Hokkaido Mathematical Journal Vol. 21 (1992) p. 475-482

Singular extensions and restrictions of order continuous functionals[†]

Dedicated to Professor T. Andô on the occasion of his 60-th birthday

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(Received September 17, 1991, Revised April 8, 1992)

Abstract.

Let X be a vector sublattice of a vector lattice Y and let f be a positive functional on X. Generalizing a result of M. Valadier [Val] we show that for a large class of vector lattices X and Y as above, for each f there exists a singular (=localizable) extension to Y. If f is additionally anormal, then Theorem 2 asserts that there always exists an anormal extension. Let us mention also Theorem 5 which describes the situation when any positive order continuous functional has only singular positive extensions.

1. Introduction

The present article was inspired by the following unexpected result due to M. Valadier [Val]. Let λ denote the standard Lebesgue measure on $[0,1]$ and let $\psi_\lambda(x):=\int x d\lambda$, $x\in L_\infty[0,1]$. The space $C[0,1]$ of continuous functions is considered as a vector sublattice of $L_\infty[0,1]$. Valadier [Val] proved that the restriction $\psi_\lambda|C[0,1]$ of the order continuous functional ψ_λ admits a singular extension back to the whole space L_∞. In other words, the restriction of a "good" functional to a fairly large subspace $C[0,1]$ of L_∞ admits a "bad" extension back to L_∞. It was tempting to explain this result by the absence of non-trivial order continuous functionals on $C[0,1]$; however, it is not the case, and Theorem 1 shows that this phenomenon occurs rather often. After establishing this we investigate in a detailed way the problem of when each extension of a good functional is bad (Theorems 4 and 5), and find out that this is equiv-

[†]This work was made possible by a NATO Collaborative Research Grant CRG-890909 which allowed Y. A. Abramovich to visit the Queen's University of Belfast in December of 1990.

[‡]The first author takes this opportunity to thank the Queen's University of Belfast for hospitality.

alent to the opposite problem of when the restriction of each good functionlal from Y to X is bad (Theorem 4).

2. Preliminaries

We use the standard terminology regarding Banach and vector lattices; for the most part it may be found in [AB] and [Zaa]. All our vector lattices are assumed to be Archimedean. The only definitions that we need to recall in detail are those of different classes of functionals which will be used throughout the work.

Let Y be a vector lattice. Then $Y^{\sim}$ denotes the space of all order bounded functionals on Y. The band in $Y^{\sim}$ of all order continuous functionals (=normal integrals) is denoted by $Y_n^{\sim}$.

A functional $\psi \in Y^{\sim}$ is said to be anormal if there exists an order dense ideal J in Y such that $\psi|J=0$. The collections of all anormal functionals is denoted by $Y_{an}^{\sim}$. It is knows (see [Vul], Theorem IX. 4. 2) that $Y_{an}^{\sim}$ is an order dense ideal in $(Y_n^{\sim})^d$, the disjoint complement of $Y_n^{\sim}$ in $Y^{\sim}$. A well known theorem of G. Ya. Lozanovsky and Luxemburg-Zaanen (see [KA], Theorem X. 4. 6) asserts the equality $Y_{an}^{\sim}=(Y_n^{\sim})^d$ under a very mild assumption that $Y_n^{\sim}$ separates the points of Y.

Finally, following G. Ya. Lozanovsky [Loz], we say that a functional $\psi \in Y^{\sim}$ is localizable (=singular in the sense of Valadier) if there exists an increasing collection of bands B_α in Y such that $\psi|B_\alpha=0$ for each α and that the only element disjoint to each of these bands is $y=0$. We denote by $Y_{loc}^{\sim}$ the space of all localizable functionals on Y. Recall [Loz] that $Y_{loc}^{\sim}$ is an order dense ideal in $Y_{an}^{\sim}$, and in general, $Y_{loc}^{\sim} \neq Y_{an}^{\sim}$. However, a simple criterion due to Lozanovsky (see [Loz], Theorem 1) asserts that if Y is a Banach lattice and a functional $\psi \in Y_{an}^{\sim}$ is of countable type (=satisfies the countable sup property), then ψ is localizable. To sum up: in general the following inclusions hold $Y_{loc}^{\sim} \subseteq Y_{an}^{\sim} \subseteq (Y_n^{\sim})^d$, and if Y is a Banach lattice with the separating order continuous dual $Y_n^{\sim}$ and Y^* is of countable type, then all three spaces $(Y_n^{\sim})^d$, $Y_{an}^{\sim}$ and $Y_{loc}^{\sim}$ coincide. In particular, it is so for $Y=L_\infty[0,1]$ and $Y=\ell_\infty(\Gamma)$, where Γ is an arbitrary set.

At this point a word of warning is due. The term "singular functional" is overused in the existing literature. For example, according to [AB] and [Zaa] singular functionals are defined as elements of the band $(Y_c^{\sim})^d$, complementary to the band of order σ-continuous functionals, while according to [KA] the space of singular functionlals is defined as $Y_{an}^{\sim}$. And, as we mentioned above, Valadier's definition of singularity means $Y_{loc}^{\sim}$. Although the differences between these definitions are small and they often define the same object, nevertheless, to avoid any risk of misunderstanding, we refrain from using the term "singular functional" in the remainder of this paper, adhering instead to the precise and consistent terminology introduced above.

3. Results

In the first theorem, let K be a compact Hausdorff space containing a countable dense subset D; let Q be a compact Hausdorff hyperstonean space with no isolated points and let $\pi : Q \to K$ be a continuous surjection, so that we may consider $C(K)$ (or, more precisely, $C(K)\circ\pi$) as a sublattice of $C(Q)$. Recall [KA] that Q is called hyperstonean if there exists a regular Borel measure μ on Q such that $\mu(A)=0$ if and only if A is nowhere dense in Q. This implies in particular that $\mu(A)=0$ for each countable subset A of Q.

THEOREM 1. *Every bounded linear functional ϕ on $C(K)$ is the restriction of a localizable linear functional ψ on $C(Q)$. Moreover, if ϕ is positive, then ψ can also be chosen positive.*

PROOF. For each $k\in D$, choose an element $q_k\in\pi^{-1}(k)$. Let $D'=\{q_k : k\in D\}$. Since Q is hyperstonean with no isolated points, D' is nowhere dense in Q, so the ideal $J=\{f\in C(Q) : f|D'=0\}$ is order dense in $C(Q)$. The ideal J has trivial intersection with $C(K)\circ\pi$ as if $g\in C(K)$ and $g\circ\pi\in J$, then $g(\pi q_k)=0$ for all $k\in D$. That is, g vanishes on the dense subset D of K, and hence g is identically zero by continuity. We may thus consider the mapping of $C(K)\circ\pi+J$ onto $C(K)$ defined by $P : a\circ\pi+b\mapsto a$. This is clearly linear and has norm one as if $\|a\circ\pi+b\|\leq 1$, then for each $q_k\in D'$ we have $|a(\pi q_k)+b(q_k)|=|a(k)|\leq 1$. Thus $|a(k)\leq 1$ at all points of the dense subset D of K and hence, by continuity, on the whole of K. If $\phi\in C(K)^*$, then $\phi\circ P$ is a bounded linear functional defined on the subspace $C(K)\circ\pi+J$ of $C(Q)$, so extends to a bounded linear functional, ψ, on the whole of $C(Q)$, by the Hahn-Banach theorem. If $b\in J$ then $\psi(b)=\phi(Pb)=0$, so ψ is anormal, and consequently localizable. Clearly if $a\in C(K)$, then $\psi(a\circ\pi)=\phi(P(a\circ\pi))=\phi(a)$, so that ψ does indeed extend ϕ.

To finish the proof in the case when ϕ is positive, notice that the projection P is clearly a positive operator. Therefore the functional $\phi\circ P$ is also positive, and hence its extension ψ can be made positive, too. ■

Recall that $L_\infty[0,1]$ may be identified with $C(Q)$, where Q is the Stone space of the Banach lattice $L_\infty[0,1]$ (or, equivalently, Q is the maximal ideal space of the Banach algebra L_∞). It is well known (see, for example, [GJ], Chapter 10) that Q may be continuously mapped onto $K=[0,1]$. Thus the pair $Y=L_\infty[0,1]$ and $X=C[0,1]$ satisfies the conditions of the previous theorem. That is, Valadier's result is valid not only for ϕ_λ but for any $\phi\in(L_\infty)^*$.

The idea used in Theorem 1 for the concrete spaces L_∞ and $C(K)$ may be easily formalized as follows for a general setting. Let X be a closed vector sublattice of a Banach lattice Y satisfying $Y^*_{loc}=(Y^*_n)^d$ and assume that there exists a closed order dense ideal J in Y such that $X\cap J=\{0\}$ and $X+J$ is closed (the latter condition guarantees the continuity of the projection $P(x+j)=x$ from $X+J$ onto X). Then every bounded functional on X is a restriction of a localizable functional on Y. Notice that we have constructed the ideal J in Theorem 1 more or less explicitly. It is also possible to produce J with the desired properties *via* Zorn's lemma.

Our next result is a Hahn-Banach extension type theorem, which says that if a functional ϕ on a sublattice X of Y is anormal, then there always exists an anormal extension to all of Y.

THEOREM 2. *Let Y be a normed lattice and X a vector sublattice of Y. If $f\in X^*_{an}(:=X^*\cap X^\sim_{an})$, then there exists $\hat{f}\in Y^*_{an}$ with $\hat{f}|X=f$. Moreover, if $f>0$ then we may make $\hat{f}\geq 0$.*

PROOF. The proof will be divided into four simple steps.

1) If X is an order dense ideal in Y, then the statement is obvious, since we can take for $\hat{f}$ the standard minimal extension of f. Recall that if $f\geq 0$, then $\hat{f}$ is defined as follows: $\hat{f}(y)=\sup\{f(x): 0\leq x\leq y\}$ for $y\in Y^+$ and linearly for other elements in Y. If f is arbitrary, then we apply the previous procedure separately to f^+ and f^-. Clearly $\hat{f}\in Y^*_{an}$.

2) Let X be an dieal in Y. Consider $J=X+X^d$. Then J is as order dense ideal in Y and there is a trivial extension (still denoted by f) of f from X to J by letting f be zero on X^d. Obviously f is anormal on J, and hence we can apply step 1.)

3) Let X be arbitrary and assume that $f\geq 0$. Let $J=Y_X$ be the ideal in Y generated by X. It is a well known consequence of the standard Hahn-Banach extension theorem that there is a positive extension, $\bar{f}$, of f to J with $\|\bar{f}\|=\|f\|$. We claim that $\bar{f}\in J^*_{an}$. Since $N_f=\{x\in X: f(|x|)=0\}$ is an order dense ideal in X, it follows that the ideal J_{N_f} generated by N_f in J is an order dense ideal in J, and it is immediate that $\bar{f}|J_{N_f}=0$. That is, f is an anormal extension of f from X onto J. Now we can apply step 2) to $\bar{f}\in J^\sim_{an}$.

4) Let X and f be arbitrary. Applying step 3) to f^+ and f^- we

obtain a desired anormal extension. ■

REMARK. (1) We do not know whether a result similar to Theorem 2 is true for $f \in X^*_{loc}$ or $f \in (X^*_n)^d$. The only step in the proof above which does not go through is Step 3). Just to single this step out we have divided the proof of Theorem 2 into several steps; otherwise, for $f \in X^*_{an}$, the proof could be made slightly shorter.

(2) If Y is not assumed to be a normed space, or, in other words, if one deals with $f \in X^{\sim}_{an}$, then it may happen that f has no order bounded extension to Y at all. However, if we assume that some order bounded extension $\hat{f}$ exists, then basically the same proof shows that there exists an anormal extension too. The only modification we need is to apply the following version of the Hahn-Banach theorem. Let X be a vector sublattice of a vector lattice Y and let Z be a Dedekind complete vector lattice. If a regular operator $T : X \to Z$ admits a regular extension $\hat{T} : Y \to Z$, then T_+ admits a positive extension dominated by $(\hat{T})_+$.

In two previous results we claimed the existence of a particular type of extension. Now we are going to consider a situation when any extension is of this type. To this end we need a definition.

DEFINITION 3. Let Y be a Riesz space and X a vector sublattice of Y. Let us introduce the following two conditions on the pair (X, Y).

(R^+_{ns}) For any $0 \leq \psi \in Y^{\sim}_n$ its restriction $\psi|X \in (X^{\sim}_n)^d$.

(E^+_{ns}) For any $0 \leq \phi \in X^{\sim}_n$ any positive extension, $\hat{\phi}$, of ϕ to Y belongs to $(Y^{\sim}_n)^d 1$.

It is interesting that these two seemingly opposite properties are actually equivalent.

THEOREM 4. *For an arbitrary pair (X, Y) such that $X^{dd} = Y$, we have $(R^+_{ns}) \Leftrightarrow (E^+_{ns})$.*

PROOF. Suppose that (R^+_{ns}) holds and that $0 \leq \phi \in X^{\sim}_n$ has a positive extension $\hat{\phi}$ to Y which does not belong to $(Y^{\sim}_n)^d$. Let ψ denote the order continuous part of $\hat{\phi}$, so that $0 < \psi \leq \hat{\phi}$. It follows that $0 \leq \psi|X \leq \hat{\phi}|X = \phi$. Since ϕ is order continuous, it follows that $\psi|X$ is also order continuous. On the other hand, by (R^+_{ns}), the restriction $\psi|X$ is disjoint to $X^{\sim}_n$, so that $\psi|X = 0$. The positivity of ψ now forces ψ to be zero on the ideal generated by X in Y and, by order continuity, on the band generrated by X, which is the whole of Y. This contradicts the assumption that $\psi \neq 0$.

Now suppose that (E^+_{ns}) holds and let $0 \leq \psi \in Y^{\sim}_n$. Assume that $\phi = \psi|X$ does not belog to $(X^{\sim}_n)^d$, then we may write $\phi = \phi_n + \phi_s$, where $0 < \phi_n$

$\in X_n^\sim$ and $0\leq\phi_s\in(X_n^\sim)^d$. Since $0\leq\phi_n\leq\psi|X$, we may extend ϕ_n to a positive linear functional on the whole of Y dominated by ψ, which is therefore order continuous. This contradicts (E_{ns}^+). ■

REMARK. (1) Without the assumption that $X^{dd}=Y$ the implication $(\mathrm{R}_{ns}^+)\Rightarrow(\mathrm{E}_{ns}^+)$ clearly does not hold. For example, we can take any pair (X, Y) satisfying (R_{ns}^+) and replace Y by the direct sum of Y with $\boldsymbol{R}$ and X by $\{(x,0):x\in X\}$.

(2) If $Y_n^\sim$ separates the points of Y, then (E_{ns}^+) actually implies that $X^{dd}=Y$, since otherwise any positive order continuous functional on Y vanishing on X would be a counter-example.

(3) If $X_n^\sim=\{0\}$, then the pair (X, Y) automatically satisfies the condition (R_{ns}^+).

(4) There is a simple way of generalizing Definition 3 and Theorem 4 to operators. Let X, Y be as in Definition 3 and let Z be an arbitrary Dedekind complete vector lattice . We say that the pair (X, Y) satisfies $R_{ns}^+(Z)$ if for each positive order continuous operator $T\in L_n^\sim(Y,Z)$ its restriction $T|X$ is disjoint from the band $L_n^\sim(X,Z)$. Similarly, (X, Y) satisfies $E_{ns}^+(Z)$ if for any $0\leq T\in L_n^\sim(X,Z)$ any positive extension of T to Y is disjoint from $L_n^\sim(Y,Z)$. Now, imitating the proof of Theorem 4 we can easily show that $R_{ns}^+(Z)\Leftrightarrow E_{ns}^+(Z)$.

Next we are ready to present an example of a pair (X, Y) satisfying either of the equivalent conditions of the previous theorem.

THEOREM 5. *Let $X=C(Q)$ and $Y=\ell_\infty(Q)$, where Q is an arbitrary totally disconnected space without isolated points. Then the pair (X, Y) satisfies (E_{ns}^+) and (R_{ns}^+). Also, if $0\leq\phi\in X_n^\sim$ has an order continuous extension $\psi\in Y^\sim$, then $\psi=0$ (and hence, of course, $\phi=0$)*† *On the other hand, for each $0\leq\phi\in X_n^\sim$ there exists an extension ψ to Y which is not localizable.*

REMARK. As we mentioned in Section 2, for the space $Y=\ell_\infty(Q)$ all three spaces $(Y_n^\sim)^d$, $Y_{an}^\sim$ and $Y_{loc}^\sim$ coincide.

PROOF. In view of Theorem 4 it is enough to verify, say, (E_{ns}^+). Take any $0\leq\phi\in X_n^\sim$, and let ψ denote any positive extension of ϕ to Y. We will show that ψ is disjoint to the band $\ell_\infty(Q)_n^\sim$ of order continuous functionals on $\ell_\infty(Q)$. Recall that $\ell_\infty(Q)_n^\sim=\ell_1(Q)$. Suppose, contrary to

†Notice that ψ is not assumed to be positive, otherwise the desired conclusion that $\psi=0$ is immediate in view of (E_{ns}^+).

what we claim, that $\psi \notin (\ell_\infty(Q)_n^\sim)^d$. Then ψ dominates a non-zero multiple of evaluation at some point q of Q, that is, $\psi \geq \alpha\delta_q$. Consider the net $\{\chi_E\}$ of characteristic functions of all clopen subsets E of Q containing q. Obviously the infimum of this net in $X = C(Q)$ is zero, so the order continuity of ϕ guarantees that for any $\varepsilon > 0$ there is some E such that $0 \leq \phi(\chi_E) < \varepsilon$. As $0 \leq \chi_{\{q\}} \leq \chi_E$, we have $0 \leq \alpha = (\alpha\delta_q)(\chi_{\{q\}}) \leq \psi(\chi_{\{q\}}) \leq \psi(\chi_E) = \phi(\chi_E) < \varepsilon$. Thus $\alpha = 0$, a contradiction.

To prove our second statement suppose that ψ is an order continuous extension of ϕ to $\ell_\infty(Q)$. Assume that $\psi \neq 0$. Hence $\psi = \psi^+ - \psi^-$ and $\psi^+ \neq 0$ since otherwise it would imply that $\phi < 0$. Clearly $\phi \leq \psi^+ | C(Q)$. So the Hahn-Banach-Kantorovich extension theorem gives us a positive extension of ϕ to the whole of $\ell_\infty(Q)$ which is dominated by ψ^+, and hence this extension is order continuous, contradicting the part already proven.

To prove the last statement of the theorem, fix $q \in Q$. The functional $\delta_q \in C(Q)^*$ is anormal, so, by Theorem 2, it extends to an anormal functional $\hat{\delta}_q \in \ell_\infty(Q)^*_{an}$. It also allows a trivial order continuous extension, namely the very same evaluation at q which we will again denote by δ_q. Clearly δ_q and $\hat{\delta}_q$ are disjoint in $\ell_\infty(Q)^*$. If ϕ is any order continuous functional on $C(Q)$, then it extends to a positive functional, ψ, on $\ell_\infty(Q)$ which is localizable by (i). Now $\psi - \delta_q + \hat{\delta}_q$ is another extension of ϕ to $\ell_\infty(Q)$, and the order continuous part of this extension equals $-\delta_q$ which is non-zero. ∎

Note that we may also introduce analogues of both (R_{ns}^+) and (E_{ns}^+) which do not mention positivity. Let us denote them by (R_{ns}) and (E_{ns}). It is simple to verify that (R_{ns}) is equivalent to (R_{ns}^+) so it comes as a slight surprise to observe that condition (E_{ns}) is not equivalent to (E_{ns}^+). Indeed, the pair $(C(Q), \ell_\infty(Q))$ considered in Theorem 5 satisfies (E_{ns}^+), but does not satisfy (E_{ns}) by the concluding statement of the same theorem. However, this is not actually so surprising since as our last result shows condition (E_{ns}) is actually almost impossible to fulfill, except in rather trivial cases, such as when $Y_n^\sim = \{0\}$.

THEOREM 6. *If both $Y_n^\sim$ and $X_n^\sim$ separate points, then (X, Y) does not satisfy (E_{ns}).*

PROOF. Assume that we are wrong. Take $0 < \psi \in Y_n^\sim$ and let $\phi = \psi | X$. Since clearly $(E_{ns}) \Rightarrow (E_{ns}^+) \Rightarrow (R_{ns}^+)$ we have that $\phi \in (X_n^\sim)^d$. By the condition, $X_n^\sim$ separates the points of X, so $(X_n^\sim)^d = X_{an}^\sim$. Therefore, by Theorem 2, we may extend ϕ to $\hat{\phi} \in Y_{an}^\sim$. Consider now the linear functional $\psi - \hat{\phi} \in Y^\sim$. It extends the (order continuous) zero functional on X, and $\psi - \hat{\phi} \notin (Y_n^\sim)^d$, contradicting (E_{ns}). ∎

One concluding remark. Let X be vector sublattice of a vector lattice Y. A glance at the list of problems discussed in this article reveals that one very natural question has not been considered. Namely, when does any order continuous functional on X admit an order continuous extension to all of Y? In general the answer to this question is negative, since the existence of such an extension implies a strong relationship between X and Y. We refer to [And] for some deep positive results in this direction.

The authors would like to thank the referee for his valuable suggestions.

References

[AB] C. D. ALIPRANTIS and O. BURKINSHAW, *Positjve Operators*, Pure and Applied Mathematics Series #119, Academic Press, New York & London, 1985.

[And] T. ANDO, Extensions of linear functionals on Riesz spaces, *Indag. Math.*, **27** (1965), 388-395.

[GJ] L. GILLMAN and M. JERISON, *Rings of Continuous functions*, Van Nostrand, New York, 1960.

[KA] L. V. KANTOROVICH, G. P. AKILOV, *Functional Analysis*, Moskow, 1973.

[Loz] G. Ya. LOZANOVSKY, On localizable functionals in vector lattices, *Theory of Functions, Funct. Anal. and their Appl.*, Kharkov 1974, No 19, 66-80.

[Val] M. VALADIER, Une singulière forme linéaire singulière sur L^∞, *Seminaire d'analyse convexe*, Monpollier 1987, expose 4, 4.1-4.3.

[Vul] B. Z. VULIKH, *Introduction to the theory of partially ordered spaces*, Walters-Noordoff Sci. Publication, Groningen, 1967.

[Zaa] A. C. ZAANEN, *Riesz spaces II*, North Holland, Amsterdam, 1983.

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