $\qquad \tau^{-1}(\tau((f_n \circ g)^{-1}(S))) = (f_n \circ g)^{-1}(S)$ and
$\qquad \tau^{-1}(\tau(h_k \circ g)^{-1}(S))) = (h_k \circ g)^{-1}(S)$

for $n \geq 0$, $k \geq 1$ and $g \in G$, where $f_n \circ g(x) = f_n(g \cdot x)$ and $h_k \circ g(x) = h_k(g \cdot x)$.

In fact, it suffices to show that $\tau^{-1}(\tau((f_n \circ g)^{-1}(S))) = (f_n \circ g)^{-1}(S)$. And we may show that $\tau^{-1}(\tau((f_n \circ g)^{-1}(S))) \subset (f_n \circ g)^{-1}(S)$ because the reverse inclusion relation is trivial. Let $x \in \tau^{-1}(\tau((f_n \circ g)^{-1}(S)))$. Then $\tau(x) \in \tau((f_n \circ g)^{-1}(S))$. Hence $\tau(x) = \tau(x_*)$ for some $x_* \in (f_n \circ g)^{-1}(S)$. Since $x \sim x_*$, we have $f_n \circ g(x) = f_n \circ g(x_*) \in S$, and so $x \in (f_n \circ g)^{-1}(S)$. Thus (5) holds.

We first show that (i) holds. Let $\tau(x_1)$ and $\tau(x_2)$ be different elements in $X/\sim$. Then there exist f_n (or h_k) and $g \in G$ such that $f_n \circ g(x_1) \neq f_n \circ g(x_2)$. Let W_1 and W_2 be disjoint open sets in $\boldsymbol{C}$ such that $f_n \circ g(x_1) \in W_1$ and $f_n \circ g(x_2) \in W_2$. Define a function $\widetilde{f_n \circ g}$ on $X/\sim$ by

(6) $\qquad \widetilde{f_n \circ g}(\tau(x)) = f_n \circ g(x)$

for $x \in X$. This definition is well defined. In fact, if $\tau(x)=\tau(y)$, then $f_n \circ g(x)=f_n \circ g(y)$, and so $\widetilde{f_n \circ g}(\tau(x))=\widetilde{f_n \circ g}(\tau(y))$. It is obvious that $\widetilde{f_n \circ g}$ is a continuous function on $X/\sim$ (see Fig. I).

$$\begin{tikzcd} X \arrow[r, "f_n \circ g"] \arrow[d, "\tau"'] & C \\ X/\sim \arrow[ur, dash, "\widetilde{f_n \circ g}"'] & \end{tikzcd}$$

Fig. I

Hence $(\widetilde{f_n \circ g})^{-1}(W_1)$ and $(\widetilde{f_n \circ g})^{-1}(W_2)$ are disjoint open sets in $X/\sim$ such that $\tau(x_1) \in (\widetilde{f_n \circ g})^{-1}(W_1)$ and $\tau(x_2) \in (\widetilde{f_n \circ g})^{-1}(W_2)$, which shows that $X/\sim$ is a Hausdorff space. For $\tau(x) \in X/\sim$, there exists X_n such that $x \in X_n$. Then, by (5), we can verify that $\tau\left(h_n^{-1}\left(\left[\frac{1}{2}, 2\right]\right)\right)$ is a compact neighborhood of $\tau(x)$. Hence $X/\sim$ is a locally compact Hausdorff space. Next we show that $X/\sim$ is metrizable. For $f \in C_0(X)$, we note that $g \to f \circ g$ is a continuous mapping from G into $C_0(X)$. Hence $A=\bigcup_{n=0}^{\infty}\{f_n \circ g : g \in G\} \cup \bigcup_{k=1}^{\infty}\{h_k \circ g : g \in G\}$ is a σ-compact set in $C_0(X)$, and so it is separable. Hence there exists a countable dense subset $\{F_n\}$ of A. Define a function $\tilde{F}_n$ on $X/\sim$ by $\tilde{F}_n(\tau(x))=F_n(x)$ for $x \in X$. Then $\tilde{F}_n$ is a continuous function on $X/\sim$. Since $F_n \in C_c(X)$, we have $\tilde{F}_n \in C_c(X/\sim) \subset C_0(X/\sim)$.

Let $\mathscr{A}$ be a subalgebra of $C_0(X/\sim)$ generated by $\tilde{F}_n$ and $\bar{\tilde{F}}_n$ ($n=1, 2, 3, \cdots$). Then $\mathscr{A}$ separates points and is closed under complex conjugate. Moreover, for any $\tau(x) \in X/\sim$, there exists $L \in \mathscr{A}$ such that $L(\tau(x)) \neq 0$. In fact, there exists $k \in \boldsymbol{N}$ such that $x \in X_k$. Then $h_k(x)=1$. Hence there exists F_n such that $F_n(x) \neq 0$. Then $\tilde{F}_n \in \mathscr{A}$ and $\tilde{F}_n(\tau(x))=F_n(x) \neq 0$. Hence, by the Stone-Weierstrass theorem, $\mathscr{A}$ is dense in $C_0(X/\sim)$. By construction of $\mathscr{A}$ $C_0(X/\sim)$ is separable. Hence $X/\sim$ is metrizable, and (i) holds.

Next we show that (ii) holds. By (4), (3. 1) is well defined. We note that

$$\tau^{-1}(g \cdot \tilde{V})=g \cdot \tau^{-1}(\tilde{V}) \tag{7}$$

for $g \in G$ and a subset $\tilde{V}$ of $X/\sim$. For $g \in G$ and $x \in X$, let $\tilde{U}$ be an open set in $X/\sim$ containing $g \cdot \tau(x)$. Then there exists a compact neighborhood $\tilde{V}_x$ of $\tau(x)$ with $g \cdot \tilde{V}_x \subset \tilde{U}$ such that $\tau^{-1}(\tilde{V}_x)$ is a compact set in X. In fact, let n be a natural number such that $x \in X_n$. Then, by (5),

$\tau\left(h_n^{-1}\left(\left[\frac{1}{2}, 2\right]\right)\right)$ is a compact neighborhood of $\tau(x)$. It follows from (7) that $(-g)\cdot\tilde{U}$ is an open neighborhood of $\tau(x)$. Let $\tilde{U}(\tau(x))$ be a compact neighborhood of $\tau(x)$ such that $\tilde{U}(\tau(x))\subset(-g)\cdot\tilde{U}$. Set $\tilde{V}_x=\tilde{U}(\tau(x))\cap\tau\left(h_n^{-1}\left(\left[\frac{1}{2}, 2\right]\right)\right)$. Then $\tilde{V}_x$ is the desired one.

Let $y\in\tau^{-1}(\tilde{V}_x)$. Since $g\cdot y\in g\cdot\tau^{-1}(\tilde{V}_x)=\tau^{-1}(g\cdot\tilde{V}_x)\subset\tau^{-1}(\tilde{U})$, there exist an open neighborhood W_y of y and an open neighborhood $U_y(g)$ of g such that $U_y(g)\cdot W_y\subset\tau^{-1}(\tilde{U})$. Since $\tau^{-1}(\tilde{V}_x)$ is compact, there exist $y_1, y_2, \cdots, y_m\in\tau^{-1}(\tilde{V}_x)$ such that $\tau^{-1}(\tilde{V}_x)\subset\bigcup_{i=1}^{m} W_{y_i}$. Put $U(g)=\bigcap_{i=1}^{m} U_{y_i}(g)$. Then $U(g)$ is an open neighborhood of g, and $U(g)\cdot\tau^{-1}(\tilde{V}_x)$ is contained in $\tau^{-1}(\tilde{U})$. Hence we have, by (7),

$$\tau^{-1}(U(g)\cdot\tilde{V}_x)=U(g)\cdot\tau^{-1}(\tilde{V}_x)\subset\tau^{-1}(\tilde{U}),$$

which yields $U(g)\cdot\tilde{V}_x\subset\tilde{U}$. This shows that $(g, \tilde{x})\to g\cdot\tilde{x}$ is a continuous mapping from $G\times X/\sim$ onto $X/\sim$. It is easy to verify that

(8) $\tilde{x}\to g\cdot\tilde{x}$ is a homeomorphism on $X/\sim$ for each $g\in G$ and $0\cdot\tilde{x}=\tilde{x}$;

(9) $g_1\cdot(g_2\cdot\tilde{x})=(g_1+g_2)\cdot\tilde{x}$ for $g_1, g_2\in G$ and $\tilde{x}\in X/\sim$.

Hence $(G, X/\sim)$ becomes a transformation group, and (ii) holds.

Next we prove that (iii) holds. Define a function $\tilde{f}_0$ on $X/\sim$ by $\tilde{f}_0(\tau(x))=f_0(x)$. Then, as seen in the proof of (i), $\tilde{f}_0$ belongs to $C_0(X/\sim)$ and

$$\begin{aligned}\tau(\mu_1)(\tilde{f}_0)&=\int_{X/\sim}\tilde{f}_0(\tilde{x})d\tau(\mu_1)(\tilde{x})\\&=\int_X\tilde{f}_0(\tau(x))d\mu_1(x)\\&=\int_X f_0(x)d\mu_1(x)\\&\neq 0, \qquad\qquad \text{(by (1))}\end{aligned}$$

which shows that $\tau(\mu_1)\neq 0$. Thus (iii) holds.

Finally we prove that (iv) holds. Define functions $\tilde{f}_n$ on $X/\sim$ by $\tilde{f}_n(\tau(x))=f_n(x)$ $(n=1, 2, 3, \cdots)$. Then $\tilde{f}_n\in C_0(X/\sim)$, and we get

$$\begin{aligned}\|\tau(\mu_2)\|+\|\tau(\sigma_2)\|&\geq\|\tau(\mu_2)-\tau(\sigma_2)\|\\&\geq\sup_{n\geq 1}|(\tau(\mu_2)-\tau(\sigma_2))(\tilde{f}_n)|\\&=\sup_{n\geq 1}|(\mu_2-\sigma_2)(f_n)|\end{aligned}$$

$$=\|\mu_2\|+\|\sigma_2\| \qquad \text{(by (2))}$$
$$=\|\tau(\mu_2)\|+\|\tau(\sigma_2)\|.$$

Hence we have $\|\tau(\mu_2)-\tau(\sigma_2)\|=\|\tau(\mu_2)\|+\|\tau(\sigma_2)\|$, which shows that $\tau(\mu_2)\perp\tau(\sigma_2)$ because $\tau(\mu_2)$ and $\tau(\sigma_2)$ are positive measures. This completes the proof.

Now we prove Theorem 2.1. Let μ be a measure in $M(X)$, and let σ be a positive Radon measure on X that is quasi-invariant. Since μ is bounded and regular, there exist a σ-compact open set X_0 in X with $G\cdot X_0=X_0$ and a quasi-invariant measure $\sigma'\in M^+(X)$ satisfying the following:

(3.2) μ is concentrated on X_0,

(3.3) $\sigma'|_{X_0}$ and $\sigma|_{X_0}$ are mutually absolutely continuous.

Let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Then $\mu=\mu_a+\mu_s$ is also the Lebesgue decomposition of μ with respect to σ'. Thus, considering X_0 and σ' instead of X and σ if necessary, we may assume that X is a σ-compact locally compact Hausdorff space and σ is a quasi-invariant measure in $M^+(X)$.

Let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset P$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. In order to prove the first assertion, it suffices to prove that $\mathrm{sp}(\mu_s)\subset P$ because of [14, Remark 1.1 (II)]. We may assume that $\mu_s\neq 0$. Suppose there exists $\gamma_0\in\mathrm{sp}(\mu_s)$ such that $\gamma_0\notin P$. Then $\gamma_0*\mu_s\neq 0$. Hence, by Lemma 3.1, there exists an equivalence relation " $\sim$ " on X with the following properties:

(3.4) $X/\sim$ is a (σ-compact) metrizable, locally compact Hausdorff space with respect to the quotient topology;

(3.5) $(G, X/\sim)$ becomes a transformation group by the action $g\cdot\tau(x)=\tau(g\cdot x)$ for $g\in G$ and $x\in X$, where $\tau: X\to X/\sim$ is the canonical map;

(3.6) $\tau(\gamma_0*\mu_s)\neq 0$;

(3.7) $\tau(|\mu_s|)\perp\tau(\sigma)$.

By Lemma 2.1, $\tau(\sigma)$ is a quasi-invariant measure in $M^+(X/\sim)$. Since $\tau(|\mu_a|)\ll\tau(\sigma)$, (3.7) yields that $\tau(\mu_s)$ is the singular part of $\tau(\mu)$ with respect to $\tau(\sigma)$. It follows from Lemma 2.2 that $\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu)\subset P$. Hence, by (3.4) and Theorem 1.1, we have $\mathrm{sp}(\tau(\mu_s))\subset P$. On the other hand, by (3.6) and Lemma 2.1, we have $\gamma_0\in\mathrm{sp}(\tau(\mu_s))$, and so $\gamma_0\in P$. This contradicts the choice of γ_0. Hence $\mathrm{sp}(\mu_s)\subset P$.

Next we prove the second half of the theorem. It is sufficient to prove that $0\notin\mathrm{sp}(\mu_s)$. Suppose $0\in\mathrm{sp}(\mu_s)$. Then $1*\mu_s\neq 0$, where 1 is the constant function on G with value one. Hence, by Lemma 3.1, there exists an equivalence relation " $\approx$ " on X such that

(3.8) $X/\approx$ is a σ-compact metrizable, locally compact Hausdorff space with respect to the quotient topology,

(3.9) $(G, X/\approx)$ becomes a transformation group by the action $g\cdot\tau'(x) = \tau'(g\cdot x)$ for $g\in G$ and $x\in X$, where $\tau' : X\to X/\approx$ is the canonical map,

(3.10) $\tau'(1*\mu_s)\neq 0$, and

(3.11) $\tau'(|\mu_s|)\perp\tau'(\sigma)$.

Since $\tau'(|\mu_a|)\ll\tau'(\sigma)$, it follows from (3.11) that $\tau'(\mu)=\tau'(\mu_a)+\tau'(\mu_s)$ is the Lebesgue decomposition of $\tau'(\mu)$ with respect to $\tau'(\sigma)$. By Lemma 2.1, $\tau'(\sigma)$ is quasi-invariant. And, by Lemma 2.2, we have $\mathrm{sp}(\tau'(\mu))\subset P$. Let $\tilde{\pi} : X/\approx\to(X/\approx)/G$ be the canonical map. Then, by the hypothesis and Lemma 2.3, we have

$$\tilde{\pi}(|\tau'(\mu)|)\ll\tilde{\pi}(\tau'(|\mu|))\ll\tilde{\pi}(\tau'(\sigma)).$$

Since $X/\approx$ is metrizable, it follows from Theorem 1.1 that

$$\mathrm{sp}(\tau'(\mu_s))\subset P\backslash\{0\},$$

which yields

$$1*\tau'(\mu_s)=0$$

because $0\notin\mathrm{sp}(\tau'(\mu_s))$. Since $\tau'(1*\mu_s)=1*\tau'(\mu_s)$, this contradicts (3.10). Hence $0\notin\mathrm{sp}(\mu_s)$, and the proof is complete.

Next we prove Theorem 2.2. As seen in the proof of Theorem 2.1, we may assume that X is a σ-compact, locally compact Hausdorff space. Suppose μ is not quasi-invariant. Then there exists $g_0\in G$ such that $|\mu|$ is not absolutely continuous with respect to $\delta_{g_0}*|\mu|$. Let $\mu=\nu_1+\nu_2$ be the Lebesgue decomposition of μ with respect to $\delta_{g_0}*|\mu|$, where $\nu_1\ll\delta_{g_0}*|\mu|$ and $\nu_2\perp\delta_{g_0}*|\mu|$. Then $\nu_2\neq 0$. By Lemma 3.1, there exists an equivalence relation " $\sim$ ' on X satisfying (i)—(iv) in Lemma 3.1 with $\mu_1=\nu_2$, $\mu_2=|\nu_2|$ and $\sigma_2=\delta_{g_0}*|\mu|$.
By (iv) in Lemma 3.1, we have

(3.12) $\tau(|\nu_2|)\perp\tau(\delta_{g_0}*|\mu|)$,

where $\tau : X\to X/\sim$ is the canonical map. Since $|\nu_1|\ll\delta_{g_0}*|\mu|$, it follows from (3.12) that

(3.13) $\tau(|\nu_1|)\perp\tau(|\nu_2|)$.

By (3.12) and Lemma 2.1, we have

(3.14) $\quad |\tau(\nu_2)|\perp\delta_{g_0}*|\tau(\mu)|.$

Since $X/\sim$ is metrizable and $\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu)\subset E$, it follows from Theorem 1.2 that

(3.15) $\quad |\tau(\mu)|\ll\delta_{g_0}*|\tau(\mu)|.$

On the other hand, since $\tau(\mu)=\tau(\nu_1)+\tau(\nu_2)$, it follows from (3.13) that $|\tau(\nu_2)|\ll|\tau(\mu)|$. Hence, by (iii) in Lemma 3.1 and (3.15), we have $0\neq|\tau(\nu_2)|\ll\delta_{g_0}*|\tau(\mu)|$, which contradicts (3.14). Thus μ is quasi-invariant, and the proof is complete.

§ 4. Proofs of Theorems 2.3 and 2.4.

In this section we prove Theorems 2.3 and 2.4. We prepare a lemma.

LEMMA 4.1. *Let (G, X) be a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Let μ be a measure in $M_{aG}(X)$. Then $|\mu|\ll m_G*|\mu|$.*

PROOF. For a neighborhood V of 0 in G, let h_V be a nonnegative function in $L^1(G)$ with $\|h_V\|_1=1$ and $\mathrm{supp}(h_V)\subset V$. Then

(1) $\quad \lim_V\|h_V*|\mu|-|\mu|\|=0.$

In fact, for any $\varepsilon>0$, there exists a neighborhood V_0 of 0 in G such that $\|\delta_g*|\mu|-|\mu|\|<\varepsilon$ for all $g\in V_0$. Let V be a neighborhood of 0 with $V\subset V_0$. Then, for $f\in C_0(X)$ with $\|f\|_\infty\leq 1$, we have

$$\begin{aligned}
&|(h_V*|\mu|-|\mu|)(f)|\\
&=\Big|\int_G\int_X f(g\cdot x)d|\mu|(x)h_V(g)dm_G(g)\\
&\qquad-\int_G\int_X f(x)d|\mu|(x)h_V(g)dm_G(g)\Big|\\
&=\Big|\int_V\int_X f(x)d(\delta_g*|\mu|-|\mu|)(x)h_V(g)dm_G(g)\Big|\\
&\leq\|f\|_\infty\sup_{g\in V}\|\delta_g*|\mu|-|\mu|\|\\
&\leq\varepsilon,
\end{aligned}$$

which shows $\|h_V*|\mu|-|\mu|\|\leq\varepsilon$. Thus (1) holds. Since $h_V\in L^1(G)$, we get

(2) $\quad h_V*|\mu|\ll m_G*|\mu|.$

Hence the lemma follows from (1) and(2).

Now we prove Theorem 2.3. We may assume that X is σ-compact.

Let E be a Riesz set in $\hat{G}$, and let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset E$. Suppose that μ does not translate G-continuously. Let $\mu=\mu_1+\mu_2$, where $\mu_1\in M_{aG}(X)$ and $\mu_2\in M_{aG}(X)^{\perp}$. Then $\mu_2\neq 0$ and $|\mu_2|\perp m_G * |\mu_2|$. Hence, by Lemma 3.1, there exists an equivalence relation " $\sim$ " on X such that

(4.1) $X/\sim$ is a σ-compact metrizable, locally compact Hausdorff space with respect to the quotient topology,

(4.2) $(G, X/\sim)$ becomes a transformation group by the action $g\cdot\tau(x)=\tau(g\cdot x)$, where $\tau: X\to X/\sim$ is the canonical map,

(4.3) $\tau(\mu_2)\neq 0$, and

(4.4) $\tau(|\mu_2|)\perp\tau(m_G*|\mu_2|)$.

By Lemma 2.1, we have

(4.5) $\tau(M_{aG}(X))\subset M_{aG}(X/\sim)$.

Claim. $\tau(\mu)\notin M_{aG}(X/\sim)$.

By (4.5), it suffices to prove that $\tau(\mu_2)\notin M_{aG}(X/\sim)$. Suppose $\tau(\mu_2)\in M_{aG}(X/\sim)$. It follows from Lemma 4.1 that

(4.6) $|\tau(\mu_2)|\ll m_G*|\tau(\mu_2)|$.

Since $\tau(m_G * |\mu_2|)=m_G * \tau(|\mu_2|)$, (4.6) contradicts (4.3) and (4.4). Thus the claim holds.

Since $X/\sim$ is metrizable and $\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu)\subset E$, it follows from Theorem 1.3 that $\tau(\mu)$ translates G-continuously. Hence $\tau(\mu)$ belongs to $M_{aG}(X/\sim)$, which contradicts Claim. Hence μ translates G-continuously. This completes the proof of Theorem 2.3.

Finally we prove Theorem 2.4. As seen in the proof of Theorem 2.1, we may assume that X is σ-compact and $\sigma\in M^+(X)$. Let E be a Riesz set in $\hat{G}$. Let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset E$. Put $E_0=\mathrm{sp}(\mu)$. Let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. We may assume that $\mu_s\neq 0$. Suppose there exists $\gamma_0\in\mathrm{sp}(\mu_s)\backslash E_0$. Then $\gamma_0*\mu_s\neq 0$. By Lemma 3.1, there exists an equivalence relation " $\sim$ " on X satisfying (i)—(iv) in Lemma 3.1 with $\mu_1=\gamma_0 * \mu_s$, $\mu_2=|\mu_s|$ and $\sigma_2=\sigma$. Hence we have

(4.7) $\tau(\gamma_0*\mu_s)\neq 0$, and

(4.8) $\tau(|\mu_s|)\perp\tau(\sigma)$,

where $\tau: X\to X/\sim$ is the canonical map. Since $\tau(\mu_a)\ll\tau(\sigma)$, it follows from (4.8) that $\tau(\mu)=\tau(\mu_a)+\tau(\mu_s)$ is the Lebesgue decomposition of $\tau(\mu)$

with respect to $\tau(\sigma)$. By Lemma 2.1, $\tau(\sigma)$ is quasi-invariant. Since $X/\sim$ is metrizable and $\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu)=E_0$, it follows from Theorem 1.4 that

(4.9) $\quad \mathrm{sp}(\tau(\mu_s))\subset E_0.$

On the other hand, by (4.7), we have $\gamma_0*\tau(\mu_s)\neq 0$, which yields $\gamma_0\in\mathrm{sp}(\tau(\mu_s))$. Hence $\gamma_0\in E_0$, by (4.9). But this contradicts the choice of γ_0. Hence $\mathrm{sp}(\mu_s)\subset E_0=\mathrm{sp}(\mu)$, and so $\mathrm{sp}(\mu_a)=\mathrm{sp}(\mu-\mu_s)\subset\mathrm{sp}(\mu)$. This completes the proof.

§ 5. Appendix.

Let $\boldsymbol{T}$ and $\boldsymbol{Z}$ be the circle group and the integer group respectively. Let $\Phi: L^1(\boldsymbol{R})\to L^1(\boldsymbol{T})$ be a linear operator defined by

$$\Phi(f)(e^{ix})=\sum_{k\in\boldsymbol{Z}}2\pi f(x+2\pi k) \qquad (x\in[0,2\pi))$$

for $f\in L^1(\boldsymbol{R})$. Then $\|\Phi(f)\|_1=\frac{1}{2\pi}\int_0^{2\pi}|\Phi(f)(e^{ix})|dx\leq\int_{-\infty}^{\infty}|f(x)|dx=\|f\|_1$ for every $f\in L^1(\boldsymbol{R})$. Moreover $\Phi(f)^{\wedge}(n)=\hat{f}(n)$ for all $n\in\boldsymbol{Z}$.

Let $(\boldsymbol{T}, X)$ be a transformation group, in which $\boldsymbol{T}$ acts on a locally compact Hausdorff space X. Since the mapping $t\to e^{it}$ is a continuous homomorphism from $\boldsymbol{R}$ onto $\boldsymbol{T}$, we have a transformation group $(\boldsymbol{R}, X)$ by the action $t\cdot x=e^{it}\cdot x$ for $t\in\boldsymbol{R}$ and $x\in X$. Let μ be a measure in $M(X)$. For $f\in L^1(\boldsymbol{T})$ and $g\in L^1(\boldsymbol{R})$, convolutions $f*\mu\in M(X)$ and $g*\mu\in M(X)$ are defined as follows:

$$f*\mu(h)=\int_X\int_{\boldsymbol{T}}h(e^{it}\cdot x)f(e^{it})dm_{\boldsymbol{T}}(e^{it})d\mu(x) \text{ for } h\in C_0(X);$$

$$\begin{aligned} g*\mu(k)&=\int_X\int_{\boldsymbol{R}}k(t\cdot x)g(t)dtd\mu(x)\\ &=\int_X\int_{\boldsymbol{R}}k(e^{it}\cdot x)g(t)dtd\mu(x) \text{ for } k\in C_0(X). \end{aligned}$$

Put $J(\mu:\boldsymbol{T})=\{f\in L^1(\boldsymbol{T}): f*\mu=0\}$ and $J(\mu:\boldsymbol{R})=\{g\in L^1(\boldsymbol{R}): g*\mu=0\}$. Then $J(\mu:\boldsymbol{T})$ and $J(\mu:\boldsymbol{R})$ become closed ideals in $L^1(\boldsymbol{T})$ and $L^1(\boldsymbol{R})$ respectively. We define $\mathrm{sp}_{\boldsymbol{T}}(\mu)$ and $\mathrm{sp}_{\boldsymbol{R}}(\mu)$ as follows:

$$\mathrm{sp}_{\boldsymbol{T}}(\mu)=\bigcap_{f\in J(\mu:\boldsymbol{T})}\hat{f}^{-1}(0);$$

$$\mathrm{sp}_{\boldsymbol{R}}(\mu)=\bigcap_{g\in J(\mu:\boldsymbol{R})}\hat{g}^{-1}(0).$$

For $g\in L^1(\boldsymbol{R})$ and $k\in C_0(X)$, we have

$$\begin{aligned} g*\mu(k) &= \int_X \int_{\boldsymbol{R}} k(e^{it}\cdot x)g(t)dtd\mu(x) \\ &= \int_X \int_{\boldsymbol{T}} k(e^{it}\cdot x)\Phi(g)(e^{it})dm_T(e^{it})d\mu(x) \\ &= \Phi(g)*\mu(k). \end{aligned}$$

Thus we have

(5.1) $\quad g*\mu=\Phi(g)*\mu$ for $g\in L^1(\boldsymbol{R})$ and $\mu\in M(X)$.

If $\mathrm{sp}_{\boldsymbol{T}}(\mu)\subset \boldsymbol{Z}^+$, then (5.1) yields $\mathrm{sp}_{\boldsymbol{R}}(\mu)\subset\boldsymbol{R}^+$, where $\boldsymbol{Z}^+=\{n\in\boldsymbol{Z} : n\geq 0\}$ and $\boldsymbol{R}^+=\{x\in\boldsymbol{R} : x\geq 0\}$. Hence, by [5, Theorem 4] and the fact that $\delta_{e^{is}}*\mu=\delta_s*\mu$ for $s\in\boldsymbol{R}$, we have

(5.2) $\quad \lim_{t\to 0}\|\mu-\delta_{e^{it}}*\mu\|=0.$

(Of course, by [5, Theorem 3], μ is quasi-invariant.)

Let $\{n_k\}$ be a sequence of positive integers with $n_{k+1}/n_k>3$ ($k=1, 2, 3, \cdots$). Put $E=\boldsymbol{Z}^+\cup\{-n_k : k\in\boldsymbol{N}\}$. Let μ be a measure in $M(X)$ with $\mathrm{sp}_{\boldsymbol{T}}(\mu)\subset E$. Then we cannot get (5.2) from [5, Theorem 4]. On the other hand, it is known that E is a Riesz set (cf. [11, Corollary 4]). Hence we can get (5.2) from Theorem 2.3.

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