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A new one-parameter family of 2×2 matrix bialgebras

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Introduction

Kobozev gives a nice construction of the quantum matrix bialgebra $M_q(2)$ (in case $q^2\neq -1$) starting with two quantum planes $A^{2|0}$ and $A^{0|2}$ (Manin [M, Theorem 1.4]). The first author [T1] generalizes this construction as follows. Let us work over an algebraically closed field k with char$(k)\neq 2$. Let F be the free matric bialgebra on $e_{ij}^{\vee}, 1<i, j\leq 2$ (see Section 1), and consider the 2-dimensional F-comodule $V=ke_1+ke_2$ with coaction $e_j\mapsto\sum_i e_i\otimes e_{ij}^{\vee}$. The tensor algebra $T(V)=k\langle e_1, e_2\rangle$ has an extended F-comodule algebra structure. The results of [T1, Section 3] tell that if W is a subspace of $T(V)$, there is a least subspace $N(W)$ of F such that the comodule structure map sends W into $W\otimes F+T(V)\otimes N(W)$ and that $N(W)$ is *coideal*.

Assume we are given a decomposition $V\otimes V=W_1\oplus W_2$. We will define a quadratic bialgebra $F(W_1, W_2)$ as the quotient bialgebra of F by the ideal generated by $N(W_1)$ and $N(W_2)$.

Kobozev's previously mentioned theorem tells that $M_q(2)$ is of this type, with W_1, W_2 of dimensions 1 and 3. The quantum matrix bialgebra $M_q(2)$ is well-studied, and it is known to be cosemisimple unless q is a root of 1.

In this paper we study a new one-parameter family of quadratic bialgebras B_λ of the form $F(W_1, W_2)$ with both W_i 2-dimensional. Explicitly, let λ be a parameter in k such that $\lambda\neq 0$, $\lambda^4\neq 1$ (this assumption is technically needed), and put $B_\lambda=F(V^+, V_\lambda^-)$, where

$$V^+=\langle e_1\otimes e_2+e_2\otimes e_1,\ e_1\otimes e_1+e_2\otimes e_2\rangle$$
$$V_\lambda^-=\langle e_1\otimes e_2-e_2\otimes e_1,\ (\lambda+\lambda^{-1}-2)e_1\otimes e_1-(\lambda+\lambda^{-1}+2)e_2\otimes e_2\rangle.$$

The first remarkable property is that B_λ is associated with a Yang-Baxter operator R_λ (see 1.7). Recall that $M_q(2)$ is associated with Jimbo's R-matrix. So we are in a similar situation.

In this paper we are mainly interested in representations and corepresentations. We prove that B_λ is cosemisimple if λ is not a root of 1. In case of $M_q(2)$, the cosemisimplicity is proved by using the complete reducibilty of $U_q(sl(2))$ and the duality of $M_q(2)$ and $U_q(sl(2))$ (see [T2]). Interestingly enough, in the present case of B_λ, the role of $U_q(sl(2))$ will be played by B_λ itself!

In Section 1 we give two presentations of B_λ by generators and relations. The first one follows directly from definition and the second one follows from it by a linear transformation of generators.

In Sections 2 and 3 we develop representation theory of B_λ. Theorem 2.1 tells that all simple B_λ-modules have dimensions 1 or 2 and that the 2-dimensional simple modules are classified by parameters ξ, $\eta \in k$ with $\xi^2 \neq \eta \neq 0$. Proposition 3.1 gives the decomposition rule for the tensor product of simple modules.

Looking at the decomposition rule carefully, we deduce that there is a nondegenerate bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda \times B_\mu \to k$ if λ, μ are not roots of 1. This pairing yields to embed B_λ into a subbialgebra of the dual bialgebra B_μ°, so that corepresentation theory for B_λ can be thought of as part of representation theory for B_μ. Using this, we claim that B_λ is cosemisimple and all nontrivial simple comodules are 2-dimensional (Section 5).

Section 6 deals with a realization of all nontrivial simple comodules as the homogeneous components of the syzygies of the algebra $T(V)/(V_\lambda^-) \cong k[x, y]/(xy)$. Tensor product decompositions of simple comodules are also described.

The main results were announced at the Tsukuba International Conference on Representation of Algebras and Related Topics in August 1990 and at the San Fransisco meeting of the AMS in January 1991.

Convensions. The dual space of a vector space V is denoted by V^*. The dual bialgebra of a bialgebra B in the sense of [S] is denoted by B°. B-Mod and Comod-B stand for the categories of left B-modules and right B-comodules, respectively.

1. The bialgebra B_λ

We fix a parameter $\lambda \in k$, $\lambda^4 \neq 0$, 1 and a square root $\sqrt{\lambda}$ throughout. Let $V = k^2$ with basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then $V \otimes V$ is the direct sum of subspaces:

$$V^+ = \langle e_1 \otimes e_2 + e_2 \otimes e_1,\ e_1 \otimes e_1 + e_2 \otimes e_2 \rangle$$
$$V_\lambda^- = \langle e_1 \otimes e_2 - e_2 \otimes e_1,\ (\lambda + \lambda^{-1} - 2) e_1 \otimes e_1 - (\lambda + \lambda^{-1} + 2) e_2 \otimes e_2 \rangle.$$

We define a bialgebra B_λ associated to these subspaces as follows [M], [T1].

Let F be the free algebra on $e_{ij}^\vee$, $1 \leq i$, $j \leq 2$, with the following bialgebra structure:

$$\Delta(e_{ij}^{\vee})=\sum_{k} e_{ik}^{\vee}\otimes e_{kj}^{\vee}, \quad \varepsilon(e_{ij}^{\vee})=\delta_{ij}.$$

The map $e_j\mapsto\sum_i e_i\otimes e_{ij}^{\vee}$ makes V into a right F-comodule. We define $B_\lambda=F/I_\lambda$, the quotient bialgebra, with I_λ the smallest bi-ideal of F such that V^+, V_λ^- are F/I_λ-subcomodules of $V\otimes V$.

With x_{ij} the image of $e_{ij}^{\vee}$, we have the following defining relations of B_λ:

(1.1)
$$\begin{aligned}
&(\mathrm{i}) && x_{11}x_{22}=x_{22}x_{11}\\
&(\mathrm{ii}) && x_{12}x_{21}=x_{21}x_{12}\\
&(\mathrm{iii}) && x_{11}x_{12}-\mu x_{12}x_{11}=(1-\mu)x_{21}x_{22}\\
&(\mathrm{iv}) && x_{22}x_{21}+\mu x_{21}x_{22}=(1+\mu)x_{12}x_{11}\\
&(\mathrm{v}) && x_{11}x_{21}-\mu x_{21}x_{11}=-(1+\mu)x_{12}x_{22}\\
&(\mathrm{vi}) && x_{22}x_{12}+\mu x_{12}x_{22}=-(1-\mu)x_{21}x_{11}\\
&(\mathrm{vii}) && x_{11}^2+x_{12}^2=x_{21}^2+x_{22}^2\\
&(\mathrm{viii}) && (1+\mu)x_{12}^2=-(1-\mu)x_{21}^2
\end{aligned}$$

where $\mu=(\lambda+\lambda^{-1})/2$.

There is a more convenient presentation. Put

(1.2)
$$\begin{aligned}
f&=\frac{1}{2}(x_{11}+x_{22})\\
g&=\frac{1}{2}(x_{11}-x_{22})\\
s&=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}+\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right)\\
t&=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}-\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right).
\end{aligned}$$

We have

$$\begin{aligned}
&(1.1)(\mathrm{i}) \iff fg=gf\\
&(1.1)(\mathrm{ii}) \iff st=ts.
\end{aligned}$$

(1.1)(iii)–(vi) are equivalent to

$$f\begin{pmatrix}x_{12}\\ x_{12}\end{pmatrix}=\begin{pmatrix}\mu & -(1-\mu)\\ 1+\mu & \mu\end{pmatrix}\begin{pmatrix}x_{12}\\ x_{21}\end{pmatrix}g$$

$$g\begin{pmatrix}x_{12}\\ x_{21}\end{pmatrix}=\begin{pmatrix}\mu & 1-\mu\\ -(1+\mu) & \mu\end{pmatrix}\begin{pmatrix}x_{12}\\ x_{21}\end{pmatrix}f,$$

which are rewritten respectively as

$$f\begin{pmatrix}s\\ t\end{pmatrix}=\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}\begin{pmatrix}s\\ t\end{pmatrix}g$$

$$g\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} f.$$

Assuming these relations, we have

$$(1.1)(\text{vii}),\ (\text{viii}) \Longleftrightarrow fg = s^2 + t^2,\ st = 0.$$

It follows that B_λ is defined by generators f, g, s, t and relations

$$(1.3)\quad \begin{array}{ll} fg = gf = s^2 + t^2 & \\ st = ts = 0 & \\ fs = \lambda sg & gs = \lambda^{-1} sf \\ ft = \lambda^{-1} tg & gt = \lambda tf. \end{array}$$

With these new generators, we have

$$\begin{aligned} &\Delta(f) = f \otimes f + g \otimes g + (\lambda - \lambda^{-1})(s \otimes s - t \otimes t) \\ &\Delta(g) = f \otimes g + g \otimes f + (\lambda - \lambda^{-1})(t \otimes s - s \otimes t) \\ &\Delta(s) = f \otimes s + g \otimes t + s \otimes f - t \otimes g \\ &\Delta(t) = f \otimes t + g \otimes s + t \otimes f - s \otimes g \\ &\varepsilon(f) = 1, \varepsilon(g) = \varepsilon(s) = \varepsilon(t) = 0. \end{aligned}$$

REMARK 1.4. We have $B_\lambda = B_{\lambda^{-1}}$, because $V_\lambda^- = V_{\lambda^{-1}}^-$. We have the bialgebra automorphism $\sigma: B_\lambda \to B_\lambda = B_{\lambda^{-1}}$ taking x_{ij} to $(-1)^{i-j} x_{ij}$, because the automorphism $e_i \mapsto (-1)^i e_i$ of V preserves the decomposition $V \otimes V = V^+ \oplus V_\lambda^-$. If we write $s = s(\sqrt{\lambda})$, $t = t(\sqrt{\lambda})$, then

$$\sigma: f \mapsto f,\ g \mapsto g,\ s(\sqrt{\lambda}) \mapsto t(\sqrt{\lambda^{-1}}),\ t(\sqrt{\lambda}) \mapsto s(\sqrt{\lambda^{-1}}).$$

This symmetry will be useful in later computations.

REMARK 1.5. B_λ has k-bases

$$\{x_{11}^a x_{22}^b x_{12}^c x_{21}^d \mid a, b \geq 0, 0 \leq c, d \leq 1\}$$

and

$$\begin{aligned} &\{f^i s^j \mid i, j \geq 0\} \cup \{f^i t^j \mid i \geq 0, j > 0\} \\ &\cup \{g^i s^j \mid i > 0, j \geq 0\} \cup \{g^i t^j \mid i, j > 0\}. \end{aligned}$$

REMARK 1.6. The $GL(V)$-orbits of the pairs (V^+, V_λ^-) for all λ occupy a dense subset of the set of pairs (U, U') of subspaces of $V \otimes V$ such that $\dim U = \dim U' = 2$ and $U \subset \mathrm{Sym}^2(V)$, $U' \supset \mathrm{Alt}^2(V)$.

REMARK 1.7. Let R_λ be the linear automorphism of $V \otimes V$ such that $R_\lambda = 1$ on V^+ and $R_\lambda = -\lambda^2$ on V_λ^-. Explicitly

$$\begin{aligned}R_\lambda=&[1-\frac{(\lambda-1)^2}{2}]e_{11}\otimes e_{11}+[1-\frac{(\lambda+1)^2}{2}]e_{22}\otimes e_{22}\\&+\frac{(\lambda-1)^2}{2}e_{12}\otimes e_{12}+\frac{(\lambda+1)^2}{2}e_{21}\otimes e_{21}\\&+\frac{1-\lambda^2}{2}(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})\\&+\frac{1+\lambda^2}{2}(e_{12}\otimes e_{21}+e_{21}\otimes e_{12})\end{aligned}$$

where e_{ij} are the matrix units. Then R_λ satisfies the Yang-Baxter equation

$$(R_\lambda\otimes 1)(1\otimes R_\lambda)(R_\lambda\otimes 1)=(1\otimes R_\lambda)(R_\lambda\otimes 1)(1\otimes R_\lambda)$$

in $\mathrm{End}(V\otimes V\otimes V)$. This is verified directly or as a consequence of [T, Proposition 8.3]. Since B_λ is defined by the relation $(X\otimes X)R_\lambda=R_\lambda(X\otimes X)$, with $X=(x_{ij})_{ij}$, B_λ has a braid structure in the sense of [H], [LT]. This braid sructure will be studied in detail in [T, Section 8].

2. Representation of B_λ

In this section we construct all simple B_λ-modules of dimension>1. One will see that the following algebra maps $\pi_{\lambda s}(\alpha,\beta)$, $\pi_{\lambda t}(\alpha,\beta)$ for $\alpha,\beta\in k$ are well-defined.

$$\pi_{\lambda s}(\alpha,\beta): B_\lambda\to M_2(k)$$

$$f\mapsto\begin{pmatrix}\alpha+\beta & 0\\ 0 & \alpha-\beta\end{pmatrix}\frac{1}{2}\qquad g\mapsto\begin{pmatrix}\alpha-\beta & 0\\ 0 & \alpha+\beta\end{pmatrix}\frac{\lambda}{2}$$

$$s\mapsto 0\qquad t\mapsto\begin{pmatrix}0 & \alpha-\beta\\ \alpha+\beta & 0\end{pmatrix}\frac{\sqrt{\lambda}}{2}$$

$$\pi_{\lambda t}(\alpha,\beta): B_\lambda\to M_2(k)$$

$$f\mapsto\begin{pmatrix}\alpha+\beta & 0\\ 0 & \alpha-\beta\end{pmatrix}\frac{1}{2}\qquad g\mapsto\begin{pmatrix}\alpha-\beta & 0\\ 0 & \alpha+\beta\end{pmatrix}\frac{\lambda^{-1}}{2}$$

$$s\mapsto\begin{pmatrix}0 & \alpha-\beta\\ \alpha+\beta & 0\end{pmatrix}\frac{\sqrt{\lambda}^{-1}}{2}\qquad t\mapsto 0$$

THEOREM 2.1. (i) *Every irreducible representation of B_λ has dimension 1 or 2.*

(ii) *Choose a square root $\sqrt{\eta}$ for each $\eta\in k$. Then $\{\pi_{\lambda s}(\xi,\sqrt{\eta}),\pi_{\lambda t}(\xi,\sqrt{\eta})\,|\,\xi^2\neq\eta\neq 0\}$ gives a complete list of 2-dimensional irreducible representations of B_λ.*

PROOF: By (1.3) we see that $B_\lambda s$, $B_\lambda t$ are ideals annihilating each other. Hence any irreducible representation of B_λ kills s or t. So it is enough to consider representations of $B_\lambda/(s)$ and $B_\lambda/(t)$.

Let K be the algebra defined by generators u, v, w and relations

$$\begin{aligned} &uv=vu\\ &uw=wu\\ &vw=-wv\\ &u^2-v^2=w^2. \end{aligned}$$

The following is easily verified.

PROPOSITION 2.2. *We have algebra isomorphisms*

$$\begin{aligned} &B_\lambda/(s) \quad \cong K \cong B_\lambda/(t)\\ &\bar{f}+\lambda^{-1}\bar{g} \leftrightarrow u \leftrightarrow \bar{f}+\lambda\bar{g}\\ &\bar{f}-\lambda^{-1}\bar{g} \leftrightarrow u \leftrightarrow \bar{f}-\lambda\bar{g}\\ &2\sqrt{\lambda}^{-1}\bar{t} \leftrightarrow w \leftrightarrow 2\sqrt{\lambda}\,\bar{s} \end{aligned}$$

where bar means the residue class.

The centre of K is the polynomial algebra $C=k[u, v^2]$ and K is a quaternion C-algebra generated by v and w. Therefore all irreducible representations of K have dimensions 1 or 2, and any 2-dimensional one is equivalent to the representation

$$\begin{aligned} &u \mapsto \begin{pmatrix} \xi & \\ & \xi \end{pmatrix}\\ &v \mapsto \begin{pmatrix} \sqrt{\eta} & \\ & -\sqrt{\eta} \end{pmatrix}\\ &w \mapsto \begin{pmatrix} & \xi-\sqrt{\eta} \\ \xi+\sqrt{\eta} & \end{pmatrix} \end{aligned}$$

for a unique ξ, $\eta \in k$ such that $\xi^2 \neq \eta \neq 0$. Through the isomorphisms of Proposition 2.2, the above map induces the representations $\pi_{\lambda s}(\xi, \sqrt{\eta})$, $\pi_{\lambda t}(\xi, \sqrt{\eta})$ of B_λ. This proves the theorem.

REMARK 2.3. The quaternion algebra K is split over C. Indeed we have an injective C-algebra map

$$\begin{aligned} &K \to M_2(C)\\ &v \mapsto \begin{pmatrix} u & u^2-v^2 \\ -1 & -u \end{pmatrix}\\ &w \mapsto \begin{pmatrix} 0 & u^2-v^2 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

whose cokernel is annihilated by $v^2(u^2-v^2)$.

This enables us to improve the parametrization of irreducible representations of B_λ as follows. Let

$$\rho_s : B_\lambda \to B_\lambda/(s) \cong K \to M_2(C)$$
$$\rho_t : B_\lambda \to B_\lambda/(t) \cong K \to M_2(C)$$

be the composites of the natural surjections, the isomorphisms of Proposition 2.2 and the above map. For $\xi,\ \eta \in k$, composing $\rho_s,\ \rho_t$ with the evaluation map $u \mapsto \xi,\ v^2 \mapsto \eta$, we have algebra maps $\rho_s(\xi, \eta),\ \rho_t(\xi, \eta)$: $B_\lambda \to M_2(k)$. Then $\rho_s(\alpha, \beta^2) \cong \pi_{\lambda s}(\alpha, \beta),\ \rho_t(\alpha, \beta^2) \cong \pi_{\lambda t}(\alpha, \beta)$ if $\beta \neq 0$, and $\{\rho_s(\xi, \eta),\ \rho_t(\xi, \eta) \mid \xi^2 \neq \eta \neq 0\}$ is a complete list of 2-dimensional irreducible representations of B_λ.

REMARK 2.4. Since t is not a zero divisor in $B_\lambda/(s)$, we have $(s) \cap (t) = 0$. Thus we have a pullback diagram of rings:

$$\begin{tikzcd} B_\lambda \arrow[r] \arrow[d] & B_\lambda/(s) \arrow[d] \\ B_\lambda/(t) \arrow[r] & B_\lambda/(s, t). \end{tikzcd}$$

The ideal $(s, t) = (x_{12}, x_{21})$ is a bi-ideal of B_λ and the quotient bialgebra $B_\lambda/(x_{12}, x_{21})$ is generated by group-like elements $\bar{x}_{11},\ \bar{x}_{22}$ with relations $\bar{x}_{11}\bar{x}_{22} = \bar{x}_{22}\bar{x}_{11},\ \bar{x}_{11}^2 = \bar{x}_{22}^2$.

3. Tensor products of simple B_λ-modules

In this section we study tensor products of simple B_λ-modules. For α, $\beta \in k$ let $M_{\lambda s}(\alpha, \beta),\ M_{\lambda t}(\alpha, \beta)$ be the B_λ-modules with underlying space k^2 and actions $\pi_{\lambda s}(\alpha, \beta),\ \pi_{\lambda t}(\alpha, \beta)$ respectively. We write $e_1 = (1, 0),\ e_2 = (0, 1) \in k^2$. The subscript λ is omitted in this section.

PROPOSITION 3.1. *Let $\alpha \neq \pm\beta,\ \alpha' \neq \pm\beta',\ \alpha\alpha' \neq \pm\beta\beta'$. Then we have isomorphisms of B_λ-modules*

(i) $M_s(\alpha, \beta) \otimes M_s(\alpha', \beta') \cong M_s(\alpha\alpha', \beta\beta') \oplus M_t(\lambda^2\alpha\alpha', \lambda^2\beta\beta')$

$$\begin{aligned} &e_1 \otimes e_1 + e_2 \otimes e_2 && \leftrightarrow (e_1, 0) \\ &e_1 \otimes e_2 + e_2 \otimes e_1 && \leftrightarrow (e_2, 0) \\ &(\alpha-\beta)(\alpha'-\beta')e_1 \otimes e_1 - (\alpha+\beta)(\alpha'+\beta')e_2 \otimes e_2 && \leftrightarrow (0, e_1) \\ &(\alpha-\beta)(\alpha'+\beta')e_1 \otimes e_2 - (\alpha+\beta)(\alpha'-\beta')e_2 \otimes e_1 && \leftrightarrow (0, e_2) \end{aligned}$$

(ii) $M_s(\alpha, \beta) \otimes M_t(\alpha', \beta') \cong M_s(\alpha\alpha', \beta\beta') \oplus M_t(\alpha\alpha', \beta\beta')$

$$\begin{aligned} &(\alpha\alpha'+\beta\beta')((\alpha-\beta)e_1 \otimes e_1 + (\alpha+\beta)e_2 \otimes e_2) && \leftrightarrow (e_1, 0) \\ &(\alpha\alpha'-\beta\beta')((\alpha-\beta)e_1 \otimes e_2 + (\alpha+\beta)e_2 \otimes e_1) && \leftrightarrow (e_2, 0) \\ &(\alpha\alpha'+\beta\beta')((\alpha'-\beta')e_1 \otimes e_1 - (\alpha'+\beta')e_2 \otimes e_2) && \leftrightarrow (0, e_1) \end{aligned}$$

$$(\alpha\alpha'-\beta\beta')((\alpha'+\beta')e_1\otimes e_2-(\alpha'-\beta')e_2\otimes e_1) \qquad \leftrightarrow (0,\ e_2)$$

(iii) $M_t(\alpha,\ \beta)\otimes M_s(\alpha',\ \beta')\cong M_t(\alpha\alpha',\ \beta\beta')\oplus M_s(\alpha\alpha',\ \beta\beta')$

(iv) $M_t(\alpha,\ \beta)\otimes M_t(\alpha',\ \beta')\cong M_t(\alpha\alpha',\ \beta\beta')\oplus M_s(\lambda^{-2}\alpha\alpha',\ \lambda^{-2}\beta\beta')$,

where the maps in (iii), (iv) *are the same as the ones in* (ii), (i) *respectively.*

PROOF : (i) With respect to the basis $e_1\otimes e_1,\ e_1\otimes e_2,\ e_2\otimes e_1,\ e_2\otimes e_2$, the actions of $g,\ s,\ t$ on $M_s(\alpha,\ \beta)\otimes M_s(\alpha',\ \beta')$ are represented as follows.

$$g\mapsto\begin{pmatrix} \alpha\alpha'-\beta\beta' & 0 & 0 & 0 \\ 0 & \alpha\alpha'+\beta\beta' & 0 & 0 \\ 0 & 0 & \alpha\alpha'+\beta\beta' & 0 \\ 0 & 0 & 0 & \alpha\alpha'-\beta\beta' \end{pmatrix}\frac{\lambda}{2}$$

$$s\mapsto\begin{pmatrix} 0 & \gamma^-\gamma'^- & -\gamma^-\gamma'^- & 0 \\ \gamma^-\gamma'^+ & 0 & 0 & -\gamma^-\gamma'^+ \\ -\gamma^+\gamma'^- & 0 & 0 & \gamma^+\gamma'^- \\ 0 & -\gamma^+\gamma'^+ & \gamma^+\gamma'^+ & 0 \end{pmatrix}\frac{\lambda\sqrt{\lambda}}{4}$$

$$t\mapsto\begin{pmatrix} 0 & \gamma^+\gamma'^- & \gamma^-\gamma'^+ & 0 \\ \gamma^+\gamma'^+ & 0 & 0 & \gamma^-\gamma'^- \\ \gamma^+\gamma'^+ & 0 & 0 & \gamma^-\gamma'^- \\ 0 & \gamma^+\gamma'^- & \gamma^-\gamma'^+ & 0 \end{pmatrix}\frac{\sqrt{\lambda}}{4},$$

where $\gamma^{\pm}=\alpha\pm\beta,\ \gamma'^{\pm}=\alpha'\pm\beta'$. Therefore $\operatorname{Ker} s$ has the basis

$$v_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad v_2=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and $\operatorname{Ker} t$ has the basis

$$w_1=\begin{pmatrix} \gamma^-\gamma'^- \\ 0 \\ 0 \\ -\gamma^+\gamma'^+ \end{pmatrix},\quad w_2=\begin{pmatrix} 0 \\ \gamma^-\gamma'^+ \\ -\gamma^+\gamma'^- \\ 0 \end{pmatrix}.$$

We have the B_λ-isomorphisms

$$M_s(\alpha\alpha',\ \beta\beta')\cong\operatorname{Ker} s:\quad e_i\leftrightarrow v_i$$
$$M_t(\lambda^2\alpha\alpha',\ \lambda^2\beta\beta')\cong\operatorname{Ker} t:\quad e_i\leftrightarrow w_i.$$

If $(\alpha\alpha')^2 \neq (\beta\beta')^2$, then $\operatorname{Ker} s \cap \operatorname{Ker} t = 0$. This proves (i).

(ii) For $\gamma \in k$ let $L(\gamma)$ be the B_λ-module with underlying space k and character $g \mapsto \gamma$; f, s, $t \mapsto 0$. We have B_λ-isomorphisms

$$M_s(\alpha, \beta) \otimes L(\gamma) \cong M_t(\lambda\alpha\gamma, \lambda\beta\gamma)$$
$$M_t(\alpha, \beta) \otimes L(\gamma) \cong M_s(\lambda^{-1}\alpha\gamma, \lambda^{-1}\beta\gamma)$$

given by the same correspondence

$$e_1 \otimes 1 \leftrightarrow (\alpha+\beta)e_2$$
$$-e_2 \otimes 1 \leftrightarrow (\alpha-\beta)e_1.$$

Therefore tensoring (i) with $L(\lambda^{-1})$ yields (ii).

(iii) and (iv) follow from (ii) and (i), using the formula $\pi_{\lambda s}(\alpha, \beta) = \pi_{\lambda^{-1} t}(\alpha, \beta) \circ \sigma$, where $\sigma: B_\lambda \to B_{\lambda^{-1}}$ is the bialgebra isomorphism of Remark 1.4.

COROLLARY 3.2. *Let λ, $\mu \in k - \{0\}$ with $\lambda^4 \neq 1 \neq \mu^4$. If we let B_λ act on $V = k^2$ through $\pi_{\lambda s}(1, \mu)$, then V^+, V_μ^- (Section 1) are B_λ-submodules of $V \otimes V$.*

REMARK 3.3. The computations in the above proof show also that if $\alpha\alpha' = \pm\beta\beta'$, the composition factors of the tensor product modules in the proposition are 1-dimensional.

REMARK 3.4. We consider here tensor products of the generic representations ρ_s, ρ_t: $B_\lambda \to M_2(C)$, where $C = k[u, v^2]$ (Remark 2.3). Let N_s, N_t be the (B_λ, C)-bimodules with underlying right C-module C^2 and left B_λ-actions through ρ_s, ρ_t respectively. An algebra map $\varphi: C \to C \otimes C$ induces the scalar extension functor Mod-$C \to$ Mod-$C \otimes C$, which we denote by $(-)^\varphi$. Define algebra maps Δ, Δ': $C \to C \otimes C$ by

$$\Delta: u \mapsto u \otimes u, \quad v^2 \mapsto v^2 \otimes v^2$$
$$\Delta': u \mapsto \lambda^2 u \otimes u, \quad v^2 \mapsto \lambda^2 v^2 \otimes v^2.$$

Then there exist injective $(B_\lambda, C \otimes C)$-bimodule maps

$$N_s \otimes N_s \leftarrow N_s^\Delta \oplus N_t^{\Delta'}$$
$$N_s \otimes N_t \leftarrow N_s^\Delta \oplus N_t^\Delta$$

with cokernels annihilated by $(v^2 \otimes 1)(u^2 \otimes u^2 - v^2 \otimes v^2) \in C \otimes C$.

4. Faithfulness of $\oplus_n M_s(\alpha, \beta)^{\otimes n}$

In this section we show that the representation $\pi_s(\alpha, \beta): B_\lambda \to M_2(k)$ factors through no proper quotient bialgebra of B_λ if λ, α, β are general.

PROPOSITION 4.1. *Let $\alpha, \beta \in k$. Assume $\beta \neq 0$ and $\alpha^n \neq \beta^n$ for all $n > 0$. Then we have isomorphisms of B_λ-modules*

$$M_s(\alpha, \beta)^{\otimes n}$$
$$\cong \oplus_m M_s(\lambda^{2m}\alpha^n, \lambda^{2m}\beta^n)^{\oplus\binom{n-1}{2m}} \oplus \oplus_m M_t(\lambda^{2m}\alpha^n, \lambda^{2m}\beta^n)^{\oplus\binom{n-1}{2m-1}}$$

for all $n > 0$.

PROOF: This follows from Proposition 3.1 easily by induction.

PROPOSITION 4.2. *Let $\alpha, \beta \in k$, $\beta \neq 0$. Assume that neither λ nor α/β is a root of 1. Then the B_λ-module $\oplus_{n>0} M_s(\alpha, \beta)^{\otimes n}$ is faithful.*

PROOF: Set

$$Q_s = \bigoplus_{\substack{m, n \in \mathbf{N} \\ m \leq (n-1)/2}} M_s(\lambda^{2m}\alpha^n, \lambda^{2m}\beta^n), \quad Q_t = \bigoplus_{\substack{m, n \in \mathbf{N} \\ 1 \leq m \leq n/2}} M_t(\lambda^{2m}\alpha^n, \lambda^{2m}\beta^n).$$

In view of Proposition 4.1 and the fact that $(s) \cap (t) = 0$, it is enough to show that the $B_\lambda/(s)$-module Q_s and the $B_\lambda/(t)$-module Q_t are faithful. Set $u = \bar{f} + \lambda^{-1}\bar{g}$, $v = \bar{f} - \lambda^{-1}\bar{g} \in B_\lambda/(s)$. Any nonzero ideal of $B_\lambda/(s)$ has a nonzero intersection with the centre $k[u, v^2]$. Therefore Q_s is a faithful $B_\lambda/(s)$-module if it is a faithful $k[u, v^2]$-module, namely if there is no nonzero polynomial $P(X, Y)$ such that $P(\lambda^{2m}\alpha^n, \lambda^{4m}\beta^{2n}) = 0$ for all $m, n \in \mathbf{N}$ with $m \leq (n-1)/2$. But this is verified in an elementary way by using the Vandermonde determinant and the assumption that λ, α/β are not roots of 1. The faithfulness of Q_t is proved similarly.

REMARK 4.3. A similar argument shows that the conclusion of Proposition 4.2 holds if $\alpha\beta \neq 0$ and α, β are independent in the multiplicative group of k.

5. The bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda \times B_\mu \to k$

A bialgebra pairing means a bilinear map $A \times B \to k$ with A, B bialgebras such that the adjoint maps $A \to B^*$, $B \to A^*$ are algebra maps. In this section we construct a bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda \times B_\mu \to k$, which is nondegenerate if λ, μ are not roots of 1. Using this, we prove that B_μ is cosemisimple if μ is not a root of 1.

The bialgebra pairing arises as a consequence of Corollary 3.2. Let $\lambda, \mu \in k - \{0\}$ with $\lambda^4 \neq 1 \neq \mu^4$. We consider $V = k^2$ which is a left B_λ-module through $\pi_{\lambda s}(1, \mu)$, hence a right B_λ°-comodule. Corollary 3.2 means that the subspaces V^+, V_μ^- of $V \otimes V$ are B_λ°-subcomodules. Hence the universality of B_μ yields a bialgebra map $B_\mu \to B_\lambda^\circ$, or a bialgebra pairing $\omega_{\lambda,\mu}$: $B_\lambda \times B_\mu \to k$.

The pairing $\omega=\omega_{\lambda,\mu}$ is described as follows:

$$\omega(x_{11}, x_{11})=\frac{1}{2}(1+\lambda+\mu-\lambda\mu)$$
$$\omega(x_{11}, x_{22})=\frac{1}{2}(1+\lambda-\mu+\lambda\mu)$$
$$\omega(x_{22}, x_{11})=\frac{1}{2}(1-\lambda+\mu+\lambda\mu)$$
$$\omega(x_{22}, x_{22})=\frac{1}{2}(1-\lambda-\mu-\lambda\mu)$$
$$\omega(x_{12}, x_{12})=-\frac{1}{2}(1-\lambda)(1-\mu)$$
$$\omega(x_{12}, x_{21})=-\frac{1}{2}(1-\lambda)(1+\mu)$$
$$\omega(x_{21}, x_{12})=-\frac{1}{2}(1+\lambda)(1-\mu)$$
$$\omega(x_{21}, x_{21})=-\frac{1}{2}(1+\lambda)(1+\mu)$$
$$\omega(x_{ij}, x_{kl})=0 \text{ for the other } i, j, k, l.$$

In particular $\omega_{\lambda,\mu}(a, b)=\omega_{\mu,\lambda}(b, a)$ for all $a\in B_\lambda$, $b\in B_\mu$.

THEOREM 5.1. *If neither λ nor μ is a root of* 1, *then the pairing $\omega_{\lambda,\mu}$ is nondegenerate.*

PROOF: By symmetry it is enough to show that the map ${}^{\#}\omega: B_\lambda\to B_\mu^*$ adjoint to ω is injective. Let p: $T(\mathrm{End}(V)^*)\to B_\mu$ be the natural surjection. The map $p^*\circ{}^{\#}\omega$ is identified with the map $B_\lambda\to\prod_{n\geq 0}\mathrm{End}(V^{\otimes n})$ coming from the B_λ-module structures of $V^{\otimes n}$. As $V=M_{\lambda s}(1, \mu)$, the B_λ-module $\oplus_n V^{\otimes n}$ is faithful by Proposition 4.2. Thus ${}^{\#}\omega$ is injective.

Right B_μ-comodules are viewed as left B_λ-modules through the algebra map ${}^{\#}\omega$: $B_\lambda\to B_\mu^*$. This functor is denoted by j: Comod-$B_\mu\to B_\lambda$-Mod. If λ, μ are not roots of 1, then by Theorem 5.1 ${}^{\#}\omega$ has a dense image. In this case j is fully faithful and $\mathrm{Im}\, j$ is closed under submodules.

THEOREM 5.2. *If μ is not a root of* 1, *then B_μ is cosemisimple.*

PROOF : Take $\lambda=\mu$. Proposition 4.1 means that the B_λ-modules $j(V^{\otimes n})=j(V)^{\otimes n}=M_{\lambda s}(1, \mu)^{\otimes n}$ are semisimple for all n. Therefore the B_μ-comodules $V^{\otimes n}$ are semisimple for all n. Thus B_μ is cosemisimple.

COROLLARY 5.3. *If μ is not a root of* 1, *all simple B_μ-comodules are* 2-*dimensional except for the trivial one. In particular* 1 *is the unique grouplike element in B_μ.*

6. The Koszul complex and simple B_μ-comodules

If μ is not a root of 1, all nontrivial simple B_μ-comodules are 2-dimensional. In this section we give an explicit construction of them by means of the Koszul complex ([M, Section 9]). We consider the quadratic graded algebra

$$S_\mu=\oplus_n S_{\mu n}=T(V)/(V_\mu^-)$$

and its Manin dual

$$S^!_\mu=\oplus_n S^!_{\mu n}=T(V^*)/((V_\mu^-)^\perp).$$

The subscript μ will be omitted if no confusion may arise. $(S^!_n)^*$ is canonically identified with the subspace $\bigcap_{p+q=n-2} V^{\otimes p}\otimes V_\mu^-\otimes V^{\otimes q}$ of $V^{\otimes n}$ for each $n\geq 0$. The Koszul complex of S_μ consists of the differential maps

$$\partial_{m,n}:\ S_{m-1}\otimes(S^!_{n+1})^*\xrightarrow{1\otimes(\mathrm{incl})}S_{m-1}\otimes V\otimes(S^!_n)^*\xrightarrow{(\mathrm{mult})\otimes 1}S_m\otimes(S^!_n)^*.$$

The canonical right B_μ-comodule structure of V makes S (resp. $S^!$) into a right (resp. left) comodule algebra. Hence the Koszul complex consists of right B_μ-comodules and comodule maps. We define $S_{\mu,m,n}=S_{m,n}=\mathrm{Im}\ \partial_{m,n}$ for $m,\ n\geq 0$.

THEOREM 6.1. *If μ is not a root of 1, then $S_{m,n}$ for $m>0,\ n\geq 0$ form a complete list of nontrivial simple B_μ-comodules.*

PROPOSITION 6.2. *If μ is not a root of 1, then we have isomorphisms of B_μ-comodules*

$$S_{m,n}\otimes S_{m',n'}\cong S_{m+m',n+n'}\oplus S_{m+m'-1,n+n'+1}$$

for $m,\ m'>0,\ n,\ n'\geq 0$.

REMARK 6.3. The graded algebra $R=T(V)/(V^+)$ is also a right B_μ-comodule algebra. The compositions $(S^!_n)^*\hookrightarrow T(V)_n\twoheadrightarrow R_n$ of the natural maps are isomorphisms for all n if μ is not a root of 1.

Before proving these results, we give an explicit description of $S_{m,n}$. For elements $x,\ y$ of an algebra we write

$$\{x,y\}_n=\begin{cases}(xy)^{n/2} & \text{if } n \text{ is even}\\ (xy)^{(n-1)/2}x & \text{if } n \text{ is odd.}\end{cases}$$

Set

$$f_1=(\mu-1)e_1+(\mu+1)e_2$$
$$f_2=(\mu-1)e_1-(\mu+1)e_2.$$

The algebra S is defined by generators f_1, f_2 and relations $f_1f_2=f_2f_1=0$. Hence S_n has the basis f_1^n, f_2^n and $(S_n^!)^*$ is identified with the linear span of $\{f_1, f_2\}_n$, $\{f_2, f_1\}_n$ in $V^{\otimes n}$ for $n>0$. We see easily that

$$\begin{aligned}\partial_{m,n}:\ & f_i^{m-1}\otimes\{f_i, f_{i'}\}_{n+1}\mapsto f_i^m\otimes\{f_{i'}, f_i\}_n\\ & f_i^{m-1}\otimes\{f_{i'}, f_i\}_{n+1}\mapsto 0\end{aligned}$$

where $i'=3-i$. Hence $S_{m,n}$ has the basis $f_1^m\otimes\{f_2, f_1\}_n$, $f_2^m\otimes\{f_1, f_2\}_n$ for $m>0$, $n\geq 0$.

Let $y_{ij}\in B_\mu$ be such that the comodule structure of V is given by $f_i\mapsto\sum_i f_i\otimes y_{ij}$. The condition that $V_{\bar{\mu}}$ is a subcomodule of $V\otimes V$ amounts to the relations $y_{i1}y_{i2}=y_{i2}y_{i1}=0$ for $i=1, 2$. It follows that the comodule structure of $S_{m,n}$ is given by

$$f_j^m\otimes\{f_{j'}, f_j\}_n\mapsto\sum_i (f_i^m\otimes\{f_{i'}, f_i\}_n)\otimes y_{ij}^m\{y_{i'j'}, y_{ij}\}_n.$$

We have

$$(y_{ij})=\begin{pmatrix} f+\sqrt{\mu}s+\sqrt{\mu}^{-1}t & g-\sqrt{\mu}^{-1}s-\sqrt{\mu}t\\ g+\sqrt{\mu}^{-1}s+\sqrt{\mu}t & f-\sqrt{\mu}s-\sqrt{\mu}^{-1}t\end{pmatrix}.$$

The following is proved easily by induction.

LEMMA 6.4. *We have*

$$\begin{aligned} y_{ii}^m\{y_{jj}, y_{ii}\}_n &\equiv y_{ii}(f+\mu^{-1}g)^{m-1}(f-\mu^{-1}g)^n && \bmod s\\ &\equiv y_{ii}(f+\mu g)^{m-1}(f-\mu g)^n && \bmod t\\ y_{ij}^m\{y_{ji}, y_{ij}\}_n &\equiv y_{ij}(\mu f+g)^{m-1}(-\mu f+g)^n && \bmod s\\ &\equiv y_{ij}(\mu^{-1}f+g)^{m-1}(-\mu^{-1}f+g)^n && \bmod t\end{aligned}$$

for $i\neq j$.

Theorem 6.1 and Proposition 6.2 are proved by means of the pairing $\omega_{\lambda,\mu}$. Let λ, μ be not roots of 1. We have the embedding j: Comod-$B_\mu\to B_\lambda$-Mod induced by $\omega_{\lambda,\mu}$ (Section 5). For $m>0$, $n\geq 0$ set $\phi_1=f_1^m\otimes\{f_2, f_1\}_n$, $\phi_2=f_2^m\otimes\{f_1, f_2\}_n$, the basis of $S_{\mu,m,n}$.

PROPOSITION 6.5. *We have isomorphisms of* B_λ*-modules*

$$\begin{aligned} jS_{\mu,m,n} &\cong M_{\lambda s}(\lambda^n, \lambda^n\mu^m) && \textit{if } n \textit{ is even}\\ &\quad (1+\mu^{m+n})(\phi_1+\phi_2)\leftrightarrow e_1\\ &\quad -(1-\mu^{m+n})(\phi_1-\phi_2)\leftrightarrow e_2\\ jS_{\mu,m,n} &\cong M_{\lambda t}(\lambda^{n+1}, \lambda^{n+1}\mu^m) && \textit{if } n \textit{ is odd}\end{aligned}$$

$$\phi_1+\phi_2 \leftrightarrow e_1$$
$$\phi_1-\phi_2 \leftrightarrow e_2.$$

PROOF: Let $\varphi: B_\lambda \to \mathrm{End}(S_{\mu,m,n})$ be the module structure of $jS_{\mu,m,n}$. Identify $\mathrm{End}(S_{\mu,m,n})=M_2(k)$ by the basis ϕ_1, ϕ_2. Then

the (i, j) entry of the matrix $\varphi(x_{kl})$
$=\omega_{\lambda,\mu}(x_{kl}, y_{ij}^m\{y_{i'j'}, y_{ij}\}_n)$
$=$the (k, l) entry of the matrix $\pi_{\mu s}(1, \lambda)(y_{ij}^m\{y_{i'j'}, y_{ij}\}_n)$.

One can compute the last matrix using Lemma 6.4. It results that φ is given by

$$x_{11} \mapsto \begin{pmatrix} 1+\lambda & (-1)^n(1-\lambda)\mu^{m+n} \\ (-1)^n(1-\lambda)\mu^{m+n} & 1+\lambda \end{pmatrix}\frac{\lambda^n}{2}$$
$$x_{22} \mapsto \begin{pmatrix} 1-\lambda & (-1)^n(1+\lambda)\mu^{m+n} \\ (-1)^n(1+\lambda)\mu^{m+n} & 1-\lambda \end{pmatrix}\frac{(-\lambda)^n}{2}$$
$$x_{12} \mapsto \begin{pmatrix} (-1)^n & -\mu^{m+n} \\ \mu^{m+n} & -(-1)^n \end{pmatrix}\frac{\lambda^n(1-\lambda)}{2}$$
$$x_{21} \mapsto \begin{pmatrix} (-1)^n & -\mu^{m+n} \\ \mu^{m+n} & -(-1)^n \end{pmatrix}\frac{(-\lambda)^n(1+\lambda)}{2}.$$

Comparing this with the representations of Section 2, the proposition follows.

Now let us prove Theorem 6.1 and Proposition 6.2. Take any $\lambda \in k$ which is not a root of 1. By Propositions 6.5 and 4.1, the B_λ-modules $jS_{\mu,m,n}$ for $m>0$, $n>0$ are in one to one correspondence with the isomorphism classes of the simple direct summands of $j(V^{\otimes l})$ for $l>0$. This proves Theorem 6.1. Proposition 6.2 follows from Propositions 6.5 and 3.1.

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