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Blow-up solutions to a finite difference analogue of $u_t=\Delta u+u^{1+\alpha}$ in N-dimensional balls*

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§ 1. Introduction.

In this paper we consider asymptotic behaviours of difference solutions for a semilinear parabolic equation

(E) $$u_t=\Delta u+u^{1+\alpha},\ (t, x)\in(0, T)\times\Omega$$

with the boundary condition

(BC) $$(1-\sigma)u+\sigma\frac{\partial u}{\partial \boldsymbol{n}}=0 \text{ for } (t, x)\in(0, T)\times\partial\Omega$$

and initial value

(IV) $$u(0, x)=u_0(x),\ x\in\bar{\Omega}.$$

Here, $u_0\in C^1(\bar{\Omega})$, $\Omega=B(R)=\{x ; |x|<R\}(0<R<+\infty)$ is a ball in $\boldsymbol{R}^N$ and $\boldsymbol{n}$ is the outward normal of $\partial\Omega$, while $\sigma\in[0,1]$ and $\alpha>0$ are fixed constants. For convenience we refer to (BC) as (DBC) if $\sigma=0$ which gives the Dirichlet boundary condition, or refer to it as (NBC) if $\sigma=1$ which leads to the Neumann boundary condition.

It is well-known that a classical solution u of (E) may *blow up* in finite time, which means that its maximal existence time $T=\sup\{s ; u(t, x)$ is bounded in $[0, s]\times\Omega\}$ is finite and thus its maximum norm tends to infinity as $t\to T$. In this case, T is called the *blow-up time* of the solution and a *blow-up point* is a point in $\bar{\Omega}$ such that $u(t, x)$ is unbounded in any neighbourhood of it for $t\in[0, T)$. There are many works on the blow-up problem for semilinear parabolic equations (for instance, see [Fu1], [Fu2], [FuC], {C2], [FrM], [GK] and [W]).

On the other hand, numerical solutions and analogues for the equation

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(E) are also studied for the purpose of computing the blow-up solutions with computers, such as the finite difference methods and the finite element methods ([N], [NU]). Recently, the asymptotic behaviours of blow-up solutions of difference analogues are discussed ([C1], [BK]), for the one-dimensional problem of (E). The author studied in [C1] the asymptotic behaviours of the blow-up difference solutions near the blow-up point for a difference analogue of (E) with a variable time increment which was essentially presented in [N] and improved in [C1]. Later, a rescaling algorithm for the blow-up difference solutions was studied in [BK].

In [C1], the author proved that even if a difference solution blows up, its values will remain bounded up to the moment of blow-up except at the maximum point and its adjacent points; moreover, the number of blow-up (net) points depends in a way on the value of the parameter α provided that the initial value is not a constant and only has one maximum point.

Here, we are going to extend the results in [C1] to the multi-dimensional case $N \geq 2$, showing that the blow-up points of a difference solution will concentrate to its maximum points. We also present a difference scheme for (E) in the ball $B(R) \in \boldsymbol{R}^N$ ($N \geq 2$). We note that from the results of numerical experiments with this scheme we obtained important information for the investigation of the blow-up set of a classical solution (see [C2]).

For simplicity, we assume throughout this article that the initial value is radially symmetric, namely

$$u_0(x) = \phi(|x|) \text{ for } x \in \bar{\Omega},$$

where $\phi(r)$ is a nonnegative function satisfying the compatibility conditions needed. This implies that a unique classical solution $u(t, x)$ of (E) exists (at least locally), and is nonnegative and radially symmetric according to the uniqueness. Thus the solution can be written as $u(t, r)$ with $r = |x|$. Using the polar (spherical) coordinates we can rewrite the equation (E) into

$$\text{(E}') \qquad u_t = u_{rr} + \frac{N-1}{r} u_r + u^{1+\alpha}, \ (t, r) \in (0, T) \times (0, R)$$

with the boundary condition

$$\text{(BCO)} \qquad u_r(t, 0) = 0 \quad \text{for} \quad t \in (0, T),$$
$$\text{(BC)} \qquad (1-\sigma) u(t, R) + \sigma u_r(t, R) = 0 \quad \text{for } t \in (0, T)$$

and initial value

$$\text{(IV)} \qquad u(0, r) = \phi(r), \ r \in [0, R].$$

Here, (BCO) is obtained by noting the radial symmetry of the solution.

Our main results are stated in Theorem 3.2, including that if the initial value is radially decreasing and the difference solution for (E) blows up, namely, $\lim_{n\to\infty}\|u^n\|_\infty=+\infty$ and $\sum_{n=0}^{\infty}\tau_n<+\infty$, then the solution blows up in a sharp shape. Furthermore, if $0<\alpha\leq 1$ then the solution also blows up at the points adjacent to the maximum point which is the central point of Ω; while if $\alpha>1$ then there is only a single point for the solution to blow up. In particular, if $\alpha=1$, then the solution just blows up at the maximum point and the points around (adjacent to) it, but remains bounded at all of the rest points.

By the way, we note that our difference scheme has a good approximate accuracy in that the difference analogue for the Laplacian operator in a radially symmetric domain has an error estimate of order $O(h^2)$ uniformly up to the origin $r=0$.

Our difference scheme for (E′) is introduced in § 2 and the main theorem on the asymptotic behaviours of the difference solution is proved in § 3, with the analysis of error estimates for the difference approximation in § 4. Finally, we show several illustrations of numerical experiments for blow-up solutions with a personal computer by our difference scheme.

§ 2. The difference scheme and corresponding lemmas.

We state our finite difference scheme for the equation (E′).

Denoting by u_j^n the value of the differene solution at the n-th time step t_n and the spatial net point r_j, our difference scheme which is referred to as (S), is given by the following five equations, (S0)—(S2), (SB) and (SI):

$$\text{(S0)}\qquad \frac{u_0^{n+1}-u_0^n}{\tau_n}=2N\cdot\frac{u_1^{n+1}-u_0^{n+1}}{h^2}+(u_0^n)^{1+\alpha},$$

$$\text{(S1)}\qquad \frac{u_j^{n+1}-u_j^n}{\tau_n}=N\cdot\frac{u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}}{h^2}+(u_j^n)^{1+\alpha},\ 1\leq j<N_0,$$

$$\text{(S2)}\qquad \frac{u_j^{n+1}-u_j^n}{\tau_n}=\frac{u_{j-1}^{n+1}-2u_i^{n+1}+u_{j+1}^{n+1}}{h^2}+\frac{N-1}{r_j}\cdot\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h}$$
$$+(u_j^n)^{1+\alpha},\ N_0\leq j\leq m-1,\ n=0,1,2,\cdots;$$

$$\text{(SB)}\qquad (\sigma+(1-\sigma)h)u_m^n-\sigma u_{m-1}^n=0,\ n=1,2,\cdots;$$

$$\text{(SI)}\qquad u_j^0=\phi(r_j),\ j=0,\cdots,m.$$

Here, several notations have been introduced and will be used hereafter, as below.

It has been assumed that $h=R/m$ is the spatial mesh size of the division where m is the number of subintervals in the uniform division of the interval $[0,R]$, and $r_j=jh$ is the j-th net point on $[0,R](j=0,1,\cdots,m)$, with $N_0=[(N+1)/2]$ being the integral part of $(N+1)/2$. And t_n is assumed to be the n-th discrete time step and $\tau_n=t_{n+1}-t_n$ is the time increment.

The approximate relation between the difference solution and the corresponding classical solution is given by

$$u_j^n: \text{ approximate value of } u(t_n, r_j),\ j=0, 1, \cdots, m;$$
$$\frac{u_j^{n+1}-u_j^n}{\tau_n}: \text{ approximation of } u_t(t_n, r_j),\ 0\leq j<m,\ n\geq 0;$$
$$\frac{u_{j+1}^n-u_{j-1}^n}{2h}: \text{ approximation of } u_r(t_n, r_j),\ 1\leq j\leq m-1,\ n>0;$$
$$\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}: \text{ approximation of } u_{rr}(t_n, r_j),\ 0\leq j\leq m-1.$$

And for $j=0$ (i.e., $r_0=0$), the approximation of $\frac{u_r}{r}$ is taken as that of u_{rr} because $\lim_{r\to 0}\frac{u_r}{r}=u_{rr}$ (with $u_{-1}^n=u_1^n$ by the radial symmetry).

Note that the difference scheme (S) is implicit with respect to u_j^{n+1}. Here, it is easy to get the difference equations (S0), (S2), (SB) and (SI) in the scheme (S), by a backward Euler discretization with respect to the variable t. However, the equation (S1) seems to be unreasonable. The reason for introducing (S1) is that if (S1) is replaced by (S2) (for $j=1, \cdots, m-1$), then the maximum principle holding for a solution of (E) will not hold for the solution of (S) and we can give a counter-example indicating that the scheme is no more stable for $N>3$. The trouble appears from the fact that the discretization (the difference analogue) of the Laplacian operator turns out to be unstable since the coefficient matrix has no positive definiteness. The details of this problem will be discussed in § 4.

In the scheme (S), t_n is given by

$$t_0=0, \text{ and } t_n=t_{n-1}+\tau_{n-1}=\sum_{k=0}^{n-1}\tau_k \text{ for } n\geq 1$$

and the variable time increment τ_n is determined by

$$\tau_n=\tau\cdot\min(1, \|u^n\|_p^{-\alpha}) \text{ for a fixed } p\in[1, \infty] \tag{2.1}$$

where $\tau=\lambda h^2>0$ and $\lambda>0$ is a fixed constant, and the analogue of the L^p-norm $\|\cdot\|_p$ here is defined by

$$\|u^n\|_p=\begin{cases}\left(\sum_{j=0}^{m-1} hr_{j+1}^{N-1}(u_j^n)^p\right)^{1/p}, & \text{if } 1\leq p<\infty,\\ \max_j|u_j^n|, & \text{if } p=\infty.\end{cases}$$

It is easy to show that the coeficient matrix of $\{u_j^{n+1}\}$ in (S) is regular and the solution can be solved uniquely. And actually, we can prove a discrete version of the strong maximum principle and the comparison theorem for the solutions of the difference equation (S).

LEMMA 2.1. *Let $\{u_j^n\}$ and $\{v_j^n\}$ be two solutions of (S).*
(i) If $u_j^0 \geq v_j^0$ $(j=0, 1, \cdots, m)$ then

$$u_j^n \geq v_j^n \quad \text{for} \quad j=0, 1, \cdots, m-1 ; \ n=1, 2, \cdots.$$

(ii) The equality part of the inequality in (i) holds for a pair of j and n $(0 \leq j < m,\ n > 0)$ if and only if $u_j^0 = v_j^0$ for $j=0, 1, \cdots, m$.

The proof of Lemma 2.1 will be given in § 4.

In practical computation, the parameter p in the definition of τ_n is to be chosen suitably from 1, 2 or ∞, according to the problem concerned. We note that all norms $\|\cdot\|_p$ $(p \geq 1)$ here are equivalent for a fixed h because they are all norms in a finite-dimensional linear space; actually we have an evaluation of

$$C\|u^n\|_p \leq \|u^n\|_\infty \leq h^{-N/p}\|u^n\|_p \ (1 \leq p < \infty),$$

where C is a constant only depending on R.

On the other hand, we should also note the fact that for a fixed $p \in [1, \infty)$

$$\sup\{\|v\|_\infty / \|v\|_p ;\ v \in C[0, R]\} = h^{-N/p} \to \infty \text{ as } h \to 0$$

holds, where in the definition of $\|v\|$ the discrete function v_j is derived from the continuous function $v(r)$ by $v_j = v(r_j)$.

For convenience we refer to the difference scheme (S) as (SD) if $\sigma=0$, or we call it (SN) if $\sigma=0$. Under these assumptions, we have the following lemma which is concerned with the local convergence of the difference solutions to the corresponding solutions of (E).

LEMMA 2.2. *Let $u=u(t, r)$ be the classical solution of (E) in the domain $Q=[0, T)\times(0, R]$, and let u_j^n be the solution of (S). Assume that $0<S<T$ and $\lambda=\tau h^{-2}$ are fixed. If t_n lies in the interval $[O, S]$ and h is sufficiently small, then the following estimates*

$$\max_{0 \leq j < m} |u_j^k - u(t_k, r_j)| \leq C_0 h^2, \ k=0, 1, \cdots, n; \tag{2.2}$$

$$\max_{0 < j \leq m} \left| \frac{u_j^k - u_{j-1}^k}{h} - \frac{\partial u}{\partial r}(t_k, r_j) \right| \leq C_1 h, \ k=0, 1, \cdots, n; \tag{2.3a}$$

$$\max_{0 \leq j < m} \left| \frac{u_{j+1}^k - u_j^k}{h} - \frac{\partial u}{\partial r}(t_k, r_j) \right| \leq C_1 h, \ k=0, 1, \cdots, n; \tag{2.3b}$$

hold true, where C_0 and C_1 are constants depending only on u and S.

We shall give the proof of Lemma 2.2 in § 4.

For the case $N=1$, we can get the convergence of the numerical blow-up time to the blow-up time of the corresponding classical solution if the variable time increment is determined by (2. 1), as in [N] and [C1]. We just state this fact in the following Proposition 2.3 which was proved in [N] for $\alpha=1$ and $p=2$ under the Dirichlet boundary condition and in [C1] for $\alpha>0$ and $p=1$ under the Neumann boundary condition, for $N=1$. This is why we take the variable time increment as (2. 1).

PROPOSITION 2. 3. *Suppose the solution of (E) blows up at the blow-up time T. Assume(2. 1) with a fixed $\lambda>0$. Then*

$$\lim_{\tau\to 0}\tilde{T}(\tau)=T,$$

where $\tilde{T}(\tau)=\sum_{n=0}^{\infty}\tau_n$ is the blow-up time of the difference solution which depends on the parameter τ.

The proof of Proposition 2.3 and the detailed discussion on this problem will be omitted here.

We want to discuss the behaviour of the blow-up solutions of difference scheme (S) which are radially decreasing, and here we make the following assumption (A).

ASSUMPTION (A).

(1) $\phi(r)$ is nonnegative and radically monotone decreasing in $[0, R]$, i.e.,

$$\phi(r_1)\geq\phi(r_2) \text{ for } 0\leq r_1\leq r_2\leq R\,;$$

(2) $\phi(r)$ is not a constant.

Thus, the solution of (E) is also radially decreasing and it is easy to obtain.

LEMMA 2. 4. *Let $\{u_j^n\}$ be a solution of (S).*

(i) The assumption (A) implies $0<u_{j+1}^n<u_j^n$ for $j=0, \cdots, m-1$; $n\geq 1$;

(ii) If $\phi(r)\geq 0$ and $\phi(r)$ is not constant in $[0, R]$ then the solution of (SN) blows up, namely $\lim_{n\to\infty}\|u^n\|_\infty=+\infty$ but $\sum_{n=0}^{\infty}\tau_n<+\infty$, while the occurrence of blow-up ot the solution for (SD) depends on its initial value.

The proof of Lemma 2. 4. will be given in § 4, and as a consequence of (i), we can immediately get

COROLLARY 2. 5. *Assume (A), then*

$$u_0^n=\|u^n\|_\infty=\max_j u_j^n,$$
$$u_1^n=\max_{j\neq 0} u_j^n, \quad u_2^n=\max_{1<j\leq m} u_j^n$$

for $n=1, 2, \cdots$.

§ 3. The asymptotic behaviours of the blow-up difference solution.

We discuss in this section the asymptotic behaviour of a discrete solution $\{u_j^n\}$ computed by the scheme (S) for the case the solution blows up. Lemma 2.4 gives a blow-up condition for solutions of (SN), and for more information one can see {N] and [C1].

Before stating the main results, we introduce some notation to be used in the analysis of asymptotic behaviours of the difference solutions.

DEFINITION 3.1. Regarding a difference solution $\{u_j^n\}$, two sequences $\{a_n\}$ and $\{b_n\}$ are defined as

$$a_n=\frac{u_1^n}{u_0^n},\ b_n=\frac{(u_0^n)^\alpha}{\|u^n\|_p^\alpha},\ n=0, 1, \cdots.$$

Our main theorem is

THEOREM 3.2. *Let $\{u_j^n\}$ be a blow-up solution of (SN) or (SD). Then, under the assumption (A), the solution has the following properties (i)—(iii):*

(*i*) *For all $\alpha>0$, the blow-up takes place in a sharp shape, namely, the ratio $u_1^n/u_0^n=\max_{1\leq j\leq m}|u_j^n|/\max_{0\leq j\leq m}|u_j^n|$ tends to zero,*

$$\lim_{n\to\infty}(u_1^n/u_0^n)=\lim_{n\to\infty}a_n=0 \tag{3.1}$$

with

$$\lim_{n\to\infty}b_n=b>0, \tag{3.2}$$

where $b=h^{-N\alpha/p}$.

Moreover, if $\alpha\geq 1$ then

$$0\leq\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{1+\tau b}<1. \tag{3.3}$$

(*ii*) *If $\alpha\leq 1$, then the solution blows up even at the points adjacent to the maximum point, namely, the value u_1^n of the solution at the net point $r=r_1$ also tends to infinite:*

$$\lim_{n\to\infty}u_1^n=+\infty. \tag{3.4}$$

(iii) *If $\alpha\geq 1$, then the solution is bounded at net points apart from the maximum point. Actually, for $\alpha=1$ the solution is bounded except at the central (maximum) point and the points adjacent to (or, in other words, on the spherical surface around) it, namely, there is a constant $M=M(u_0, h)<\infty$ such that*

$$u_2^n=\max_{j\neq 0,1}\{u_j^n\}\leq M \text{ for all } n\geq 0, \tag{3.5}$$

However, if $\alpha>1$ then the solution is bounded except at the maximum point, namely, there is a constant $M=M(u_0, h)<\infty$ such that

$$u_1^n=\max_{j\neq 0}\{u_j^n\}\leq M \text{ for all } n\geq 0. \tag{3.6}$$

REMARK 3.3. It is indicated by Theorem 3.2 for the difference solution $\{u_j^n\}$ that if $\alpha>1$ then the discrete blow-up set consists of a single net point, while if $\alpha=1$ then the blow-up set consists of a single net point and the net points around it, in the circumstances.

PROOF OF THEOREM 3.2. First, we prove the statement (*ii*). Rewriting (S0)—(S2) in the scheme (S) yields

$$\text{(S0}'\text{)} \qquad (1+2N\lambda_n)u_0^{n+1}-2N\lambda_n u_1^{n+1}=(1+\tau_n(u_0^n)^\alpha)u_0^n,$$

$$\text{(S1}'\text{)} \qquad -N\lambda_n u_{j-1}^{n+1}+(1+2N\lambda_n)u_j^{n+1}-N\lambda_n u_{j+1}^{n+1}=(1+\tau_n(u_j^n)^\alpha)u_j^n,\ 1\leq j<N_0;$$

$$\text{(S2}'\text{)} \qquad -(1-\frac{N-1}{2j})\lambda_n u_{j-1}^{n+1}+(1+2\lambda_n)u_j^{n+1}-(1+\frac{N-1}{2j})\lambda_n u_{j+1}^{n+1}$$
$$=(1+\tau_n(u_j^n)^\alpha)u_j^n,\ N_0\leq j<m;\ n=0\,1,2,\cdots;$$

where $\lambda_n=\tau_n h^{-2}=\lambda\cdot\min\{1, \|u^n\|_p^{-\alpha}\}$. By (S0′) and Lemma 2.4, we have

$$u_0^{n+1}=\frac{2N\lambda_n u_1^{n+1}+(1+\tau_n(u_0^n)^\alpha)u_0^n}{1+2N\lambda_n} \geq\frac{(1+\tau_n(u_0^n)^\alpha)u_0^n}{1+2N\lambda_n}. \tag{3.7}$$

If $N\geq 3$ (namely, $N_0\geq 2$), then (S1′) applies and is followed by

$$u_1^n=\frac{N\lambda_n(u_0^{n+1}+u_2^{n+1})+(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2N\lambda_n} \geq\frac{N\lambda_n u_0^{n+1}+(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2N\lambda_n}. \tag{3.8}$$

From these inequalities it is easy to get

$$u_1^{n+1}\geq\frac{N\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{(1+2N\lambda_n)^2} \geq\frac{N\lambda_n u_0^n+u_1^n}{(1+2N\lambda_n)^2} \text{ for } n\geq 0 \text{ (if } N_0\geq 2). \tag{3.9}$$

If $N=1$ or 2 (namely, $N_0=1$), then merely (S0′) and (S2′) apply and the latter implies

$$(3.10)\qquad u_1^{n+1}=\frac{(1-\frac{N-1}{2})\lambda_n u_0^{n+1}+(1+\frac{N-1}{2})\lambda_n u_2^{n+1}+(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2\lambda_n}$$

$$\geq\frac{\frac{3-N}{2}\lambda_n u_0^{n+1}+(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2\lambda_n}$$

which with (3.7) yields

$$(3.11)\qquad u_1^{n+1}\geq\frac{\frac{3-N}{2}\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{(1+2N\lambda_n)(1+2\lambda_n)}$$

$$\geq\frac{\lambda_n u_0^n/2+u_1^n}{(1+2N\lambda_n)(1+2\lambda_n)},\quad n\geq 0\ (\text{if } N_0=1).$$

Either (3.9) or (3.11) leads to (3.4) because

$$\lambda_n u_0^n=\lambda b_n(u_0^n)^{1-\alpha}\geq\text{constant}>0\ \ (\text{for large } n),$$

by the definition of λ_n and b_n.

Next, we show the assertion (*i*). For the case $N_0=1$, from the equalities of (3.7) and (3.10) we can get

$$u_1^{n+1}\leq\frac{\frac{3-N}{2}\lambda_n(2N\lambda_n u_1^{n+1}+(1+\tau_n(u_0^n)^\alpha)u_0^n)}{(1+2\lambda_n)(1+2N\lambda_n)}+\frac{\frac{N+1}{2}\lambda_n u_1^{n+1}+(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2\lambda_n}$$

Solving u_1^{n+1} from this inequality yields

$$(3.12)\qquad u_1^{n+1}\leq\frac{\frac{3-N}{2}\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+\frac{3}{2}(N+1)\lambda_n}.$$

Similarly, if $N_0\geq 2$ then equalities of (3.7) and (3.8) lead to

$$(3.13)\qquad u_1^{n+1}\leq\frac{N\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+3N\lambda_n}$$

On the other hand, from the equality of (3.7) and inequality of (3.10), it follows when $N_0=1$ that

$$u_1^{n+1}\geq\frac{\frac{3-N}{2}\lambda_n(2N\lambda_n u_1^{n+1}+(1+\tau_n(u_0^n)^\alpha)u_0^n)}{(1+2\lambda_n)(1+2N\lambda_n)}+\frac{(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2\lambda_n},$$

and solving u_1^{n+1} from this inequality leads to

$$(3.14)\qquad u_1^{n+1} \geq \frac{\frac{3-N}{2}\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{1+2(N+1)\lambda_n+N(N+1)\lambda_n^2}$$

By the definition of a_n and the equality of (3.7), it is easy to see

$$a_{n+1}=\frac{1+2N\lambda_n}{2N\lambda_n+(1+\tau_n(u_0^n)^\alpha)u_0^n/u_1^{n+1}}.$$

Here, using (3.12) we get

$$a_{n+1}\leq\frac{(1+2N\lambda_n)(\frac{3-N}{2}\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n)}{(1+\frac{3}{2}(N+1)\lambda_n+N(3-N)\lambda_n^2)(1+\tau_n(u_0^n)^\alpha)u_0^n+2N\lambda_n(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}$$

$$\leq\frac{\frac{3-N}{2}\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n}{(1+\frac{3-N}{2}\lambda_n)(1+\tau_n(u_0^n)^\alpha)u_0^n+2N\lambda_n(1+\tau_n(u_1^n)^\alpha)u_1^n},$$

and thus for n sufficiently large,

$$(3.15)\qquad a_{n+1}\leq\frac{\frac{3-N}{2}\lambda_n(1+\tau b_n)+(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)a_n}{(1+\frac{3-N}{2}\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^\alpha)a_n}$$

since $\|u^n\|_p\geq 1$ and $\tau_n=\tau\|u^n\|_p^{-\alpha}$. Similarly, by using (3.14) we can obtain

$$(3.16)\qquad a_{n+1}\geq\frac{\frac{3-N}{2}\lambda_n(1+\tau b_n)+(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)a_n}{(1+2\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^\alpha)a_n}$$

for large n,

However, if $N_0\geq 2$, then by a similar argument we can derive from (3.7) and (3.8) the following

$$(3.17)\qquad a_{n+1}\leq\frac{N\lambda_n(1+\tau b_n)+(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)a_n}{(1+N\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^\alpha)a_n}$$

$$(3.18)\qquad a_{n+1}\geq\frac{N\lambda_n(1+\tau b_n)+(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)a_n}{(1+2N\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^\alpha)a_n}$$

Let $C=C(N)$ be a constant defined by

$$(3.19)\qquad C=C(N)=\begin{cases}\frac{3-N}{2}, & \text{if } N_0=1,\\ N, & \text{if } N_0\geq 2.\end{cases}$$

Then (3.15) and (3.17) can be written in a same form, as

$$a_{n+1}\leq\frac{C\lambda_n(1+\tau b_n)+(1+2N\lambda_n)(1+\tau b_n a_n^{\alpha})a_n}{(1+C\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n}. \tag{3.20}$$

And we can calculate

$$\begin{aligned}a_{n+1}-a_n&\leq\frac{C\lambda_n(1+\tau b_n)(1-a_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n(1-a_n)+\tau b_n a_n(a_n^{\alpha}-1)}{(1+C\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n}\\&\leq\frac{\tau_n(1-a_n)(C+2N)(1+\tau b_n)h^{-2}+\tau b_n a_n(a_n^{\alpha}-1)}{(1+C\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n}\end{aligned}$$

by $0<a_n<1$ for $n\geq1$.

Note the following

$$0<a_n^{\alpha}\leq a_n<1 \quad \text{if} \quad \alpha\geq1$$

and

$$0<a_n\leq a_n^{\alpha}<1,\ 1-a_n\leq(K+1)(1-a_n^{\alpha}) \text{ if } 0<\alpha<1,$$

where $K=[1/\alpha]$ (integral part of $1/\alpha$). The latter holds true because for $\alpha\in(0,1)$,

$$\begin{aligned}1-a_n&\leq1-a_n+a_n^{1-K\alpha}(1-a_n^{(K+1)\alpha-1})\\&=(1-a_n^{\alpha})(1+\sum_{i=1}^{K}a_n^{1-i\alpha})\\&\leq(K+1)(1-a_n^{\alpha}).\end{aligned} \tag{3.21}$$

For sufficiently large n, if $\alpha\geq1$ we obtain

$$a_{n+1}-a_n\leq\frac{\tau_n(1-a_n)(D(1+\tau b_n)h^{-2}-u_1^n(u_0^n)^{\alpha-1})}{(1+C\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n}; \tag{3.22}$$

while if $\alpha\in(0,1)$ we get

$$a_{n+1}-a_n\leq\frac{\tau_n(1-a_n)(D(K+1)(1+\tau b_n)h^{-2}-a_n(u_0^n)^{\alpha})}{(1+C\lambda_n)(1+\tau b_n)+2N\lambda_n(1+\tau b_n a_n^{\alpha})a_n}, \tag{3.23}$$

where $D=D(N)$ is a constant defined as

$$D(N)=C(N)+2N=\begin{cases}\dfrac{3(N+1)}{2}, & \text{if } N_0=1,\\ 3N, & \text{if } N_0\geq2.\end{cases}$$

Hence, if n is sufficiently large, then the right-hand side of (3.22) takes negative value because u_0^n tends to infinity, which means that if $\alpha\geq1$ then

$$0<a_n<a_{n+1}<1 \text{ for } n \text{ sufficiently large.}$$

Thus we see that $\lim_{n\to\infty} a_n = a$ exists, and $0 \le a = \lim_{n\to\infty} a_n < 1$ if $\alpha \ge 1$. Here, it should be noted that if the boundedness of $\{u_1^n\}$ is already known then (3.1) has already been proved; while if u_1^n can take very large value then it is monotone increasing for large n.

For the case of $\alpha \ge 1$, we can take a convergent subsequence of $\{b_n\}$, with its limit, say β being necessarily positive. Setting $n\to\infty$ and considering the limits of (3.15) and (3.16) along the corresponding subsequence, by $\lambda_n \to 0$, we see that the limit of $\{a_n\}$ satisfies

$$a = \frac{(1+\tau\beta a^\alpha)a}{1+\tau\beta}. \tag{3.24}$$

This leads to $a=0$ immediately and proves (3.1) and therefore (3.2)

Furthermore, since $\alpha \ge 1$ it follows that

$$\lim_{n\to\infty}(\lambda_n/a_n) = \lim_{n\to\infty}(\lambda b_n (u_0^n)^{1-\alpha}(u_1^n)^{-1}) = 0.$$

Thus, from (3.15) and (3.16) if $N_0=1$, or from (3.17) and (3.18) if $N_0 \ge 2$, we can obtain

$$\frac{1}{1+\tau b} \le \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \le \frac{1}{1+\tau b}$$

which proves (3.3).

It remains to prove (3.1) and (3.2) for the case of $0<\alpha<1$. We do this by reduction to absurdity. First we show the convergence of the sequence $\{a_n\}$, and then show that the limit is nothing but zero.

If $\{a_n\}$ is not a convergent sequence, then we have

$$0 \le a_* < a^* \le 1$$

where $a_* = \varliminf_{n\to\infty} a_n$ and $a^* = \varlimsup_{n\to\infty} a_n$. Thus there is a constant $\gamma \in (a_*, a^*)$ and three subsequences A, B and $\tilde{A}$ can be defined as

$$\begin{aligned} A &= \{a_n;\ a_n \le \gamma\}. \\ B &= \{a_n;\ a_n > \gamma\}, \\ \tilde{A} &= \{a_{n_i};\ a_{n_i} \in A,\ a_{n_i+1} \in B\}. \end{aligned}$$

Let $\{n_i\}$ be an index subsequence such that $a_{n_i} \in \tilde{A}$, and $\{n_k\}$ be subsequence of $\{n_i\}$ such that $\{b_{n_k}\}$ is a convergent subsequence of $\{b_{n_i}\} \subset \{b_n\}$. Substituting $\{n_k\}$ into (3.20) and setting $k\to\infty$, we get

$$\varlimsup_{n\to\infty} a_{n_k+1} \le \frac{1+\tau\beta\gamma^\alpha}{1+\tau\beta} \cdot \gamma < \gamma$$

where $\beta\in[1, h^{-N\alpha/p}]$ is the limit of $\{b_{n_k}\}$. This is a contradiction to $a_{n_k+1}\in B$, which proves the convergence of $\{a_n\}$.

Setting $n\to\infty$ in (3.15) and (3.16) if $N_0=1$, or in (3.17) and (3.18) if $N_0\geq 2$, along a subsequence $\{n_k\}$ such that $\{b_{n_k}\}$ converges to the limit $\beta>0$, we get (3.24) again. Thus, we see that either $a=1$ or $a=0$ holds true.

If $a=1$ holds, we should have

$$\lim_{n\to\infty} a_n(u_0^n)^\alpha=+\infty.$$

Then, however, by means of (3.23) we can obtain

$$a_{n+1}-a_n<0 \text{ for sufficintly large } n$$

which implies $a=\lim_{n\to\infty} a_n<1$, because $a_n<1$ for all $n\geq 1$. This is a contradiction which proves (3.1), and hence, (3.2) for $\alpha\in(0,1)$.

We note the following

$$\text{(3.25)} \qquad \lim_{n\to\infty}\frac{u_0^{n+1}}{u_0^n}=1+\tau b>1 \text{ (for all } \alpha>0)$$

and

$$\text{(3.26)} \qquad \text{if } \alpha\geq 1, \text{ then } 0<\sum_{n=0}^{\infty} a_n<\infty \text{ and } 1<\prod_{n=0}^{\infty}(1+\tilde{C}a_n)<\infty \text{ for any } \tilde{C}>0.$$

The relation (3.25) can be derived from the equation

$$1+2N\lambda_n(1-a_{n+1})=(1+\tau_n b_n\|u^n\|_p^\alpha)\frac{u_0^n}{u_0^{n+1}}$$

by (S0′). Taking the limit leads to

$$1=(1+\tau b)\cdot(\lim_{n\to\infty}(u_0^{n+1}/u_0^n))^{-1}$$

which is just (3.25). The statement (3.26) can be obtained from (3.3), noting that

$$\prod_{n=0}^{\infty}(1+\tilde{C}a_n)=\exp(\sum_{n=0}^{\infty}\log(1+\tilde{C}a_n))\leq\exp(\tilde{C}\sum_{n=0}^{\infty}a_n).$$

Now we are ready to prove (*iii*). First we consider the case when $\alpha>1$. If $N_0=1$, then by virtue of (3.12), we have

$$u_1^{n+1}\leq\frac{3-N}{2}\lambda_n(1+\tau b_n)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n \quad \text{for large } n,$$

and thus

$$u_1^{n+1}-u_1^n \le \frac{3-N}{2}\lambda_n(1+\tau b_n)u_0^n+(2N\lambda_n+\tau_n(u_1^n)^\alpha+2N\lambda_n\tau_n(u_1^n)^\alpha)u_1^n. \tag{3.27}$$

Similarly, if $N_0 \ge 2$ then using (3.13) we get

$$u_1^{n+1}-u_1^n \le N\lambda_n(1+\tau b_n)u_0^n+(2N\lambda_n+\tau_n(u_1^n)^\alpha+2N\lambda_n\tau_n(u_1^n)^\alpha)u_1^n, \tag{3.28}$$

and combining (3.27) and (3.28) we can write for all $N \ge 1$

$$u_1^{n+1}-u_1^n \le C\lambda_n(1+\tau b_n)u_0^n+(2N\lambda_n+\tau_n(u_1^n)^\alpha+2N\lambda_n\tau_n(u_1^n)^\alpha)u_1^n, \tag{3.29}$$

where $C=C(N)$ is the constant in (3.19).

Noting that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\lambda_{n+1}(1+\tau b_{n+1})u_0^{n+1}}{\lambda_n(1+\tau b_n)u_0^n}=\lim_{n\to\infty}\frac{\lambda b_{n+1}(u_0^{n+1})^{1-\alpha}}{\lambda b_n(u_0^n)^{1-\alpha}}=(1+\tau b)^{1-\alpha}<1,\\
&\lim_{n\to\infty}\frac{\lambda_{n+1}u_1^{n+1}}{\lambda_n u_1^n}=\lim_{n\to\infty}\frac{\lambda b_{n+1}(u_0^{n+1})^{-\alpha}u_1^{n+1}}{\lambda b_n(u_0^n)^{1-\alpha}u_1^n}=\lim_{n\to\infty}\frac{(u_0^{n+1})^{1-\alpha}a_{n+1}}{(u_0^n)^{1-\alpha}a_n}\\
&\quad=(1+\tau b)^{1-\alpha}\cdot(1+\tau b)^{-1}<1,\\
&\lim_{n\to\infty}\frac{\tau_{n+1}(u_1^{n+1})^{1+\alpha}}{\tau_n(u_1^n)^{1+\alpha}}=\lim_{n\to\infty}\frac{\tau b_{n+1}a_{n+1}^{1+\alpha}u_0^{n+1}}{\tau b_n a_n^{1+\alpha}u_0^n}=(1+\tau b)^{-(1+\alpha)}\cdot(1+\tau b)<1,
\end{aligned}$$

we obtain

$$\begin{aligned}
u_1^n&=\sum_{k=0}^{n-1}(u_1^{k+1}-u_1^k)+u_1^0\\
&\le\sum_{k=0}^{\infty}(C\lambda_n(1+\tau b_n)u_0^n+(2N\lambda_n+\tau_n(u_1^n)^\alpha+2N\lambda_n\tau_n(u_1^n)^\alpha)u_1^n)+u_1^0\\
&<+\infty,\quad n\ge 0.
\end{aligned} \tag{3.30}$$

Thus we have proved the boundedness of $\{u_1^n\}$ when $\alpha>1$.

For $\alpha=1$, we show the boundedness of $\{u_2^n\}$ as below. If $N_0 \le 2$, then by (S2′) it follows that

$$u_2^{n+1}\le\frac{\frac{5-N}{4}\lambda_n u_1^{n+1}+\frac{3+N}{4}\lambda_n u_2^{n+1}+(1+\tau_n(u_2^n)^\alpha)u_2^n}{1+2\lambda_n},$$

and thus, solving u_2^{n+1} from the inequality leads to

$$\begin{aligned}
u_2^{n+1}&\le\frac{\frac{5-N}{4}\lambda_n u_1^{n+1}+(1+\tau_n(u_2^n)^\alpha)u_2^n}{1+\frac{5-N}{4}\lambda_n}\\
&\le\frac{5-N}{4}\lambda_n u_1^{n+1}+(1+\tau_n(u_2^n)^\alpha)u_2^n,\quad n\ge 0.
\end{aligned} \tag{3.31}$$

By means of (3.12) and (3.13), we see

$$u_1^{n+1}\le N\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n$$

as in showing (3. 29), since $C(N)\leq N$. Substituting this inequality into (3. 31) yields

$$(3.32)\quad \begin{aligned} u_2^{n+1}+ &\leq \frac{5-N}{4}\lambda_n(N\lambda_n(1+\tau_n(u_0^n)^\alpha)u_0^n+(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n) \\ &\quad +(1+\tau_n(u_2^n)^\alpha)u_2^n \\ &=\frac{N(5-N)}{4}\lambda_n^2(1+\tau_n(u_0^n)^\alpha)u_0^n+\frac{5-N}{4}\lambda_n(1+2N\lambda_n)(1+\tau_n(u_1^n)^\alpha)u_1^n \\ &\quad +(1+\tau_n(u_2^n)^\alpha)u_2^n,\quad n\geq 0. \end{aligned}$$

Therefore, we have

$$(3.33)\quad \begin{aligned} u_2^{n+1} &\leq \frac{N(5-N)}{4}\lambda_n^2(1+\tau b_n)u_0^n+\frac{5-N}{4}\lambda_n(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)u_1^n \\ &\quad +(1+\tau b_n a_n^\alpha)u_2^n \\ &=A_n u_2^n+B_n \text{ for sufficiently large } n, \end{aligned}$$

where the sequences $\{A_n\}$ and $\{B_n\}$ are defined as

$$(3.34)\quad \begin{aligned} A_n &=1+\tau b_n a_n^\alpha, \\ B_n &=\frac{N(5-N)}{4}\lambda_n^2(1+\tau b_n)u_0^n+\frac{5-N}{4}\lambda_n(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)u_1^n. \end{aligned}$$

Noting $\sum_n a_n^\alpha<\infty$ and

$$(3.35)\quad 1<A_n\leq 1+\tau b a_n^\alpha,$$

we see

$$(3.36)\quad 1<\prod_{n=0}^{\infty}A_n<\infty.$$

On the other hand, by virtue of

$$\lim_{n\to\infty}\frac{\lambda_{n+1}^2(1+\tau b_{n+1})u_0^{n+1}}{\lambda_n^2(1+\tau b_n)u_0^n}=\lim_{n\to\infty}\frac{\lambda^2 b_{n+1}^2(u_0^{n+1})^{1-2\alpha}}{\lambda^2 b_n^2(u_0^n)^{1-2\alpha}}=(1+\tau b)^{1-2\alpha}<1$$

and

$$\begin{aligned} &\lim_{n\to\infty}\frac{\lambda_{n+1}(1+2N\lambda_{n+1})(1+\tau b_{n+1}a_{n+1}^\alpha)u_1^{n+1}}{\lambda_n(1+2N\lambda_n)(1+\tau b_n a_n^\alpha)u_1^n} \\ &=\lim_{n\to\infty}\frac{\lambda b_{n+1}a_{n+1}(u_0^{n+1})^{1-\alpha}}{\lambda b_n a_n(u_0^n)^{1-\alpha}}=(1+\tau b)^{-1}\cdot(1+\tau b)^{1-\alpha}<1, \end{aligned}$$

we get

$$(3.37)\quad \sum_{n=0}^{\infty}B_n<+\infty.$$

By the induction, we obtain

$$u_2^n \le u_2^0 \prod_{i=0}^{n-1} A_i + \left(\sum_{i=0}^{n-2} B_i \prod_{s=i+1}^{n-1} A_s + B_{n-1}\right)$$
$$\le u_2^0 \prod_{i=0}^{n} A_i + \sum_{i=0}^{n} B_i \prod_{s=0}^{n} A_s \text{ for large } n,$$

which immediately leads to

$$u_2^n \le \left(u_2^0 + \sum_{n=0}^{\infty} B_k\right) \prod_{n=0}^{\infty} A_k < +\infty \text{ for all } n \ge 0. \tag{3.38}$$

Thus we have proved (3.5) for the case of $N_0 \le 2$.

If $N_0 \ge 3$, then by (S1′) we have

$$u_2^{n+1} \le \frac{N\lambda_n(u_1^{n+1} + u_2^{n+1}) + (1 + \tau_n(u_2^n)^\alpha)u_2^n}{1 + 2N\lambda_n}.$$

Solving u_2^{n+1} from this inequality yields

$$u_2^{n+1} \le N\lambda_n u_1^{n+1} + (1 + \tau_n(u_2^n)^\alpha)u_2^n, \quad n \ge 0. \tag{3.39}$$

Hence, replacing $(5-N)/4$ by N in (3.31)—(3.34) and making the same argument as in the case of $N_0 \le 2$, we can get the same estimation (3.38) for the case of $N_0 \ge 3$, which completes the proof of Theorem 3.2. ■

REMARK 3.4: Theorem 3.2 gives very sharp estimates for the blow-up points, in that if $\alpha > 1$, the solution only has a single blow-up point, while when $\alpha = 1$ the blow-up set exactly consists of a single point and its adjacent net points.

Considering the fact that the boundary condition was not used explicitly in the proof, we can immediately get a similar result for solutions with the third boundary condition.

COROLLARY 3.5. *The conclusion of Theorem* 3.2 *remains valid when the boundary condition is replaced by* (*BC*) *with* $0 < \sigma < 1$. ■

§ 4. Proofs of lemmas and remarks.

The purpose of this section is to discuss the properties of the difference solutions and the accuracy of the difference scheme introduced in § 2.

PROOF OF LEMMA 2.1: To get the comparison principle, we consider $w_j^n = u_j^n - v_j^n$, which satisfies the following (4.1)—(4.5),

$$(1 + 2N\lambda_n)w_0^{n+1} - 2N\lambda_n w_1^{n+1} = w_0^n + \tau_n((u_0^n)^{1+\alpha} - (v_0^n)^{1+\alpha}), \tag{4.1}$$

(4.2) $$(1+2N\lambda_n)w_j^{n+1}-N\lambda_n(w_{j-1}^{n+1}+w_{j+1}^{n+1})=w_j^n+\tau_n((u_j^n)^{1+\alpha}-(v_j^n)^{1+\alpha}),$$
$1\leq j<N_0,\ n\geq 0$;

(4.3) $$-(1-\frac{N-1}{2j})\lambda_n w_{j-1}^{n+1}+(1+2\lambda_n)w_j^{n+1}-(1+\frac{N-1}{2j})\lambda_n w_{j+1}^{n+1}=w_j^n$$
$$+\tau_n((u_j^n)^{1+\alpha}-(v_j^n)^{1+\alpha}),\ N_0\leq j<m,\ n\geq 0;$$

(4.4) $$(1+(1-\sigma)h)w_m^n-\sigma w_{m-1}^n=0,\ n\geq 0;$$

(4.5) $$w_j^0=u_j^0-v_j^0\geq 0,\ j=0,1,\cdots,m.$$

First, we show

(4.6) $$w_j^n\geq 0,\ 0\leq j\leq m,\ n\geq 0$$

by the induction. By the assumption, it holds for $n=0$. Suppose it also holds for $n=k$ and assume $w_i^{k+1}=\min_j w_j^{k+1}$. If $i=0$, then

$$(1+2N\lambda_k)w_0^{k+1}-2N\lambda_k w_1^{k+1}\leq w_0^{k+1},$$

and thus (4.1) leads to

(4.7) $$w_0^{k+1}\geq w_0^k+\tau_k((u_0^k)^{1+\alpha}-(v_0^k)^{1+\alpha})\geq 0.$$

If $1\leq j<N_0$, then

$$(1+2N\lambda_k)w_i^{k+1}-N\lambda_k(w_{i-1}^{k+1}+w_{i+1}^{k+1})\leq w_i^{k+1}$$

and (4.2) implies

(4.8) $$w_i^{k+1}\geq w_i^k+\tau_k((u_i^k)^{1+\alpha}-(v_i^k)^{1+\alpha})\geq 0.$$

While if $N_0\leq i\leq m$, then because of

$$w_m^n=\frac{\sigma}{1+(1-\sigma)h}w_{m-1}^n$$

we may only consider the case when $i<m$. It is easy to see that (4.3) with

$$-(1-\frac{N-1}{2i})\lambda_k w_{i-1}^{k+1}+(1+2\lambda_k)w_i^{k+1}-(1+\frac{N-1}{2i})\lambda_k w_{i+1}^{k+1}\leq w_i^k$$

implies (4.8) again. Thus we have proved (i) of Lemma 2.1.

To verify the assertion (ii), we discuss the case of $n=1$, and the case of $n>1$ can be proved by the induction.

Let $w_i^0=u_i^0-v_i^0>0$ for some i. Then with the first part of the inequality (4.7) (when $i=0$) or (4.8) (when $i>0$) for $k=0$, we see

(4.9) $$w_i^1>0.$$

Noting $w_j^1 \geq 0$ for $j=0, 1, \cdots, m$, we can get

$$\begin{cases} w_j^1 \geq F_j^- w_{j-1}^1 > 0 & \text{if } j>1 \text{ and } w_{j-1}^1>0, \\ w_j^1 \geq \mathrm{F}_j^+ w_{j+1}^1 > 0 & \text{if } j<m \text{ and } w_{j+1}^1>0, \end{cases}$$

where

$$F^{\pm}=\begin{cases} 2N\lambda_n/(1+2N\lambda_n) & \text{if } j=0, \\ N\lambda_n/(1+2N\lambda_n) & \text{if } 1<j<N, \\ \left(1\pm\dfrac{N-1}{2j}\right)\lambda_n/(1+2\lambda_n) & \text{if } N_0\leq j<m. \end{cases}$$

Thus, by virtue of (4.9) we obtain (ii) of Lemma 2.1. ■

Before proving Lemma 2.2, we first discuss the accuracy of the analogue part of the Laplacian operator in the difference scheme. Let $v=v(|x|)=v(r)$ be a sufficiently smooth function defined on $\overline{\Omega}=\overline{B(R)}$, $v_j=v(r_j)$, and the discrete operator L be defined by

$$\mathrm{L}v_0=2N\cdot\frac{v_1-v_0}{h^2},$$

$$\mathrm{L}v_j=N\cdot\frac{v_{j-1}-2v_j+v_{j+1}}{h^2}, \quad 1\leq j<N_0,$$

$$\mathrm{L}v_j=\frac{v_{j-1}-2v_j+v_{j+1}}{h^2}+\frac{N-1}{r_j}\cdot\frac{v_{j+1}-v_{j-1}}{2h}, \quad N_0\leq j\leq m-1,$$

with

$$(\sigma+(1-\sigma)h)v_m-\sigma v_{m-1}=0,$$

where the symbols used here are similar to those introduced in § 2.

Without loss of generality we only discuss the case when the Dirichlet boundary condition is assumed. For the discrete operator L, we heve the following theorem concerning the error estimate between L and the Laplacian operator Δ.

THEOREM 4.1. *Let $h=R/m$ be fixed. If h is sufficiently small, then*

$$\max_{0\leq j\leq m-1}|\mathrm{L}v_j-\Delta v(r_j)|=O(h^2). \tag{4.10}$$

PROOF : For $j=0$, we see

$$\mathrm{L}v_0=2Nh^{-2}\left(v(0)+hv_r(0)+\frac{h^2}{2}v_{rr}(0)+\frac{h^3}{6}v_{rrr}(0)+\frac{h^4}{24}v_{rrrr}(\tilde{h})-v(0)\right)$$

$$=N\left(v_{rr}(0)+\frac{h^2}{12}v_{rrrr}(\tilde{h})\right)$$

for some $\tilde{h} \in (0, h)$, by using

$$v_r(0) = v_{rrr}(0) = 0.$$

For $j = 1, 2, \cdots, N_0 - 1$, we have

$$\begin{aligned}
Lv_j &= Nh^{-2}(v(r_j) - hv_r(r_j) + \frac{h^2}{2} v_{rr}(r_j) - \frac{h^3}{6} v_{rrr}(r_j) + \frac{h^4}{24} v_{rrrr}(\tilde{r}_j) - 2v(r_j) \\
&\qquad + v(r_j) + hv_r(r_j) + \frac{h^2}{2} v_{rr}(r_j) + \frac{h^3}{6} v_{rrr}(r_j) + \frac{h^4}{24} v_{rrrr}(\bar{r}_j)) \\
&= N(v_{rr}(r_j) + \frac{h^2}{24}(v_{rrrr}(\tilde{r}_j) + v_{rrrr}(\bar{r}_j))) \\
&= N(v_{rr}(0) + \frac{h^2}{2} v_{rr}(r_j^*) + \frac{h^2}{24}(v_{rrrr}(\tilde{r}_j) + v_{rrrr}(\bar{r}_j))),
\end{aligned}$$

where $r_{j-1} < \tilde{r}_j < r_j < \bar{r}_j < r_{j+1}$ and $r_j^* \in (0, r_j)$. For $j = N_0, \cdots, m-1$, we have

$$\begin{aligned}
Lv_j &= h^{-2}(v(r_j) - hv_r(r_j) + \frac{h^2}{2} v_{rr}(r_j) - \frac{h^3}{6} v_{rrr}(r_j) + \frac{h^4}{24} v_{rrrr}(\tilde{r}_j) - 2v(r_j) \\
&\qquad + v(r_j) + hv_r(r_j) + \frac{h^2}{2} v_{rr}(r_j) + \frac{h^3}{6} v_{rrr}(r_j) + \frac{h^4}{24} v_{rrrr}(\bar{r}_j)) \\
&\quad + \frac{N-1}{r_j} \cdot (2h)^{-1}((v(r_j) + hv_r(r_j) + \frac{h^2}{2} v_{rr}(r_j) + \frac{h^3}{6} v_{rrr}(\xi_j)) \\
&\quad - ((v(r_j) - hv_r(r_i) + \frac{h^2}{2} v_{rr}(r_j) - \frac{h^3}{12} v_{rrr}(\eta_j))) \\
&= v_{rr}(r_j) + \frac{h^2}{24}(v_{rrrr}(\tilde{r}_j) + v_{rrrr}(\bar{r}_j)) + \frac{N-1}{r_j} v_r(r_j) \\
&\quad + \frac{N-1}{j} \cdot \frac{h^2}{12}(v_{rrr}(\xi_j) + v_{rrr}(\eta_j)),
\end{aligned}$$

where $r_{j-1} < \tilde{r}_j$, $\eta_j < r_j < \bar{r}_j$, $\xi_j < r_{j+1}$.

Thus we get (4.10) by means of

$$\begin{aligned}
\Delta v(r_0) &= Nv_{rr}(0), \\
\Delta v(r_j) &= v_{rr}(r_j) + \frac{N-1}{r_j} v_r(r_j) \\
&= v_{rr}(0) + hv_{rrr}(0) + \frac{h^2}{2} v_{rrrr}(\tilde{r}_j^*) \\
&\quad + \frac{N-1}{r_j} \cdot (v_r(0) + r_j v_{rr}(0) + \frac{1}{2} r_j^2 v_{rrr}(0) + \frac{1}{6} r_j^3 v_{rrrr}(\bar{r}_j^*)) \\
&= Nv_{rr}(0) + \frac{1}{2} h^2 v_{rrrr}(\tilde{r}_j^*) + \frac{N-1}{6} \cdot j^2 h^2 v_{rrrr}(\bar{r}_j^*)
\end{aligned}$$

for $j = 1, 2, \cdots, N_0 - 1$, and

$$\Delta v(r_j)=v_{rr}(r_j)+\frac{N-1}{r_j}\cdot v_r(r_j) \text{ for } j=N_0, \cdots, m-1,$$

where $\tilde{r}_j^*$, $\bar{r}_j^* \in (0, r_j)$, since $N_0=[(N+1)/2]$ is a fixed constatnt. ■

Now, we are ready to prove Lemma 2. 2.

PROOF OF LEMMA 2. 2: For simplicity, we only show the case when $\alpha=1$ and the Dirichlet boundary condition (namely, $\sigma=0$) is concerned. Let $w_j^k=u(t_k, r_j)-u_j^k$. From the difference scheme (S) and the equation (E) we have

$$(4.11)\qquad \frac{w_0^{k+1}-w_0^k}{\tau_k}=2N\cdot\frac{w_1^{k+1}-w_0^{k+1}}{h^2}+u(t_k, 0)^{1+\alpha}-(u_0^k)^{1+\alpha}+O(h^2),$$

$$(4.12)\qquad \frac{w_j^{k+1}-w_j^k}{\tau_k}=N\cdot\frac{w_{j-1}^{k+1}-2w_j^{k+1}+w_{j+1}^{k+1}}{h^2}$$
$$+u(t_k, r_j)^{1+\alpha}-(u_j^k)^{1+\alpha}+O(h^2),\ 1\leq j<N_0,$$

$$(4.13)\qquad \frac{w_j^{k+1}-w_j^k}{\tau_k}=\frac{w_{j-1}^{k+1}-2w_j^{k+1}+w_{j+1}^{k+1}}{h^2}+\frac{N-1}{r_j}\cdot\frac{w_{j+1}^{k+1}-w_{j-1}^{k+1}}{2h}$$
$$+u(t_k, r_j)^{1+\alpha}-(u_j^k)^{1+\alpha}+O(h^2),\ N_0\leq j\leq m-1,\ 0\leq k<n;$$

$$(4.14)\qquad w_m^k=0 \text{ (since } \sigma=0),\ k=1, 2, \cdots, n;$$

$$(4,15)\qquad w_j^k=0,\ j=0, \cdots, m.$$

Rewriting (4. 11)—(4. 13) leads to

$$(1+2N\lambda_k)w_0^{k+1}-2N\lambda_k w_1^{k+1}=w_0^k+\tau_k(u(t_k, 0)^{1+\alpha}-(u_0^k)^{1+\alpha}+O(h^2)),$$
$$(1+2N\lambda_k)w_j^{k+1}-N\lambda_k(w_{j-1}^{k+1}+w_{j+1}^{k+1})=w_j^k+\tau_k(u(t_k, r_j)^{1+\alpha}-(u_j^k)^{1+\alpha}+O(h^2)),$$
$$1\leq j<N_0,\ 0\leq k<n;$$
$$-(1-\frac{N-1}{2j})\lambda_k w_{j-1}^{k+1}+(1+2\lambda_k)w_j^{k+1}-(1+\frac{N-1}{2j})\lambda_k w_{j+1}^{k+1}=w_j^k$$
$$+\tau_k(u(t_k, r_j)^{1+\alpha}-(u_j^n)^{1+\alpha}+O(h^2)),\ N_0\leq j<m,\ 0\leq k<n.$$

Thus, putting $W_k=\min_j|w_j^k|$ we can obtain

$$W_{k+1}\leq W_k+\tau_k F_k+\tau_k\tilde{R}h^2,\ k=0, 1, \cdots, n-1,$$

where $\tilde{R}$ is a constant depending only on $S\in(0, T)$ and the bound of u which is given by $U=\max\{|u(t, r)|;\ (t, r)\in[0, S]\times[0, R]\}$, and

$$F_k=\max_j|u(t_k, r_j)^{1+\alpha}-(u_j^k)^{1+\alpha}|.$$

Here, since we have assumed $\alpha=1$, it follows that

$$
\begin{aligned}
F_k &= \max_j |u(t_k, r_j)^2-(u_j^k)^2| \\
&= \max_j |(u(t_k, r_j)-u_j^k)(u(t_k, r_j)+u_j^k)| \\
&= \max_j |w_j^k(2u(t_k, r_j)-w_j^k)| \\
&\le W_k(W_k+2U), \ k=0,\ 1, \cdots,\ n-1,
\end{aligned}
$$

This implies

$$W_{k+1}\le W_k+\tau_k W_k(W_k+2U)+\tau_k\tilde{R}h^2, \ k=0,\ 1, \cdots,\ n-1,$$

By comparison of W_k to the solution of

$$\frac{dy}{dt}=y(y+2U)+\tilde{R}h^2, \ t>0, \tag{4.16}$$

$$y(0)=0, \tag{4.17}$$

we can see that if the parameter $\tau=\lambda h^2$ is suficiently small, say,

$$0<\tau\tilde{R}\le\frac{4U^2}{4+e^{2US}},$$

then

$$W_k\le y(t_k)\le C_0h^2, \ k=0,\ 1, \cdots,\ n, \tag{4.18}$$

which is exactly (2.2). This is because the solution of (4.16) and (4.17) is given by

$$y(t)=\frac{\tau\tilde{R}(\exp(2t\sqrt{U^2-\tau\tilde{R}})-1)}{U+\sqrt{U^2-\tau\tilde{R}}-(U-\sqrt{U^2-\tau\tilde{R}})\exp(2t\sqrt{U^2-\tau\tilde{R}})}$$

for $t\in[0, S]$, for which we can get the following estimation

$$y(t)\le\frac{\tau\tilde{R}}{\sigma U}(e^{2US}-1)=C_0h^2 \text{ for } t\in[0, S] \tag{4.19}$$

where $C_0=\frac{\lambda\tilde{R}}{\sigma U}(e^{2US}-1)$.

To prove (2.3a), we make the following estimation

$$
\begin{aligned}
\left|\frac{u_j^k-u_{j-1}^k}{h}-u_r(t_k, r_j)\right| &= \left|\frac{1}{h}(u_j^k-u(t_k, r_j))-\frac{1}{h}(u_{j-1}^k-u(t_k, r_{j-1}))\right. \\
&\qquad \left.+\frac{1}{h}(u(t_k, r_j)-u(t_k, r_{j-1}))-u_r(t_k, r_j)\right| \\
&\le\frac{1}{h}|w_j^k-w_{j-1}^k|+|u_r(t_k, \bar{r}_j)-u_r(t_k, r_j)| \\
&\le\frac{1}{h}W_j^k+|u_{rr}(t_k, \tilde{r}_j)|\cdot|r_j-\bar{r}_j|,
\end{aligned}
$$

where $\bar{r} \in (r_{j-1}, r_j)$ and $\tilde{r}_j \in (\bar{r}_j, r_j)$. Then noting

$$|r_j - \bar{r}_j| < |r_j - r_{j-1}| = h$$

and (4. 18), we obtain (2. 3a), with the constant C_1 given by

$$C_1 = C_1(u, S) = 2C_0 + \max\{|u_{rr}(t, r)|;\ (t, r) \in [0, S] \times [0, R]\}.$$

The estimation (2. 3b) can be shown in the same way. Thus, we have completed the proof of Lemma 2. 2. ■

Next, we give an outline of the proof of Lemma 2. 4.

OUTLINE OF THE PROOF OF LEMMA 2.4: The assertion (i) can be shown by a same argument as in the proof of Lemma 2, 1, with $\{w_j^n\}$ be defined by $w_j^n = u_j^n - u_{j+1}^n$ $(0 \leq j < m)$. While (ii) can be proved by the methods used in [N] and [C1], which is omitted here. ■

Finally, we give a counter-example which indicates the significance of the equation (S1) for the difference scheme in the case when $N \geq 3$. It is introduced in order to ensure the accuracy, the stability (i. e., the local boundedness), as well as the validity of the maximum principle for a difference solution even in dealing with the linear problems. To show the key point of the problem in a simpler way, we consider a corresponding explicit difference scheme here.

EXAMPLE 4. 2: Let $\{w_j^n\}$ be a solution of difference equation

(S0″) $$\frac{w_0^{n+1} - w^n{}_0}{\tau_n} = 2N \cdot \frac{w_1^n - w_0^n}{h^2},$$

(S2″) $$\frac{w_j^{n+1} - w_j^n}{\tau_n} = \frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{h_2} + \frac{N-1}{r_j} \cdot \frac{w_{j+1}^n - w_{j-1}^n}{2h}$$

$$1 \leq j < m,\quad n \geq 0;$$

(SB″) $$w_m^n = 0, \quad n \geq 0;$$

(SI″) $$w_j^0 = \begin{cases} 1, & j = 0, \\ 0, & j = 1, \cdots, m. \end{cases}$$

The solution $\{w_j^n\}$ can be solved explicitly as

(4. 20) $$w_0^{n+1} = (1 - 2N\lambda_n) w_0^n + 2N\lambda_n w_1^n,$$

(4. 21) $$w_j^{n+1} = \left(1 - \frac{N-1}{2j}\right)\lambda_n w_{j-1}^n + (1 - 2\lambda_n) w_j^n + \left(1 + \frac{N-1}{2j}\right)\lambda_n w_{j+1}^n,$$

$$1 \leq j < m,\quad n \geq 0.$$

The scheme is stable if $N\leq 3$ and $\lambda_n\leq 1/(2N)$. But for $N>3$, the coefficient of w_{j-1}^n on the right-hand side of (4. 21) turns out to be negative if $1\leq j<N_0=[(N+1)/2]$. Consequently, assuming $0<\lambda<1/(2N)$ we get

$$w_0^1=(1-2N\lambda_0)w_0^0+2N\lambda_0 w_1^0=1-2N\lambda_0>0,$$

while

$$w_1^1=(1-\frac{N-1}{2})\lambda_0 w_0^0+(1-2\lambda_0)w_1^0+(1+\frac{N-1}{2})\lambda_0 w_2^0=\frac{3-N}{2}\lambda_0<0,$$

which shows absurdity of the maximum principle. For the convergence, one is referred to [E1] and [E2].

REMARK 4. 3: Although for $N=3$ the solution in the example satisfies the maximum principle, it does not satisfy the strong maximum principle because

$$w_0^n>0,\ w_j^n=0\ (1\leq j\leq m)\ \text{for}\ n\geq 0$$

under present circumstances. This is why we have applied (S1) for $j=1,\cdots,N_0-1$, which appears even for $N=3$.

REMARK 4. 4: There are several other works on the difference approximate schemes for elliptic equations and parabolic equations in radial domains which ought to be mentioned here. But to our knowledge, the scheme (S), or merely the linear part of it, is the first one which has an accuracy of $O(h^2)$ and is stable for all $N\geq 1$, concerning a radial domain and radial solutions. The difference approximations of elliptic equations in radial domains were discussed in [D], [Fry], [SN], [SS] and [Sw] for $N\leq 3$. On the other hand, the difference approximations for parabolic equations in radial domains were studied in [A], [E1], [E2], [FaM], [Fra], [I], [Sa] and [Sm], mostly for $N\leq 3$. The only works dealing with the case of $N>3$ are [E1] and [E2] which are concerning the linear parabolic problems, so far as we know. However, in [E1] it was assumed that $2\leq N\leq 4$ or N is even to get the uniform stability and convergence; while for the difference scheme in [E2] the stability was proved for maximum norm but the consistency was obtained with respect to L^p-norm. It was also pointed out in [E1] and [E2] that straightforward replacement of derivatives by corresponding difference quotients could often lead to unbounded difference operators, even with respect to the L^2-norm.

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Illustrations of numerical computation of blow-up solutions.

Figure 1. Assume $N=2$, $R=1$ and $\alpha=1$ with the Dirichlet boundary condition ($\sigma=0$). The initial value is $\phi(r)=1000\cos\left(\frac{\pi r}{2}\right)$. Taking $m=50$ and $\lambda=10$, the graph shows the behaviour of the blow-up solution in (t, r, u)-space.

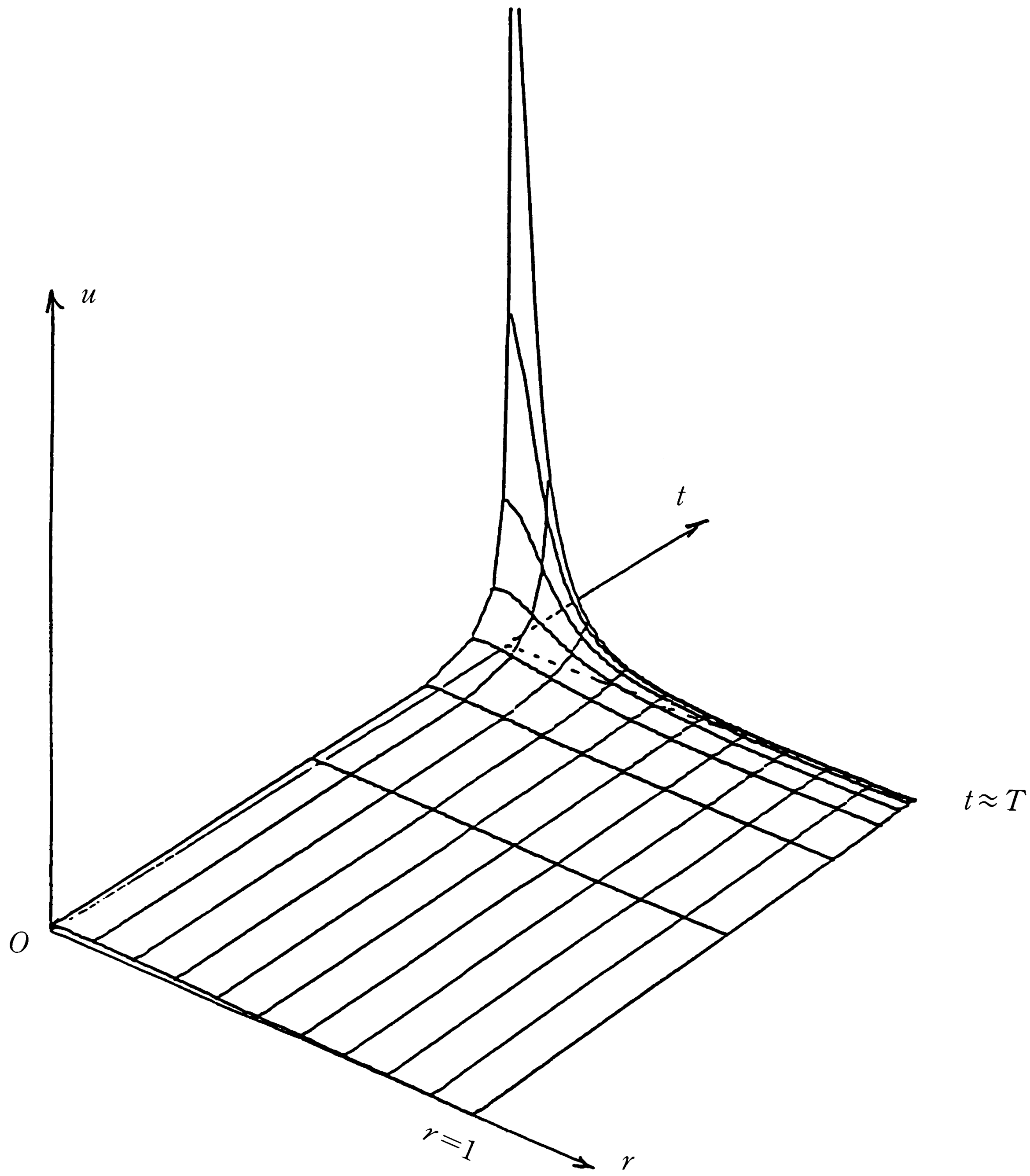

Figure 2. Assume $N=5$, $R=1$ and $\alpha=3$ with the Neumann boundary condition ($\sigma=1$). The initial value is $\phi(r)=1000+500\cos(\pi r)$. Taking $m=50$ and $\lambda=5$, the graph shows the behaviour of the blow-up solution in (t, r, u)-space.

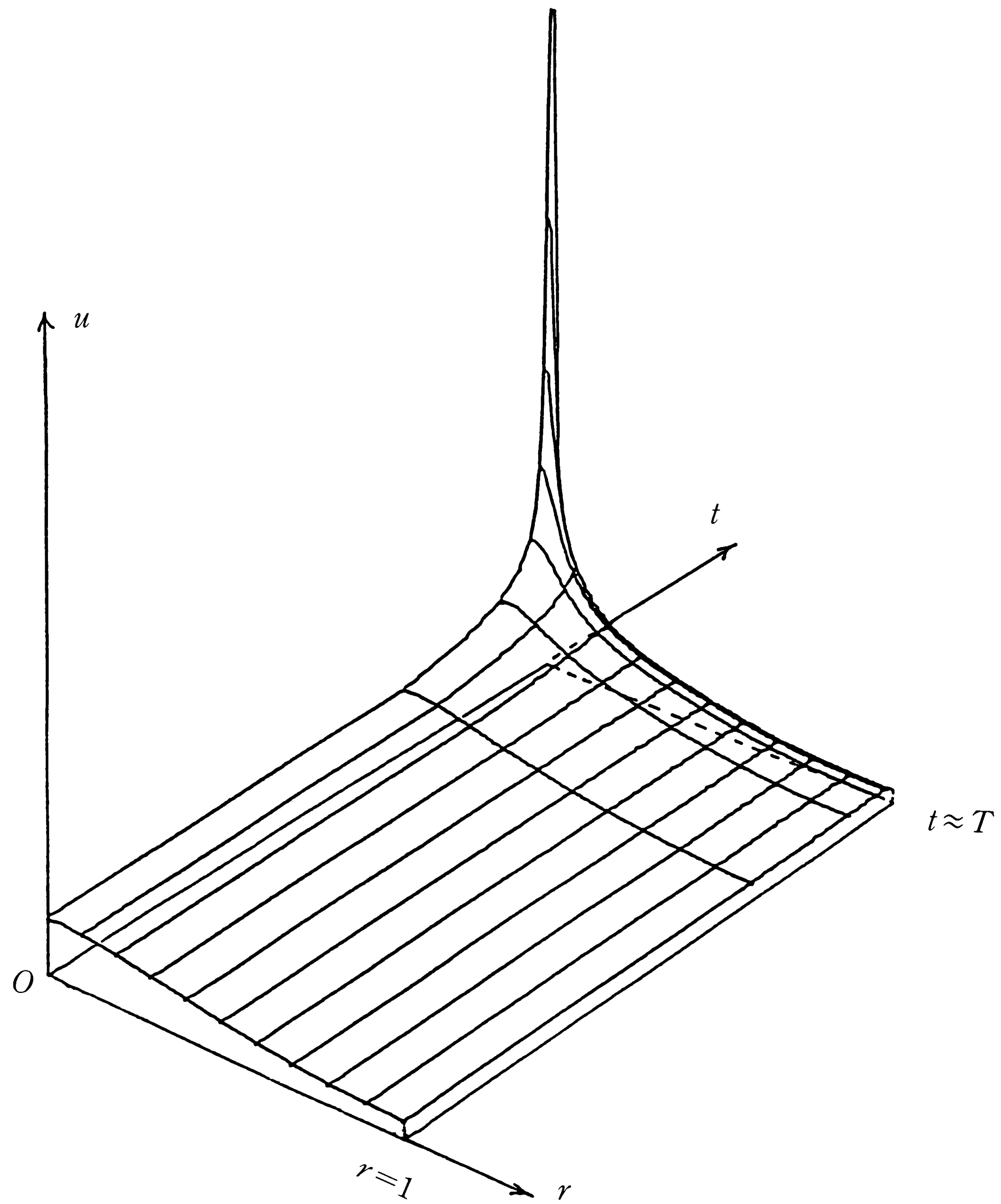

Figure 3. The shape of the blow-up solution in Figure 1 at time $t \approx T$ in (x_1, x_2, u)-space.