Hokkaido Mathematical Journal Vol. 21(1992) p. 195-220

Simple graded Lie algebras of finite depth

Tomoaki YATSUI

(Received November 15, 1989, Revised December 10, 1991)

Introduction

In this paper we classify the infinite dimensional simple graded Lie algebras of finite depth over an algebraically closed field K of characteristic zero.

A graded Lie algebra (GLA) $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is a Lie algebra $\mathfrak{g}$ endowed with a gradation $\{\mathfrak{g}_p\}_{p\in Z}$ such that dim $\mathfrak{g}_p<\infty$ and $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$. It is called simple if the underlying Lie algebra $\mathfrak{g}$ is simple. We say a GLA $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is of finite depth if the negative part $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$ is finite dimensional. Note that a GLA of finite depth having at least dimension two and no proper graded ideal is simple (see § 1). Note also that a simple GLA is necessarily transitive (a GLA $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is called transitive if for $x\in\mathfrak{g}_p$ $(p\geq 0)$, $[x, \mathfrak{g}_-]=0$ implies $x=0$).

Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a simple GLA of finite depth. According to Cartan's classification of the simple infinite transitive pseudogroups, or rather according to its algebraic version, i. e., classification of the primitive infinite Lie algebras, completed by many authors (in particular, Singer-Sternberg [SS65], Kobayashi-Nagano [KN66], Guillemin-Quillen-Sternberg [GQS66], Morimoto-Tanaka [MT70]), we see that the underlying Lie algebra $\mathfrak{g}$ is isomorphic to one of the following series of simple Lie algebras:

1) $W(m)$: the Lie algebra of all the polynomial vector fields $\sum_{i=1}^{m}P_i\partial/\partial x_i$ with $P_i\in K[x_i,\dots,x_m]$.

2) $S(m)$: the subalgebra of $W(m)$ consisting of the vector fields which preserve the differential form $dx_1\wedge\dots\wedge dx_m$ $(m\geq 2)$;

3) $H(n)$: the subalgebra of $W(m)$ consisting of vector fields which preserve the differential form $\sum_{i=1}^{n}dx_i\wedge dx_{n+i}$, $m=2n$;

4) $K(n)$: the subalgebra of $W(m)$ consisting of vector fields which preserve the differential form $dx_m-\sum_{i=1}^{n}x_{i+n}dx_i$ $(m=2n+1)$ up to the multicative factors in $K[x_1,\dots,x_m]$.

(Thus our initial problem is reduced to determining all the possible gradations of the above Lie algebras.)

It is well-known that each of them has a standard gradation, that is, a primitive gradation uniquely determined up to isomorphism (we say a gradation $\{\mathfrak{g}_p\}_{p\in Z}$ is primitive if $\bigoplus_{p\geq 0}\mathfrak{g}_p$ is maximal subalgebra of $\mathfrak{g}$). There also exist other gradations defined as follows ([Kac70]): Let $\boldsymbol{s}=(s_1,\ldots,s_m)$ be an m-tuple of positive integers. We define the subspaces $\mathfrak{g}_k$ ($k\in \boldsymbol{Z}$) of the Lie algebras (1)-(4) to consist of the vector fields $\sum_{i=1}^{m}P_i\partial/\partial x_i$ such that each polynomial P_i is a linear combination of monomials $x_1^{a_1}\ldots x_m^{a_m}$ with $\sum_{i=1}^{m}a_i s_i=s_i+k$. Then the family of subspaces $\{\mathfrak{g}_k\}_{k\in Z}$ defines a gradation on $W(m)$ and $S(m)$ for each $\boldsymbol{s}$, and on $H(n)$ (resp. $K(n)$) if and only if $(s_1,\ldots,s_m)=(t_1,\ldots, t_n, \mu-t_1,\ldots,\mu-t_n)$ (resp. $(s_1,\ldots,s_m)=(t_1,\ldots ,t_n, \mu-t_1,\ldots,\mu-t_n, \mu)$) for an $(n+1)$-tuple $(\boldsymbol{t}, \mu)=(t_1,\ldots ,t_n, \mu)$ with $\mu\geq 2$. We will denote by $W(m:\boldsymbol{s})$, $S(m:\boldsymbol{s})$, $H(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$ the GLAs equipped with the above gradation. Note that the standard gradations are obtained by letting $s_1=\ldots=s_m=1$ for $W(m)$, $S(m)$, and $t_1=\ldots=t_n=1$, $\mu=2$ for $H(n)$, $K(n)$.

Viewed geometrically, non-standard gradations on the Lie algebras (1)-(4) might seem rather artificial, however they are closely related with the geometry of differential systems developed by N. Tanaka. For example, if we let $s_1=\ldots=s_\ell=1$ and $s_{\ell+1}=\ldots=s_n$ for some $\ell\geq 1$ (or resp. $t_1=\ldots.=t_n=1$), then we have the GLAs $W(n:\boldsymbol{s})$ (or resp. $K(n:\boldsymbol{t}:\mu)$) that appear in higher order contact geometry (cf. [Yam82], [Yam83]). More generally all the gradations introduced as above on the Lie algebras (1)-(4) can be interpreted geometrically by using the notion, due to T. Morimoto, of weighted jet bundles associated with differential systems.

Now our main theorem may be stated as follows (which was announced in [Yat89]):

THEOREM. *Any infinite dimensional simple GLA of finite depth over an algebraically closed field K of characteristic zero is isomorphic to one of the following GLAs*: $W(n:\boldsymbol{s})$, $S(n:\boldsymbol{s})$, $H(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$.

In [Kac70]. V. G. Kac conjectured that any simple GLA of finite growth over an algebraically closed field of characteristic zero is isomorphic to one of the GLAs (1)-(4), finite dimensional simple GLAs of type $(s_1,\ldots,s_n)$, affine Lie algebras with the gradation of type $(s_1,\ldots,s_n)$ and a Lie algebra of Witt. Thus our result gives an answer to a particular case of Kac's conjecture.

Now let us explain the outline of the proof of the theorem and describe breifly the contents of this paper.

Let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be an infinite dimensional simple GLA of finite depth. Let E be the derivation of $\mathfrak{g}$ defining the gradation (i. e., $Ex=px$ for all $x\in\mathfrak{g}_p$). First we shall construct a standard gradation $\{G_k\}_{k\in \boldsymbol{Z}}$ on $\mathfrak{g}$ such that each G_k is E-stable. The existence of a primitive gradation is assured by the Cartan's classification. In order to construct an E-stable one, we follow [But67], slightly generalizing her arguement to fit in our purpose. Then again by Cartan's classification, we can identify $\mathfrak{g}=\bigoplus_{k\in \boldsymbol{Z}} G_k$ with a simple infinite Lie algebra of Cartan type with the standard gradation. We shall then determine how the element E is expressed explicitly in $\bigoplus_{k\in \boldsymbol{Z}} G_k$, by using the detailed structure of $\bigoplus_{k\in \boldsymbol{Z}} G_k$ and the method of root systems, to obtain finally our result.

In Section 1, we prepare basic notions and facts needed in the sequel. In particular, we prove some fundamental properties of reductive GLAs. Then we construct an E-stable primitive gradation as mentioned above.

Section 2 is devoted to the proof of the main theorem.

In Section 3, from some geometric motiviation, we study the prolongation of the associated truncated GLAs to GLAs of Cartan type and finite dimensional simple GLAs.

§ 1. Preliminaries.

This section contains some definitions and results used throughout the paper. Throughout the entire paper the ground field K is assumed to be an algebraically closed field of characteristic zero. Each homogeneous component of a graded vector space considered in this paper is assumed to be of finite dimension. We use the following notation: $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{b})$ denotes the centralizer of $\mathfrak{b}$ in $\mathfrak{a}$, $\mathfrak{z}(\mathfrak{a})$ denotes the center of $\mathfrak{a}$, Der $\mathfrak{a}$ denotes the derivation algebra of $\mathfrak{a}$ and $\boldsymbol{Z}$ denotes the ring of integers.

1.1. Graded Lie algebras.

A graded Lie algebra (GLA) in a graded vector space $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ equipped with a Lie bracket such that $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$. Also the family of subspaces $\{\mathfrak{g}_p\}_{p\in \boldsymbol{Z}}$ of $\mathfrak{g}$ is called a gradation on $\mathfrak{g}$. Two GLAs $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ and $\mathfrak{g}'=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}'_p$ are considered to be isomorphic if there exists an isomorphism φ: $\mathfrak{g}\longrightarrow\mathfrak{g}'$ such that $\varphi(\mathfrak{g}_p)\subset \mathfrak{g}'_p$. A GLA $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ is called transitive if for $x\in$

$\mathfrak{g}_p$ $(p\geq 0)$, $[x, \mathfrak{g}_-]=0$ implies $x=0$, where $\mathfrak{g}_-=\bigoplus_{p<0}\mathfrak{g}_p$, and irreducible if the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is so. Moreover a GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is said to be of finite depth if $\dim \mathfrak{g}_-<\infty$. In particular, if $\mathfrak{g}_{-\mu}\neq\{0\}$ and $\mathfrak{g}_p=\{0\}$ for $p<-\mu$, then $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is said to be of depth μ. In this paper we will always assume $\mu\geq 1$.

REMARK. Let $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ be a GLA of finite depth such that $\dim \mathfrak{g}\geq 2$. For $\mathfrak{g}$ to be simple, it is neccesary and sufficient that $\mathfrak{g}$ has no graded ideal. In fact, the condition is obviously neccesary. Conversely, let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{g}$. As in the proof of Proposition 1.6.1 in [Wei78], we set $\mathfrak{a}_p^{\#}=\{x\in\mathfrak{g}_p : {}^{\exists}x_i\in\mathfrak{a}_i,\ i<p, \text{ with } x+\sum x_i\in\mathfrak{a}\}$. Then $\mathfrak{a}^{\#}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p^{\#}$ is a graded ideal of $\mathfrak{g}$. If $\mathfrak{a}^{\#}$ is equal to $\mathfrak{g}$, then so does $\mathfrak{a}$. This proves our assertion.

Next let us define a truncated graded Lie algebra (truncated GLA) and its prolongation ([Mor88]). Let $k\in\boldsymbol{Z}\cup\{\infty\}$. A graded vector space $\mathfrak{g}(k)=\bigoplus_{p\leq k}\mathfrak{g}_p$ is called a truncated GLA of order k if one has a bracket operation (skew-symmetric bilinear mapping) $[\ ,\ ]: \mathfrak{g}_p\times\mathfrak{g}_q\longrightarrow\mathfrak{g}_{p+q}$ for $p, q, p+q\leq k$ such that the Jacobi identity holds whenever it makes sense. A transitive truncated GLA and a truncated GLA of finite depth are defined as in the case of GLAs. Note that a truncated GLA of order ∞ is just a GLA. If $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is a GLA, then for each integer k, $\mathrm{Trun}_k(\mathfrak{g}):=\bigoplus_{p\leq k}\mathfrak{g}_p$ becomes a truncated GLA of order k with respect to the induced bracket operation, which is called the associated truncated GLA of order k to the GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$. Let $\mathfrak{g}(k)=\bigoplus_{p\leq k}\mathfrak{g}_p (k\geq -1)$ be a transitive truncated GLA of finite depth. Then there exists, uniquely up to isomorphism, a transitive GLA $\mathrm{Prol}\,\mathfrak{g}(k)$ of finite depth satisfying the following conditions ([Tan70]):

(i) $\mathrm{Trun}_k(\mathrm{Prol}\,\mathfrak{g}(k))=\mathfrak{g}(k)$.

(ii) If $\mathfrak{h}$ is a transitive GLA and if there exists an isomorphism ψ of $\mathrm{Trun}_k(\mathfrak{h})$ onto $\mathfrak{g}(k)$, then there exists a monomorphism φ of $\mathfrak{h}$ into $\mathrm{Prol}\,\mathfrak{g}(k)$ such that $\varphi|\mathrm{Trun}_k(\mathfrak{h})=\psi$.

We call the transitive GLA $\mathrm{Prol}\,\mathfrak{g}(k)$ the prolongation of $\mathfrak{g}(k)$. We say also that a transitive GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ of finite depth is the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ if $\mathfrak{g}=\mathrm{Prol}\,\mathrm{Trun}_k(\mathfrak{g})$. Note that $\mathfrak{g}$ can be always identified with a graded subalgebra of $\mathrm{Prol}\,\mathrm{Trun}_k(\mathfrak{g})$. Finally let E be an element of a Lie algebra $\mathrm{Der}_0(\mathfrak{g})$ of all the derivations preserving the gradation of a GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ such that $[E, x]=px$ for all $x\in\mathfrak{g}_p$. We call this element E the defining element of a GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$.

1.2. Gradations of finite dimensional simple Lie algebras.

Let $\mathfrak{g}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{g}_p$ be a finite dimensional simple GLA. Then its defining element E is contained in $\mathfrak{g}_0$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$. Let Δ be a root system of $(\mathfrak{g},\mathfrak{h})$. We will usually denote by e_α the root vector corresponding to a root $\alpha\in\Delta$. Given a $\mathfrak{h}$-stable subspace $\mathfrak{a}$ of $\mathfrak{g}$, we set $\Delta(\mathfrak{a})=\{\alpha\in\Delta:\ e_\alpha\in\mathfrak{a}\}$. In particular we set $\Delta_p=\Delta(\mathfrak{g}_p)$ and $\Sigma=\bigcup_{p\geq 0}\Delta_p$. Since $\Delta=\Sigma\cup(-\Sigma)$, there are a simple root system $\Pi=\{\alpha_1,\dots,\alpha_\ell\}$ of$(\mathfrak{g},\mathfrak{h})$ and an ℓ-tuple of non-negative integers $(s_1,\dots,s_\ell)$ such that $\Delta_p=\{\alpha=\sum_i k_i\alpha_i\in\Delta:\ \sum_i s_ik_i=p\}$ (cf. [Bou75]). The ℓ-tuple $(s_1,\dots,s_\ell)$ of non-negative integers is determined only by the gradation of $\mathfrak{g}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{g}_p$ up to the ordering of $(\alpha_1,\dots,\alpha_\ell)$. In what follows, we assume that the ordering of $(\alpha_1,\dots,\alpha_\ell)$ is as in the table of [Bou68]. Moreover we assume that $s_{i_0}>s_{\mu(i_0)}$ for all the automorphisms μ of the Dynkin diagram such that $s_i\neq s_{\mu(i)}$ for some i, where $i_0=\min\{i:\ s_i\neq s_{\mu(i)}\}$. Then the above gradation of $\mathfrak{g}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{g}_p$ is called the gradation of type $(s_1,\dots,s_\ell)$. Furthermore for $k>0$, a finite dimensional simple GLA $\mathfrak{g}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{g}_p$ is called of general type of order k if $\mathfrak{g}_-=\mathfrak{g}_{-k}$, and of contact type of order k if $\mathfrak{g}_-=\mathfrak{g}_{-2k}\oplus\mathfrak{g}_{-k}$ and $\dim\mathfrak{g}_{-2k}=1$. The classification of finite dimensional simple GLAs of general type or of contact type can be easily done by using the Dynkin diagram. Here we remark that $\mathfrak{g}_0$ is reductive in $\mathfrak{g}$ and $\mathfrak{g}_p$ is contragredient to $\mathfrak{g}_{-p}$ as a $\mathfrak{g}_0$-module.

Now we state a few properties of reductive GLAs, which we will use later on.

LEMMA 1. *Let $\mathfrak{l}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{l}_p$ be a finite dimensional reductive GLA of depth $\mu\geq 1$, and $\mathfrak{u}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{u}_p$ be a graded subalgebra of $\mathfrak{l}$.*

(1) *Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{l}$ with $\mathfrak{n}\cap\mathfrak{u}=\{0\}$ and $[\mathfrak{u},\mathfrak{n}]\subset\mathfrak{n}$. If $\mathfrak{u}$ is reductive and $\mathfrak{u}_0$ contains a Cartan subalgebra of $\mathfrak{l}$, then there exists a nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{l}$ such that $\mathfrak{n}_++\mathfrak{u}+\mathfrak{n}$ is a direct sum, $[\mathfrak{u},\mathfrak{n}_+]\subset\mathfrak{n}_+$ and $\mathfrak{n}_+$ is contragredient to $\mathfrak{n}$ as a $\mathfrak{u}$-module.*

(2) *Let $\mathfrak{r}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{r}_p$ be the radical of $\mathfrak{u}$. Suppose that $Trun_k(\mathfrak{l})=Trun_k(\mathfrak{u})$ $(k\geq -1)$ and $\mathfrak{u}_0$ contains a Cartan subalgebra of $\mathfrak{l}$. Then $\mathfrak{r}_p=\{0\}$ for $p\geq -k$, $p\neq 0$, and $\mathfrak{r}_0=\mathfrak{z}(\mathfrak{l})$.*

(3) *Suppose that $\mathfrak{l}_-=\mathfrak{u}_-$ and $\mathfrak{u}$ is reductive. Then $[\mathfrak{l},\mathfrak{l}]=[\mathfrak{u},\mathfrak{u}]$.*

PROOF. (1) Let $\mathfrak{s}=[\mathfrak{l},\mathfrak{l}]$; then $\mathfrak{s}$ is graded, which we write $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$ contained in $\mathfrak{u}_0$; then $\mathfrak{h}=(\mathfrak{h}\cap\mathfrak{s})\oplus\mathfrak{z}(\mathfrak{l})$ and $\mathfrak{h}\cap\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$. Let Δ be a root system of $(\mathfrak{s}, \mathfrak{s}\cap\mathfrak{h})$. Since $\mathfrak{n}$ is nilpotent, it follows that $-\alpha\notin\Delta(\mathfrak{n})$ if $\alpha\in\Delta(\mathfrak{n})$. Let σ be an automorphism of $\mathfrak{s}$ such that $\sigma(e_{\alpha})=e_{-\alpha}$ and $\sigma(h)=-h$ for $h\in\mathfrak{h}\cap\mathfrak{s}$. If we put $\sigma(z)=-z$ for $z\in\mathfrak{z}(\mathfrak{l})$, then we can extend σ an automorphism on $\mathfrak{l}$. Then we may put $\sigma(\mathfrak{n})=\mathfrak{n}_+$.

(2) If $\alpha\in\Delta(\mathfrak{r}_p)$ for $p\geq -k$, then $-\alpha\in\Delta(\mathfrak{r}_{-p})$ because $\mathfrak{r}$ is the ideal of $\mathfrak{u}$ and $\mathrm{Trun}_k(\mathfrak{l})=\mathrm{Trun}_k(\mathfrak{u})$, so $\mathfrak{r}$ contains a three dimensional simple subalgebra of $\mathfrak{l}$, which is a contradiction. Thus we have $\mathfrak{r}_0\subset\mathfrak{h}$ and $\mathfrak{r}_p=\{0\}$ for $p\geq -k$, $p\neq 0$. If there exists a non-zero element $h\in\mathfrak{r}_0$ such that $\alpha(h)\neq 0$ for some $\alpha\in\Delta$, then we can similarly reach a contradiction. Hence $\mathfrak{r}_0=\mathfrak{z}(\mathfrak{l})$.

(3) Since $\mathfrak{u}$ and $\mathfrak{l}$ are reductive, we see that $\dim \mathfrak{l}_{-p}=\dim \mathfrak{l}_p$ and $\dim \mathfrak{u}_p=\dim \mathfrak{u}_{-p}$ for $p\neq 0$. Further since $\mathfrak{u}_p=\mathfrak{l}_p$ for $p<0$, we have $[\mathfrak{l},\mathfrak{l}]=[\mathfrak{u},\mathfrak{u}]$. $\square$

1.3. Graded Lie algebras of Cartan type.

In this subsection we describe Lie algebras of Cartan type and their gradations.

Let $A(m)$ denote the monoid (under addition) of all m-tuples of non-negative integers. For $1\leq i\leq m$ let ε_i denote the m-tuple $(\delta_{i1},\dots,\delta_{im})$. For $\alpha, \beta\in A(m)$ define $\binom{\alpha}{\beta}=\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_m}{\beta_m}$ and $\alpha!=\prod\alpha_i!$. For $\boldsymbol{t}=(t_1,\dots,t_m)$ an m-tuple of positive integers and $\alpha\in A(m)$ we set $\|\alpha\|_t=\sum_{i=1}^{m}t_i\alpha_i$. Also for $\boldsymbol{s}^{(i)}=(s_1^{(i)},\dots,s_m^{(i)})$ an m_i-tuple of positive integers $(i=1,\dots,n)$ we denote by $(\boldsymbol{s}^{(1)},\dots,\boldsymbol{s}^{(n)})$ the $\sum_{i=1}^{n}m_i$-tuple $(s_1^{(1)},\dots,s_{m_1}^{(1)},\dots,s_1^{(n)},\dots,s_{m_n}^{(n)})$. Further we write $\mathbf{1}_m$ (or $\mathbf{1}$ if no confusion arises) for them-tuple $(1,\dots,1)$, and for a positive integer k we denote the m-tuple $(k,\dots,k)$ by $k\mathbf{1}$. Let $\mathfrak{A}(m)=K[x_1,\dots,x_m]$. For $\alpha\in A(m)$ define $x^{(\alpha)}=(\prod x_i^{\alpha_i})/\alpha!$. Then $x^{(\alpha)}x^{(\beta)}=\binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}$. For any m-tuple $\boldsymbol{s}$ of positive integers we define a gradation of $\mathfrak{A}(m)$ by $\mathfrak{A}(m:\boldsymbol{s})_p=\{\sum a_{\alpha}x^{(\alpha)}: \|\alpha\|_s=p\}$. Let D_i denote the i-th partial derivative defined by

$$D_ix^{(\alpha)}=x^{(\alpha-\varepsilon_i)} \text{ for } \alpha\in A(m),$$

where we put $x^{(\beta)}=0$ if $\beta\notin A(m)$.

Let $W(m)=\mathrm{Der}\,\mathfrak{A}(m)=\{\sum u_iD_i: u_i\in\mathfrak{A}(m)\}$. The Lie algebra $W(m)$ is called the general algebra. For $\boldsymbol{s}=(s_1,\dots,s_m)$ an m-tuple of positive integers $W(m)$ has a gradation $\{W(m:\boldsymbol{s})_p\}_{p\in Z}$, where $W(m:\boldsymbol{s})_p=\sum_{i=1}^{m}\mathfrak{A}$

$(m:\ \boldsymbol{s})_{p+s_k}D_k$. We will denote by $W(m:\ \boldsymbol{s})$ the GLA $W(m)$ equipped with this gradation.

We now consider the following three differential forms:

$$\omega_S = dx_1\wedge\ldots\wedge x_m,\ \ m\geq 2,$$
$$\omega_H = \sum_{i=1}^{n} dx_i\wedge dx_{i'}\ \text{ for } m=2n,\ \ n\geq 1,$$
$$\omega_K = dx_{2n+1} - \sum_{i=1}^{n} x_{i'}dx_i\ \text{ for } m=2n+1,\ \ n\geq 1,$$

where $i' = \begin{cases} i+n \text{ for } 1\leq i\leq n \\ i-n \text{ for } n< i\leq 2n. \end{cases}$

Define subalgebras $S(m)$, $CS(m)$, $H(n)$, $CH(n)$, $K(n)\subset W(m)$ by

$$\begin{aligned} S(m) &= \{D\in W(m):\ D\omega_S=0\}, \\ CS(m) &= \{D\in W(m):\ D\omega_S\in K\omega_s\}, \\ H(n) &= \{D\in W(m):\ D\omega_H=0\}, \\ CH(n) &= \{D\in W(m):\ D\omega_H\in K\omega_H\}, \\ K(n) &= \{D\in W(m):\ D\omega_K\in \mathfrak{A}(m)\omega_K\}, \end{aligned}$$

(Here the action of D on the differential forms is defined through Lie derivative.) The Lie algebras $S(m)$, $H(n)$, $K(n)$ are called the special algebra, the Hamiltonian algebra and the contact algebra respectively. Also for $X=W$, S, CS, H, CH, K, the Lie algebra $X(n)$ is called a Lie algebra of Cartan type. Then we can easily prove the following assertions ([Kac70, § 2]).

PROPOSITION 2. *Let $\boldsymbol{s}=(s_1,\ldots,s_m)$ be an m-tuple of positive integers. Then*

(1) *$S(m)$ and $CS(m)$ are graded subalgebras of $W(m:\ \boldsymbol{s})$.*

(2) *$H(n)$ and $CH(n)$ $(m=2n)$ are graded subalgebras of $W(m:\ \boldsymbol{s})$ if and only if $s_i+s_{i'}=s_j+s_{j'}$.*

(3) *$K(n)$ $(m=2n+1)$ is a graded subalgebra of $W(m:\ \boldsymbol{s})$ if and only if $s_i+s_{i'}=s_{2n+1}$ for $i=1,\ldots,n$.*

From this proposition we can define the gradations on $S(m)$, $CS(m)$, $H(n)$, $CH(n)$, $K(n)$ induced by $W(m:\ \boldsymbol{s})$ as follows. For $X=S$ or CS, and $\boldsymbol{s}=(s_1,\ldots,s_m)$ an m-tuple of positive integers, we set $X(m:\ \boldsymbol{s})_p = W(m:\ \boldsymbol{s})_p\cap X(m)$. Further for $X=H$, CH, K, $\boldsymbol{t}=(t_1,\ldots,t_n)$ an n-tuple of positive integers, and positive integer $\mu\geq 2$ such that $t_i<\mu$, we set $X(n:\ \boldsymbol{t}:\ \mu)_p = W(m:\ \boldsymbol{s})_p\cap X(n)$, where $\boldsymbol{s}=(\boldsymbol{t},\ \mu\mathbf{1}-\boldsymbol{t})$ for $X=H$, CH and $\boldsymbol{s}=(\boldsymbol{t},\ \mu\mathbf{1}-\boldsymbol{t},\ \mu)$ for $X=K$. Then we can define a gradation on $S(m)$, $CS(m)$, $H(n)$, $CH(n)$, $K(n)$ by $\{S(m:\ \boldsymbol{s})_p\}_{p\in Z}$, $\{CS(m:\ \boldsymbol{s})_p\}_{p\in Z}$, $\{H(n:\ \boldsymbol{t}:\ \mu_p)\}_{p\in Z}$,

$\{CH(n:\boldsymbol{t}:\mu)_p\}_{p\in Z}$, $\{K(n:\boldsymbol{t}:\mu)_p\}_{p\in Z}$ respectively. The Lie algebras $S(m)$, $CS(m)$, $H(n)$, $CH(n)$ and $K(n)$ equipped with this gradation will be denoted by $S(m:\boldsymbol{s})$, $CS(m:\boldsymbol{s})$, $H(n:\boldsymbol{t}:\mu)$, $CH(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$ respectively, which are called GLAs of Cartan type. In particular, if $\boldsymbol{s}=\boldsymbol{1}$ for $X=W, S, CS$, and if $\boldsymbol{t}=\boldsymbol{1}$ and $\mu=2$ for $X=H, CH, K$, then the above gradation is called the standard gradation. Here we remark the following fact: For $X=W, S, CS$ (resp. $X=H, CH, K$), $\boldsymbol{s}=(s_1,\dots,s_m)$ (resp. $\boldsymbol{s}=(t_1,\dots,t_n,\mu-t_1,\dots,\mu-t_n)$) coincides with $\boldsymbol{s}'=(s_1',\dots s_m')$ (resp. $\boldsymbol{s}=(t_1',\dots,t_n',\mu-t_1',\dots,\mu-t_n')$) as a set if and only if $X(m:\boldsymbol{s})$ (resp. $X(n:\boldsymbol{t}':\mu)$) is isomorphic to $X(m:\boldsymbol{s}')$ (resp. $X(n:\boldsymbol{t}':\mu)$) as a GLA. In what follows, we will assume that $s_1\leq\dots\leq s_m$ for $X=W, S, CS$, and $t_1\leq\dots\leq t_n$ and $t_i\leq[\mu/2]$ for $X=H, CH, K$. Moreover we remark that for $X=W, S, CS$ (resp. $X=H, CH, K$), $X(m:\boldsymbol{s})_-$(resp. $X(n:\boldsymbol{t}:\mu)_-$) is generated by $X(m:\boldsymbol{s})_{-1}$ (resp. $X(n:\boldsymbol{t}:\mu)_{-1}$) if and only if $s_1=1$ (resp. $t_1=1$).

To simplify the caluculation on $K(n)$, we will use another characterization of $K(n)$. We define a linear mapping $D_K:\mathfrak{A}(2n+1)\longrightarrow W(2n+1)$ by means of $D_K(f)=\sum_{j=1}^{2n+1}f_jD_j$, where $f_j=-D_{i'}f$ $(1\leq j\leq n)$, $f_j=D_{i'}f+x_jD_{2n+1}f$ $(n+1\leq j\leq 2n)$ and $f_{2n+1}=f-\sum_{j=1}^{n}x_{i'}\,D_{i'}f$. For f, $g\in A(2n+1)$ we put $[f,g]=fD_{2n+1}(g)-gD_{2n+1}(f)+D_K(f)g$. Then we have $[D_K(f),D_K(g)]=[f,g]$. In particular,

$$\begin{aligned}[x^{(\alpha)},x^{(\beta)}]=&\sum_{i=1}^{n}\left\{\binom{\alpha+\beta-\varepsilon_i-\varepsilon_{i'}}{\alpha-\varepsilon_i}-\binom{\alpha+\beta-\varepsilon_i-\varepsilon_{i'}}{\beta-\varepsilon_i}\right\}x^{(\alpha+\beta-\varepsilon_i-\varepsilon_{i'})}\\&+\left\{(|\beta|'-1)\binom{\alpha+\beta-\varepsilon_{2n+1}}{\beta}-(|\alpha|'-1)\binom{\alpha+\beta-\varepsilon_{2n+1}}{\alpha}\right\}x^{(\alpha+\beta-\varepsilon_{2n+1})},\end{aligned}$$

where $|\alpha|'=\sum_{i=n+1}^{2n}\alpha_i$. Then the mapping D_K is an isomorphism of $\mathfrak{A}(2n+1)$ onto $K(n)$, so the bracket $[\ ,\ ]$ induces the Lie algebra structure on $\mathfrak{A}(2n+1)$. In what follows we identify $\mathfrak{A}(2n+1)$ with $K(n)$. We have $K(n:\boldsymbol{t}:\mu)_p=\{x^{(\alpha)}:\|\alpha\|_{\boldsymbol{s}}=\mu+p\}$ for $\boldsymbol{t}=(t_1,\dots,t_n)$ an n-tuple of positive integers and a positive integer $\mu\geq 2$ such that $t_1\leq\dots\leq t_n$ and $t_i\leq[\mu/2]$, where $\boldsymbol{s}=(\boldsymbol{t},\mu\boldsymbol{1}-\boldsymbol{t},\mu)$.

Now we state the structures of $W(m:\boldsymbol{1})$, $S(m:\boldsymbol{1})$, $CS(m:\boldsymbol{1})$, $H(n:\boldsymbol{1}:2)$, $CH(n:\boldsymbol{1}:2)$ and $K(n:\boldsymbol{1}:2)$, which is investigated by many authors (e.g., [KN65], [MT70], [Kac68]). Here we describe these results according to [Kac68].

THEOREM 3. *([Kac68, Proposition 19]) Let* $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ *be one of*

$W(n:\mathbf{1})$ $(n\geq 2)$, $S(n:\mathbf{1})$, $CS(n:\mathbf{1})$, $H(n:\mathbf{1}:2)$, $CH(n:\mathbf{1}:2)$, $K(n:\mathbf{1}:2)$. *Then* $\mathfrak{g}_0=\mathfrak{g}_0'\oplus\mathfrak{z}(\mathfrak{g}_0)$, *where* $\mathfrak{g}_0'$ *is the semisimple part of* $\mathfrak{g}_0$; *the subspace* $\mathfrak{g}'=\mathfrak{g}_-\oplus\mathfrak{g}_0'\oplus\bigoplus_{p>0}\mathfrak{g}_p$ *is an ideal of codimension one of* $\mathfrak{g}=CS(n:\mathbf{1})$ (*resp.* $CH(n:\mathbf{1}:2)$), *isomorphic to* $S(n:\mathbf{1})$ (*resp.* $H(n:\mathbf{1}:2)$). *The Lie algebras* $W(n:\mathbf{1})$, $S(n:\mathbf{1})$, $H(n:\mathbf{1}:2)$ *and* $K(n:\mathbf{1}:2)$ *are simple*; *their structures are given by the following table*:

Cartan type	$\mathfrak{g}_0'$	$\mathfrak{g}_0'$-module $\mathfrak{g}_k$	dim $\mathfrak{z}(\mathfrak{g}_0)$
$W(n:\mathbf{1})$	A_{n-1}	$T_k(n)\oplus S_k(n)$	1
$S(n:\mathbf{1})$	A_{n-1}	$T_k(n)$	0
$H(n:\mathbf{1}:2)$	C_n	$P_k(n)$	0
$K(n:\mathbf{1}:2)$	C_n	$P_{-2}(n)\oplus\dots\oplus P_k(n)$ (k=even) $P_{-1}(n)\oplus\dots\oplus P_k(n)$ (k=odd)	1

In the table, $T_k(n)$ and $S_k(n)$ are irreducible representations of A_{n-1} with the following highest weight:

$$T_k(n)\quad \overset{1}{\circ}\text{---}\circ\text{-----}\circ\text{---}\overset{k+1}{\circ}\qquad S_k(n)\quad \circ\text{---}\circ\text{------}\circ\text{---}\overset{k}{\circ}$$

and $P_k(n)$ is an irreducible representation of C_n:

$$P_k(n)\quad \overset{k+2}{\circ}\text{---}\circ\text{-----}\circ\Leftarrow\circ$$

In particular, the canonical sequence $(\mathfrak{g}_0,\mathfrak{h},\Pi,(e_\alpha)_{\alpha\in\Pi},F_\Lambda)$ of $\mathfrak{g}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{g}_p$ (where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$, Π is a simple root system of $(\mathfrak{g},\mathfrak{h})$ and F_Λ is the highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ with respect to $(\mathfrak{g}_0,\mathfrak{h},\Pi)$) is described as follows:

Case 1. $\mathfrak{g}=W(n:\mathbf{1})$

$$\mathfrak{h}=\sum_{i=1}^{n}K(x_iD_i),\quad e_{\alpha_i}=-x_{i+1}D_i,\quad F_\Lambda=D_1$$

Case 2. $\mathfrak{g}=S(n:\mathbf{1})$

$$\mathfrak{h}=\sum_{i=1}^{n}K(x_iD_i-x_{i+1}D_{i+1}),\quad e_{\alpha_i}=-x_{i+1}D_i,\quad F_\Lambda=D_1$$

Case 3. $\mathfrak{g}=CS(n:\mathbf{1})$

$$\mathfrak{h}=\sum_{i=1}^{n}K(x_iD_i),\quad e_{\alpha_i}=-x_{i+1}D_i,\quad F_\Lambda=D_1$$

Case 4. $\mathfrak{g}=H(n:\mathbf{1}:2)$

$$\mathfrak{h}=\sum_{i=1}^{n}K(x_iD_i-x_{i'}D_{i'}),\quad e_{\alpha_i}=-x_{i+1}D_i+x_{i'}D_{i'+1}(1\leq i\leq n-1),$$

$e_{\alpha_n}=-x_{2n}D_{2n},\ F_\Lambda=D_1.$

Case 5. $\mathfrak{g}=CH(n:\mathbf{1}:2)$

$\mathfrak{h}=\sum_{i=1}^{n}K(x_iD_i-x_{i'}D_{i'})\oplus K(\sum_{i=1}^{2n}x_iD_i),$

$e_{\alpha_i}=-x_{i+1}D_i+x_{i'}D_{i'+1}(1\leq i\leq n-1),\ e_{\alpha_n}=-x_{2n}D_{2n},\ F_\Lambda=D_1.$

Case 6. $\mathfrak{g}=K(n:\mathbf{1}:2)$

$\mathfrak{h}=\sum_{i=1}^{n}K(x_ix_{i'}),\ e_{\alpha_i}=x_{i+1}x_i,\ (1\leq i\leq n),\ e_{\alpha_n}=-1/2x_{2n}^2,\ F_\Lambda=x_{n+1}.$

1.4. Filtered Lie algebras.

Let L be a Lie algebra and $\mathfrak{t}$ be a finite dimensional Lie algebra. Assume that the L has a $\mathfrak{t}$-module structure and any element of $\mathfrak{t}$ acts on L as a derivation. A decreasing sequence $\{L^p\}_{p\in Z}$ of $\mathfrak{t}$-stable subspaces of L is called a $\mathfrak{t}$-filtration (simply a filtration if $\mathfrak{t}=\{0\}$) on L if $[L^p, L^q]\subset L^{p+q}$ and $\dim L^p/L^{p+1}<\infty$. The Lie algebra L with this $\mathfrak{t}$-filtration is called a $\mathfrak{t}$-fitered Lie algebra (FLA), which we write $(L, \{L^p\}_{p\in Z})$. Given a $\mathfrak{t}$-filtration $\{L^p\}_{p\in Z}$ on L, clearly $\cap L^p$ is a $\mathfrak{t}$-stable ideal of L; the $\mathfrak{t}$-filtration is called separated if $\cap L^p=\{0\}$, weakly transitive if $L^{p+1}=\{x\in L^p: [x, L^p]\subset L^{p+a+1}$ for all $a<0\}$ $(p\geq 0)$, transitive if L is weakly transitive and separated, and of finite depth if $L=L^{-\mu}$ for some $\mu\geq 0$. Let $(L, \{L^p\}_{p\in Z})$ and $(L', \{L'^p\}_{p\in Z})$ be two $\mathfrak{t}$-FLAs; a homomorphism $h: L\longrightarrow L'$ is called a $\mathfrak{t}$-homomorphism of $\mathfrak{t}$-FLAs if $h(L^p)\subset L'^p$ and h is a $\mathfrak{t}$-module homomorphism.

Let $(L, \{L^p\}_{p\in Z})$ be a $\mathfrak{t}$-FLA. Then there exists a unique uniform topology on L which is compatible with the Lie algebra structure and for which the $\{L^p\}_{p\in Z}$ constitute a fundamental system of neighborhoods of zero of L. $(L, \{L^p\}_{p\in Z})$ is called complete if L is separated and L is complete with respect to the uniform topology. If we set $\hat{L}=\varprojlim L/L^k$ and $\hat{L}^p=\varprojlim L^p/L^k$, then $(\hat{L}, \{\hat{L}^p\}_{p\in Z})$ is a complete $\mathfrak{t}$-FLA called the completion of $(L, \{L^p\}_{p\in Z})$ and there is a canonical $\mathfrak{t}$-homomorphism of $(L, \{L^p\}_{p\in Z})$ onto $(\hat{L}, \{\hat{L}^p\}_{p\in Z})$ having as its kernel the closure $\cap L^p$ of $\{0\}$ in L.

REMARK. Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a transitive GLA of finite depth. We set $L^p=\bigoplus_{k\geq p}\mathfrak{g}_k$. Then $(\mathfrak{g}, \{L^p\}_{p\in Z})$ is a transitive FLA of finite depth and its completion is identified with $(\prod\mathfrak{g}_k, \{\prod_{k\geq p}\mathfrak{g}_k\}_{p\in Z})$. If $\mathfrak{g}=W(n:\boldsymbol{t})$, $S(n:\boldsymbol{t})$, $CS(n:\boldsymbol{t})$, $H(n:\boldsymbol{t}:\mu)$, $CH(n:\boldsymbol{t}:\mu)$ or $K(n:\boldsymbol{t}:\mu)$, then any choice of $\boldsymbol{t}$ and μ gives the same topology.

Let $(L, \{L^p\}_{p\in Z})$ be a t-FLA of finite depth. Let $gr(L)=\bigoplus_{p\in Z} gr(L)_p$ be its associated GLA, where $gr(L)_p=L^p/L^{p+1}$ and the bracket operation of $gr(L)$ is defined in the obvious manner. Further the graded t-module structure on $gr(L)$ is naturally defined. Note that $gr(L)$ is transitive if and only if L is weakly transitive. Let $(\hat{L}, \{\hat{L}^p\}_{p\in Z})$ be the completion of $(L, \{L^p\}_{p\in Z})$. Then the canonical homomorphism $gr(L)\longrightarrow gr(\hat{L})$ is an isomorphism both as a GLA and as a t-module.

1.5. Construction of filtered Lie algebras.

Let L be a Lie algebra and t be a finite dimensional subalgebra of Der (L). Suppose that there is a maximal t-stable subalgebra L^0 of L of finite codimension. Then the adjoint action of L on itself induces a representation of L^0 on L/L^0. Let $L\supset L^{-1}\supset L^0$ be such that L^{-1}/L^0 is an irreducible (L^0, t)-submodule of L/L^0 (that is, it has no proper subspace which are both L^0- and t-stable). Following Weisfeiler [Wei69] and Morimoto-Tanaka [MT70], we define a t-filtration of L by $L^{i-1}=[L^i, L^{-1}]+L^i$ for $i<0$ and $L^{i+1}=\{x\in L^i : [x, L^{-1}]\subset L^i\}$ for $i\geq 0$. Then the t-FLA $(L, \{L^p\}_{p\in Z})$ is a weakly transitive t-FLA of finite depth. We call this a t-FLA corresponding to the maximal t-stable subalgebra L^0. Let $gr(L)=\bigoplus_{p\in Z} gr(L)_p$ be the associated GLA. Then $gr(L)$ is a transitive GLA such that $gr(L)_-$ is generated by $gr(L)_{-1}$ and $(gr(L)_0, \mathrm{t})$-irreducible.

The following propoition deals with a somewhat more general case than that considered in [KN65] and [MT70].

PROPOSITION 4. *Let $(L, \{L^p\}_{p\in Z})$ be as above. Suppose that L is infinite dimensional and transitive. Then:*

(*a*) *$gr(L)=\bigoplus_{p\in Z} gr(L)_p$ is isomorphic to one of $W(n: \mathbf{1})$, $S(n: \mathbf{1})$, $CS(n: \mathbf{1})$, $H(n: \mathbf{1}: 2)$, $CH(n: \mathbf{1}: 2)$, $K(n: \mathbf{1}: 2)$.*

(*b*) *If there exists an element E of $gr(L)_0$ such that all the eigenvalues of ad E are negative integers, then $gr(L)=\bigoplus_{p\in Z} gr(L)_p$ is isomorphic to one of $W(n: \mathbf{1})$, $CS(n: \mathbf{1})$, $CH(n: \mathbf{1}: 2)$, $K(n: \mathbf{1}: 2)$.*

PROOF. (a) By assumption, we get a homomorphism $\mathrm{t}\rightarrow \mathrm{Der}_0(gr(L)_-)$; we denote by $\bar{\mathrm{t}}$ its image. We set $gr(L)^t=\bigoplus_{p\in Z} gr(L)^t_p$, where $gr(L)^t_p= gr(L)_p$ for $p\neq 0$ and $gr(L)^t_0=gr(L)_0+\bar{\mathrm{t}}$ (here we consider $gr(L)_0$ as a subalgebra of $\mathrm{Der}_0(gr(L)_-)$). Then $gr(L)^t=\bigoplus_{p\in Z} gr(L)^t_p$ is an infinite dimensional irreducible transitive GLA of finite depth such that $gr(L)_-$ is generated by $gr(L)_{-1}$. Then, by [KN65] and [MT70], the $[gr(L)_{-1}, gr(L)_1]$-module $gr(L)_{-1}$ is irreducible, so the $gr(L)_0$-module $gr(L)_{-1}$ is irreducible. Now suppose that $gr(L)=\bigoplus_{p\in\boldsymbol{Z}} gr(L)_p$ is of depth two and $[gr(L)_{-2}, gr(L)_1]=\{0\}$. By the similar arguement to the proof of Lemma 4.1 in [MT70], we can prove that there exists a one dimensional subspace U of L such that $L=U\oplus L^{-1}$, $[U\oplus L^0, L^0]\subset U\oplus L^0$ and $\mathfrak{t}.U\subset U\oplus L^0$. Then we have $[U\oplus L^0, U\oplus L^0]\subset U\oplus L^0$ and $\mathfrak{t}\,(U\oplus L^0)\subset U\oplus L^0$, which contradicts the maximality of L^0. Thus by [KN65] and [MT70], $gr(L)$ is isomorphic to one of $W(n:\mathbf{1})$, $S(n:\mathbf{1})$, $CS(n:\mathbf{1})$, $H(n:\mathbf{1}:2)$, $CH(n:\mathbf{1}:2)$, $K(n:\mathbf{1}:2)$.

(b) We have only to prove that $gr(L)_0$ has the defining element e of $gr(L)=\bigoplus_{p\in\boldsymbol{Z}} gr(L)_p$. To do this, it is sufficient to prove that $E\in K^{\times}e+gr(L)_0'$ in case $gr(L)=W(n:\mathbf{1})$, $CS(n:\mathbf{1})$, $CH(n:\mathbf{1}:2)$, $K(n:\mathbf{1}:2)$, where $gr(L)_0'$ is the semisimple part of $gr(L)_0$. We can deduce this from the following assertion: Let E be an element of $\mathfrak{gl}(n:\boldsymbol{K})$ (resp. $\mathfrak{csp}(n:\boldsymbol{K})$) such that all the eigenvalues of E are negative integers. Then $E\in K^{\times}id+\mathfrak{sl}(n:\boldsymbol{K})$(resp. $K^{\times}id+\mathfrak{sp}(n:\boldsymbol{K})$). Indeed, we have $E=(E-(1/n)(tr\ E)id+(1/n)(tr\ E)id$, $E-(1/n)(tr\ E)id\in\mathfrak{sl}(n:\boldsymbol{K})$(resp. $\mathfrak{sp}(n:\boldsymbol{K})$) and $tr\ E<0$. This proves (b). □

1.6. Construction of a gradation on a filtered Lie algebra.

The following proposition gives some sufficient condition for the completion of a transitive $\mathfrak{t}$-FLA of finite depth to be isomorphic to the completion of the associated GLA both as a Lie algebra and as a $\mathfrak{t}$-module.

PROPOSITION 5. *Let $(L, \{L^p\}_{p\in\boldsymbol{Z}})$ be a transitive $\mathfrak{t}$-FLA of finite depth. We write $\mathfrak{l}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{l}_p=gr(L)$. Suppose that $\mathfrak{t}$ is commutative and the $\mathfrak{t}$-module L is completely reducible. Moreover suppose that $\mathfrak{l}_0$ contains the defining element e of $\mathfrak{l}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{l}_p$. Let $(\hat{L}, \{\hat{L}^p\}_{p\in\boldsymbol{Z}})$ be the completion of $(L, \{L^p\}_{p\in\boldsymbol{Z}})$. Then there are $\mathfrak{t}$-stable subspaces $\{G_p\}_{p\in\boldsymbol{Z}}$ of $\hat{L}$ such that $\hat{L}^p=G_p\oplus\hat{L}^{p+1}$ and $[G_p, G_q]\subset G_{p+q}$. In particular, $\hat{L}\simeq\prod G_p\simeq\prod gr(L)_p$.*

PROOF. The proof can be done by an analogical method of [But67], so we will state only an outline. If we choose $\{G_p^0\}_{p\in\boldsymbol{Z}}$ such that $L^p=G_p^0\oplus L^{p+1}$ and $\mathfrak{t}.G_p^0\subset G_p^0$, then we can identify $\mathfrak{l}_p$ with G_p^0 and decompose the bracket γ of L as $\gamma=\sum_{\ell\geq 0}\gamma_\ell^{(0)}$ with $\gamma_\ell\in\mathrm{Hom}(\Lambda^2\mathfrak{l}, \mathfrak{l})_\ell$, where $\mathrm{Hom}(\Lambda^2\mathfrak{l}, \mathfrak{l})_\ell=\{\omega\in\mathrm{Hom}(\Lambda^2\mathfrak{l}, \mathfrak{l}): \omega(\mathfrak{l}_p\wedge\mathfrak{l}_q)\subset\mathfrak{l}_{p+q+\ell}\}$. We define $\mathfrak{t}$-stable subspaces $\{G_p^k\}_{k\geq 0}$ of L^p such that $L^p=G_p^k\oplus L^{p+1}$ inductively as follows: $G_p^k=\{v-(1/$

$k)\gamma_k^{(k-1)}(e, v): v\in G_p^{k-1}\}$, where $\gamma_\ell^{(k-1)}$ is the Hom $(\Lambda^2\mathfrak{l}_-, \mathfrak{l})_\ell$-component of the decomposition of γ according to the identification via $\{G_p^{k-1}\}_{p\in Z}$. Then $[e, v]=pv$ (mod L^{p+k+1}) for all $v\in G_p^k$. We define $\{G_p\}$ as the limit of the sequence $\{G_p^k\}_{k\geq 0}$ in $\hat{L}$. By construction, we have $\hat{L}^p=G_p\oplus\hat{L}^{p+1}$, $\mathfrak{t}. G_p\subset G_p$ and $[e, v]=pv$ for $v\in G_p$. From this fact, we can easily prove that $[G_p, G_q]\subset G_{p+q}$. This proves our assertion. □

REMARK. Proposition 5 was obtained by T. Morimoto [Mor88] in case $\mathfrak{t}=\{0\}$. Moreover, if $(L, \{L^p\}_{p\in Z})$ is of depth one, it is known by Kobayashi-Nagano [KN66].

§ 2. Main theorem.

We now are ready to obtain our main theorems.

THEOREM 6. *Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be an infinite dimensional GLA of finite depth satisfying the following conditions*:

(G. 1) *$\mathfrak{g}_0$ contains the defining element E of $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$.*

(G. 2) *Every nonzero ideal of $\mathfrak{g}$ contains $\mathfrak{g}_-$.*

Let $\mathfrak{t}$ be a commutative subalgebra of $Der_0(\mathfrak{g})$ such that the $\mathfrak{t}$-module $\mathfrak{g}$ is completely reducible and that $E\in\mathfrak{t}$. Further let L^0 be a maximal $\mathfrak{t}$-stable subalgebra of $\mathfrak{g}$ containing $\bigoplus_{p\geq 0}\mathfrak{g}_p$, and $(\mathfrak{g}, \{L^p\}_{p\in Z})$ be a $\mathfrak{t}$-FLA corresponding to the maximal subalgebra L^0 of $\mathfrak{g}$. Then there exist $\mathfrak{t}$-stable subspaces $\{G_p\}_{p\in Z}$ of L such that $L^p=G_p\oplus L^{p+1}$ and $[G_p, G_q]\subset G_{p+q}$, whence $\mathfrak{g}=\bigoplus_{p\in Z}G_p\simeq gr(L)$.

PROOF. We put $\underline{L}^p=\bigoplus_{p\leq k}\mathfrak{g}_k$. By (G. 2), $(\mathfrak{g}, \{\underline{L}^p\}_{p\in Z})$ is a transitive $\mathfrak{t}$-FLA. Then we can easily prove that two $\mathfrak{t}$-filtrations $\{L^p\}_{p\in Z}$ and $\{L^p\}_{p\in Z}$ give the same topology, so that the completion of $(\mathfrak{g}, \{L^p\}_{p\in Z})$ is identified with $(\prod\mathfrak{g}_p, \{\prod_{p\leq k}\mathfrak{g}_k\}_{p\in Z})$. Moreover, by separatedness, we have dim $gr(L)=\infty$. Therefore by Proposition 4, $gr(L)$ must be isomorphic to one of $W(n:\mathbf{1})$, $CS(n:\mathbf{1})$, $CH(n:\mathbf{1}:2)$, $K(n:\mathbf{1}:2)$. Thus the $\mathfrak{t}$-FLA $(\mathfrak{g}, \{L^p\}_{p\in Z})$ satisfies the assumptions of Proposition 5. Hence there exist $\mathfrak{t}$-stable subspaces $\{G_p\}_{p\in Z}$ of $\hat{L}$ such that $\hat{L}^p=G_p\oplus\hat{L}^{p+1}$ and $[G_p, G_q]\subset G_{p+q}$, where $\hat{L}^p$ is the closure of L^p in $\prod\mathfrak{g}_p$. Therefore it is sufficient to prove that $G_p\subset\mathfrak{g}$. Let $x=\sum_{p\geq p_0}x_p\in G_p$, where $x_p\in\mathfrak{g}_p$. Suppose that Card $\{p\in \boldsymbol{Z}: x_p\neq 0\}=\infty$. We define the sequence $\{x^{(k)}\}$ of G_p inductively as follows: $x^{(0)}=x$ and $x^{(k)}=(\mathrm{ad}E)x^{(k-1)}-(p_0+k-1)x^{(k-1)}$. Then $x^{(k)}\in\hat{L}^{p_0+k+1}$.

On the other hand, there is an integer ℓ such that $\hat{L}^{p+1}\supset L^{\ell}$; then for a sufficient large integer k, we have $x^{(k)}\in\underline{\hat{L}}^{\ell}$, which contradicts the fact that $G_p\cap\hat{L}^p=\{0\}$. Hence $G_p\subset\mathfrak{g}$. □

REMARK 1. Let $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ be an infinite dimensional GLA of finite depth and $\mathfrak{t}$ be as in Theorem 6. Then the condition (G. 2) is equivalent to each of the following conditions:

(G. 2)$'$ Every nonzero $\mathfrak{t}$-stable ideal of $\mathfrak{g}$ contains $\mathfrak{g}_-$.

(G. 2)$''$ There exists a maximal $\mathfrak{t}$-stable subalgebra L^0 of $\mathfrak{g}$ of finite codimension such that L^0 contains no $\mathfrak{t}$-stable ideal of $\mathfrak{g}$.

(G. 2)$'''$ Every nonzero ideal of $\mathfrak{g}$ contains $\hat{\mathfrak{g}}:=\mathfrak{g}_-\oplus\bigoplus_{p\neq0}[\mathfrak{g}_p,\mathfrak{g}_{-p}]\oplus\mathfrak{g}_+$, where $\mathfrak{g}_+=\bigoplus_{p>0}\mathfrak{g}_p$.

Indeed, suppose that (G. 2)$'$ holds. We can easily prove that $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is transitive, so we may regard $\mathrm{Der}_0(\mathfrak{g})$ as a subalgebra of $\mathrm{Der}_0(\mathfrak{g}_-)$. We set $\mathfrak{g}^t=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p^t$, where $\mathfrak{g}_0^t=\mathfrak{g}_0+\mathfrak{t}$ and $\mathfrak{g}_p^t=\mathfrak{g}_p$ for $p\neq0$ (here we regard $\mathfrak{g}$ and $\mathfrak{t}$ as subalgebras of the prolongation of $\mathfrak{g}_-$). Then $\mathfrak{g}^t=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p^t$ is a GLA satisfying (G. 1) and (G. 2). By Theorem 6, $\mathfrak{g}^t$ is isomorphic to one of $W(n)$, $CS(n)$, $CH(n)$, $K(n)$. Then the graded ideal $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ is simple, and hence $\hat{\mathfrak{g}}=\mathfrak{g}$ or $\mathfrak{g}^t=\mathfrak{g}$. This proves that (G. 2)$'$ implies (G. 2)$'''$. It follows from the proof of Theorem 7 that (G. 2) implies (G. 2)$''$. Clearly (G. 2)$'''$ implies (G. 2), and (G. 2) implies (G. 2)$'$. Finally let us show that (G. 2)$''$ implies (G. 2)$'''$. Let $\{L^p\}_{p\in\boldsymbol{Z}}$ be the $\mathfrak{t}$-filtration on $\mathfrak{g}$ corresponding to the maximal $\mathfrak{t}$-stable subalgebra L^0 of $\mathfrak{g}$. Now let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ and $\{\mathfrak{a}^p\}_{p\in\boldsymbol{Z}}$ be a filtration on $\mathfrak{a}$ induced by that on $\mathfrak{g}$. Note that the condition (G. 2) is equivalent to the condition "every nonzero graded ideal of $\mathfrak{g}$ contains $\mathfrak{g}_-$" (see [Wei78]). Thus, since we have already shown that (G. 2) implies (G. 2)$'''$, we may assume that $\mathfrak{a}$ is graded. We write $\mathfrak{a}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{a}_p$. On the other hand, by Proposition 4, $gr(L)=\bigoplus_{p\in\boldsymbol{Z}}gr(L)_p$ is isomorphic to a certain GLA of Cartan type with the standard gradation. Thus $gr(\mathfrak{a})=\{0\}$ or $gr(\mathfrak{a})=[gr(L),gr(L)]$. If $gr(\mathfrak{a})=\{0\}$, then $\mathfrak{a}\cap L^p=\mathfrak{a}\cap L^{p+1}$ for all $p\in\boldsymbol{Z}$, so $\mathfrak{a}\subset\bigcap L^p=\{0\}$. Thus $\mathfrak{a}=\{0\}$. Next we suppose $gr(L)=[gr(L),gr(L)]$. Since $gr(L)_p=gr(\mathfrak{a})_p$ and $gr(\mathfrak{a})_0=[gr(L)_0,gr(L)_0]$, we have $L^p=\mathfrak{a}^p+L^{p+1}$ $(p\neq0)$ and $[L^0,L^0]+L^1=\mathfrak{a}^0+L^1$. Moreover we can find a graded subspace Z of L^0 such that $L^0=Z\oplus([L^0,L^0]+L^1)$ and $Z+L^1/L^1=\mathfrak{z}(gr(L)_0)$. Then if we set $L_k^p=L^p\cap\mathfrak{g}_k$ and $\mathfrak{a}_k^p=\mathfrak{a}^p\cap\mathfrak{g}_k$, then we have $L_k^0=\mathfrak{a}_k^0+L_k^1$ for $k\neq0$ and $L_0^0=(\mathfrak{a}_0^0+L_0^1)\oplus Z$, which implies $\mathfrak{g}_k=\sum_{p\in\boldsymbol{Z}}\mathfrak{a}_k^p$ for $k\neq0$ and $\mathfrak{g}_0=\sum_{p\in\boldsymbol{Z}}\mathfrak{a}_0^p\oplus Z$. Hence $\hat{\mathfrak{g}}\subset\mathfrak{a}$. This proves our assertion.

From Theorem 6, we have

THEOREM 7. *Let* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *be an infinite dimensional GLA of finite depth satisfying the conditions* (G.1) *and* (G.2) *in Theorem* 6. *Then* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *is isomorphic to one of* $W(n:\boldsymbol{t})$, $CS(n:\boldsymbol{t})$, $CH(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$.

PROOF. By Theorem 6, $\mathfrak{g}$ is isomorphic to one of $W(n)$, $CS(n)$, $CH(n)$, $K(n)$ (as a Lie algebra) and $\mathfrak{g}$ has a standard gradation $\{G_p\}_{p\in\boldsymbol{Z}}$ such that $[E, G_p]\subset G_p$. By construction, we have $\oplus_{p<0}G_p\subset\mathfrak{g}_-$. We set $\mathfrak{g}_p(q)=\mathfrak{g}_p\cap G_q$; then $\mathfrak{g}_p=\bigoplus_{p\in\mathfrak{g}}\mathfrak{g}_p(q)$. We first show that $E\in\mathfrak{g}_0(0)$. Since $\mathfrak{g}_0(0)$ is reductive in G_0 and since the G_0-module G_k is completely reducible, the $\mathfrak{g}_0(0)$-module $\mathfrak{g}_-$ is completely reducible. Furthermore we can easily prove that $\mathfrak{n}:=\bigoplus_{q>0}\mathfrak{g}_0(q)$ is the largest nilpotency ideal for the $\mathfrak{g}_0$-module $\mathfrak{g}_-$ (cf. [Bou60, § 4, n°3]). Therefore the radical of $\mathfrak{g}_0$ is $\mathfrak{z}(\mathfrak{g}_0(0))\oplus\mathfrak{n}$. Now we can decompose the element E of $\mathfrak{g}_0$ as follows: $E=E_1+E_2$, $E_1\in\mathfrak{z}(\mathfrak{g}_0(0))$, $E_2\in\mathfrak{n}$. Then since $[E, E_1]=0$, we know that $\mathrm{ad}(E-E_1)|\mathfrak{g}_-$ is semisimple. On the other hand, since $\mathrm{ad}\,E_2|\mathfrak{g}_-$ is nilpotent, we have $\mathrm{ad}(E-E_1)|\mathfrak{g}_-=0$. By transitivity, we have $E=E_1$, so $E\in\mathfrak{g}_0(0)$. Moreover let e be the defining element of $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$; then $e\in\mathfrak{g}_0(0)$ because $[e, E]=0$. If $\mathfrak{g}$ isomorphic to $W(1)$, then $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is clearly isomorphic to $W(1:k\mathbf{1})$ $(k>0)$. Hence we may assume that $n\geq 2$ in case $\mathfrak{g}$ is isomorphic to $W(n)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0(0)$. Let us examine the structure of the GLA $G_0=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p(0)$. This GLA is a reductive GLA whose center is one dimensional (Theorem 3) and contained in $\mathfrak{g}_0(0)$. Then $\mathfrak{h}$ is a Cartan subalgebra of G_0. Moreover there are a simple root system $\Pi=\{\alpha_1,\dots,\alpha_\ell\}$ of $(G_0,\mathfrak{h})$ and $\boldsymbol{s}=(s_1,\dots,s_\ell)\in A(\ell)$ such that $\Delta_p=\{\alpha=\sum k_i\alpha_i\in\Delta:\sum s_ik_i=p\}$ (cf. 1.2). We suppose that $G_{-1}=\bigoplus_{p\geq s_0}\mathfrak{g}_{-p}(-1)$ and $\mathfrak{g}_{-s_0}(-1)\neq\{0\}$. Then we can easily prove that the highest weight of the $\mathfrak{g}_0(0)$-module $\mathfrak{g}_{-s_0}(-1)$ is that of the G_0-module G_{-1}, which we write $-\alpha_0$. Here we remark the following fact: let $\mathfrak{l}^{(i)}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{l}_p^{(i)}(i=1,2)$ be GLAs isomorphic to a certain GLA of Cartan type with the standard gradation, and let us consider the sequence $(\mathfrak{l}_0^{(i)}, \mathfrak{h}^{(i)}, \Pi^{(i)}, (e_\alpha^{(i)})_{\alpha\in\Pi^{(i)}}, F_\Lambda^{(i)})$ (cf. 1.3). Then there is an isomorphism φ of $\mathfrak{l}^{(1)}$ onto $\mathfrak{l}^{(2)}$ as a GLA which transform $\mathfrak{h}^{(1)}$ into $\mathfrak{h}^{(2)}$, $\Pi^{(1)}$ into $\Pi^{(2)}$, $e_\alpha^{(1)}$ into $e_{\psi\alpha}^{(2)}$(where ψ is a contragredient mapping of $\psi|\mathfrak{h}^{(1)}$), and $F_\Lambda^{(1)}$ into $F_\Lambda^{(2)}$ (cf. [Kac68]). By identifying the GLA $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}G_p$ with a certain GLA of Cartan type with the standard gradation, we may regard the sequence $(G_0, \mathfrak{h}, \Pi, (e_\alpha)_{\alpha\in\Pi}, F_\Lambda)$ an the sequence defined in 1.3. Moreover $\{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ is a basis of $\mathfrak{h}^*$. We denote by $\{\pi_0^\vee, \ldots, \pi_\ell^\vee\}$ the dual basis of $\{\alpha_i\}$. We first assume that $\mathfrak{g}=\bigoplus_{p\in Z} G_p = W(n: \mathbf{1})$. Then $n=\ell+1$ and $\pi_k^\vee=\sum_{i=k+1}^{\ell+1} x_i D_i$ $(k=0,\ldots,\ell)$. Thus we have $E=\sum_{i=0}^{\ell} s_i \pi_i^\vee=\sum_{i=0}^{\ell}\sum_{j=k+1}^{\ell} s_i x_j D_j=\sum_{k=1}^{\ell+1}(\sum_{i=0}^{k-1} s_i) x_k D_k$. We set $t_k=\sum_{i=0}^{k-1} s_i (1\le k\le \ell+1)$; then $E=\sum_{k=1}^{\ell+1} t_k x_k D_k$. Further we have $[E, x^{(\beta)}D_p]=\sum_{k=1}^{\ell+1} t_k [x_k D_k, x^{(\beta)} D_p]=\sum_{k=1}^{\ell+1} t_k (x_k x^{(\beta-\varepsilon_k)} D_p - x^{(\beta)} x^{(\varepsilon_k-\varepsilon_p)} D_k)=\sum_{k=1}^{\ell+1}(t_k\beta_k - t_p) x^{(\beta)} D_p=(\|\beta\|_t - t_p) x^{(\beta)} D_p$. Hence $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $W(n: \boldsymbol{t})$. Similarly in case $\mathfrak{g}=\bigoplus_{p\in Z} G_p = CS(n: \mathbf{1})$, we can prove that $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $CS(n: \boldsymbol{t})$. Secondly we assume that $\mathfrak{g}=\bigoplus_{p\in Z} G_p = CH(n: \mathbf{1}: 2)$; Then $n=\ell$ and $\pi_i^\vee=\sum_{k=i+1}^{\ell}(x_k D_k + x_{k'} D_{k'}) + 2\sum_{k=1}^{i} x_{k'} D_{k'}$ $(0\le i\le \ell-1)$ and $\pi_\ell^\vee=\sum_{k=1}^{\ell} x_{k'} D_{k'}$. Thus $E=\sum_{i=0}^{\ell} s_i \pi_i^\vee=\sum_{i=1}^{\ell}\{(\sum_{k=0}^{i-1} s_k) x_i D_i + (\sum_{k=0}^{i-1} s_k + 2\sum_{k=i}^{\ell-1} s_k + s_\ell) x_{i'} D_{i'}\}$. We set $t_i=\sum_{k=0}^{i-1} s_k (1\le i\le \ell)$ and $\mu=2\sum_{k=0}^{\ell-1} s_k + s_\ell$. Then $[E, x^{(\alpha)} D_p]=(\|\alpha\|_u - t_p) x^{(\alpha)} D_p$, where $\boldsymbol{u}=(\boldsymbol{t}, \mu\mathbf{1}-\boldsymbol{t})$, so $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $CH(n: \boldsymbol{t}: \mu)$. Finally we assume that $\mathfrak{g}=\bigoplus_{p\in Z} G_p = K(n: \mathbf{1}: 2)$. Then $n=\ell$ and $\pi_i^\vee=-(1/2)\delta_{i\ell}\sum_{k=1}^{i} x_k x_{k'} - \sum_{k=i+1}^{\ell} x_k x_{k'} + x_{2\ell+1}$. Thus $E=\sum_{i=0}^{\ell} s_i \pi_i^\vee = -\sum_{j=1}^{\ell}(s_0+\sum_{k=1}^{i-1} s_k) x_j x_{j'} + (\sum_{k=1}^{\ell-1} 2 s_k + s_\ell) x_{2\ell+1}$. We set $t_i=\sum_{k=0}^{i-1} s_k$ $(1\le i\le \ell)$ and $\mu=2\sum_{k=0}^{\ell-1} s_k + s_\ell$. Then $[E, x^{(\alpha)}]=(\|\alpha\|_u - \mu) x^{(\alpha)}$, where $\boldsymbol{u}=(\boldsymbol{t}, \mu\mathbf{1}-\boldsymbol{t}, \mu)$, so $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $K(n: \boldsymbol{t}: \mu)$. □

REMARK 2. Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a finite dimensional GLA of depth μ satisfying (G. 1), (G. 2) and the condition "$\mathfrak{g}_\ell\neq\{0\}$ for some $\ell\ge 1$". Then $\mathfrak{g}$ is simple. In fact, suppose that there exists a nonzero commutative ideal $\mathfrak{a}=\bigoplus_{p\in Z}\mathfrak{a}_p$ of $\mathfrak{g}$. By (G. 2), we have $\mathfrak{g}_-\subset\mathfrak{a}$, so $[\mathfrak{a}_p, \mathfrak{g}_-]=\{0\}$ for $p\ge 0$. By virtue of transitivity, we have $\mathfrak{a}_p=\{0\}$ for $p\ge 0$, and hence $\mathfrak{g}_-=\mathfrak{a}$. This being so, since $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_-)=\mathfrak{g}_{-\mu}$, we have $\mathfrak{g}_{-\mu}=\mathfrak{g}_-=\mathfrak{a}$; thus $[\mathfrak{a}, \mathfrak{g}_\ell]\subset\mathfrak{a}_{\ell-\mu}=\{0\}$, which is a contradiction. Hence $\mathfrak{g}$ is semisimple. Let $\mathfrak{g}=\mathfrak{a}^1\oplus\mathfrak{a}^2$, where $\mathfrak{a}^1$ is a non-zero simple ideal of $\mathfrak{g}$ and $\mathfrak{a}^2$ is a semisimple ideal of $\mathfrak{g}$. Then by (G. 2), we have $\mathfrak{a}^1\supset\mathfrak{g}_-$. Therefore $\mathfrak{a}^2=\{0\}$. This proves our assertion. Actually the condition (G. 1) is unnessary.

COROLLARY. *Let* $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ *be an infinite dimensional GLA of finite depth satisfying* (*G.* 2). *Then* $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ *is isomorphic to one of* $W(n:\boldsymbol{t})$, $S(n:\boldsymbol{t})$, $CS(n:\boldsymbol{t})$, $H(n:\boldsymbol{t}:\mu)$, $CH(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$. *In particular, if* $\mathfrak{g}$ *is simple, then it is isomorphic to one of* $W(n:\boldsymbol{t})$, $S(n:\boldsymbol{t})$, $H(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$. *Furthermore if* $\mathfrak{g}$ *is simple and satisfies* (*G.* 1), *then it is isomorphic to* $W(n:\boldsymbol{t})$ *or* $K(n:\boldsymbol{t}:\mu)$.

PROOF. Let E be the defining element of $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$. We set $\mathfrak{g}_0^c=\mathfrak{g}_0+KE$ and $\mathfrak{g}_p^c=\mathfrak{g}_p$ $(p\neq 0)$. Then $\mathfrak{g}^c:=\bigoplus_{p\in Z}\mathfrak{g}_p^c$ is a GLA satisfying (G. 1) and (G. 2). By Theorem 7, $\mathfrak{g}^c=\bigoplus_{p\in Z}\mathfrak{g}_p^c$ is isomorphic to one of $W(n:\boldsymbol{t})$, $CS(n:\boldsymbol{t})$, $CH(n:\boldsymbol{t}:\mu)$, $K(n:\boldsymbol{t}:\mu)$. If $\mathfrak{g}^c=\bigoplus_{p\in Z}\mathfrak{g}_p^c$ is isomorphic to $W(n:\boldsymbol{t})$ or $K(n:\boldsymbol{t}:\mu)$, then $\mathfrak{g}^c=\mathfrak{g}$. If $\mathfrak{g}^c=\bigoplus_{p\in Z}\mathfrak{g}_p^c$ is isomorphic to $CS(n:\boldsymbol{t})$ and $\mathfrak{g}\neq\mathfrak{g}^c$, then $\mathfrak{g}$ is isomorphic to $S(n)$ because $\mathfrak{g}$ is a nontrivial ideal of $\mathfrak{g}^c$. Hence $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $S(n:\boldsymbol{t})$. Similarly in case $\mathfrak{g}^c=\bigoplus_{p\in Z}\mathfrak{g}_p^c$ is isomorphic to $CH(n:\boldsymbol{t}:\mu)$ and $\mathfrak{g}^c\neq\mathfrak{g}$. we can prove that $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is isomorphic to $H(n:\boldsymbol{t}:\mu)$. This proves the first assertion. For the remaining statements, it is obvious. □

§ 3. The prolongation of the associated truncated GLAs to GLAs of Cartan type and finite dimensional simple GLAs.

3.1. Let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a GLA of finite depth isomorphic to either a certain GLA of Cartan type or a finite dimensional simple GLA. In this section, we will give necessary and sufficient conditions so that $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ is the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. We first prove the following lemma to be used later on.

LEMMA 8. *Let* $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ *be a simple GLA of finite depth and* $\mathscr{G}=\bigoplus_{p\in Z}\mathscr{G}_p$ *be the prolongation of* $\mathit{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. *Then*:

(1) $\hat{\mathscr{G}}:=\bigoplus_{p\neq 0}\mathscr{G}_p\oplus\sum_{p\neq 0}[\mathscr{G}_p,\mathscr{G}_{-p}]$ *is simple.*

(2) *If* $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ *satisfies* (*G.* 1), *then* $\mathscr{G}=\bigoplus_{p\in Z}\mathscr{G}_p$ *is isomorphic to either* $W(n:\boldsymbol{t})$, $K(n:\boldsymbol{t}:\mu)$ *or a finite dimensional simple GLA.*

(3) *If* $k\geq 0$, *then* $\mathscr{G}=\bigoplus_{p\in Z}\mathscr{G}_p$ *is simple.*

PROOF. If $\dim\mathscr{G}<\infty$, we can easily prove that $\mathfrak{g}=\mathscr{G}$. Therefore we suppose that dim $\mathscr{G}=\infty$. Since $\hat{\mathscr{G}}=\bigoplus_{p\in \boldsymbol{Z}}\hat{\mathscr{G}}_p$ satisfies (G. 2), it is isomorphic to a certain GLA of Cartan type (Corollary to Theorem 7). Since $\hat{\mathscr{G}}$ is a nonzero ideal of $\mathscr{G}$, we know that $\hat{\mathscr{G}}$ is isomorphic to one of $W(n)$, $S(n)$, $H(n)$, $K(n)$, so $\hat{\mathscr{G}}$ is simple. This proves (1). The assertion (2) follows immediately from Corollary to Theorem 7. If $k\geq 0$, then $\hat{\mathscr{G}}=\mathscr{G}$, so $\mathscr{G}$ is simple. This proves (3). □

3. 2. Let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be a GLA of finite depth satisfying (G. 1) and (G. 2). It follows from Theorem 6 that $\mathfrak{g}$ has a standard gradation $\{G_p\}_{p\in Z}$ with $[E, G_p]\subset G_p$ and $\bigoplus_{p<0} G_p\subset\mathfrak{g}_-$. We set $\mathfrak{g}_p(q)=\mathfrak{g}_p\cap G_q$. Let e be its defining element and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0(0)$. Then by the proof of Theorem 7 we have e, $E\in\mathfrak{g}_0(0)$. Let $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ be the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. Clearly $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ satisfies (G. 1) and (G. 2), so it is isomorphic to one of GLAs of Cartan type satisfying (G. 1). Then we have

LEMMA 9. *There exists a standard gradation* $\{K_p\}_{p\in Z}$ *on* $\mathscr{G}$ *such that* $[\mathfrak{h}, K_p]\subset K_p$, $\bigoplus_{p<0} K_p\subset \mathfrak{g}_-$ *and* $e^\vee\in\mathfrak{z}(\mathfrak{g}_0(0))$, *where* $e^\vee$ *is the defining element of* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}} K_p$.

PROOF. If $k\geq 0$, the assertion follows from the proof of Theorem 7. Therefore we suppose that $k=-1$. Since $\mathscr{G}_0=\mathrm{Der}_0(\mathfrak{g}_-)$, there exists a reductive subalgebra $\mathfrak{m}$ of $\mathscr{G}_0$ such that the $\mathfrak{m}$-module $\mathfrak{g}_-$ is completely reducible, that $\mathfrak{g}_0(0)\subset \mathfrak{m}$ and that $\mathscr{G}_0=\mathfrak{m}\oplus\mathfrak{n}$, where $\mathfrak{n}$ is the largest nilpotency ideal for the $\mathscr{G}_0$-module $\mathfrak{g}_-$ (cf. [Bou75, Ch. VII, § 5. Proposition 7 and Exercise 4]). Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{m}$ containing $\mathfrak{h}$. It follows from Theorem 6 that $\mathscr{G}$ has a standard gradation $\{K_p\}_{p\in Z}$ such that $[\mathfrak{h}', K_p]\subset K_p$ and $\bigoplus_{p<0} K_p\subset\mathfrak{g}_-$. Let $e^\vee$ be its defining element. We put $\mathscr{G}_p(q)=\mathscr{G}_p\cap K_q$. Then E, $e^\vee\in\mathscr{G}_0(0)$ (cf. the proof of Theorem 7). Here we remark that $\mathfrak{n}=\bigoplus_{q>0}\mathscr{G}_0(q)$. Let $e^\vee=e_1^\vee+e_2^\vee$, where $e_1^\vee\in\mathfrak{z}(\mathfrak{m})$, $e_2^\vee\in\mathfrak{n}$. Then $0=[e^\vee, \mathscr{G}_0(0)]=[e_1^\vee, \mathscr{G}_0(0)]+[e_2^\vee, \mathscr{G}_0(0)]$. Since $[e_1^\vee, \mathscr{G}_0(0)]\subset \mathscr{G}_0(0)$ and $[e_2^\vee, \mathscr{G}_0(0)]\subset\mathfrak{n}$, we have $[e_1^\vee, \mathscr{G}_0(0)]=\{0\}$, so $[e_1^\vee, e^\vee]=0$. Since ad $(e_1^\vee-e^\vee)|\mathfrak{g}_-$ is semisimple and since ad $e_2^\vee|\mathfrak{g}_-$ is nilpotent, we have $e^\vee=e_1^\vee\in\mathfrak{z}(\mathfrak{m})$, and hence $[e^\vee, \mathfrak{g}_0(0)]=\{0\}$. Thus $\mathfrak{g}_0(0)\subset\mathscr{G}_0(0)$. Since $[\mathfrak{h}, e^\vee]=\{0\}$ and $\mathscr{G}_-=\mathfrak{g}_-$, we have $[e^\vee, \mathfrak{g}_p(q)]\subset\mathfrak{g}_p(q)$ for $p<0$, so $[e^\vee, G_p]\subset G_p$ for $p<0$. Since $[e^\vee, \mathfrak{g}_0(0)]=\{0\}$, we can easily prove that there exists an element h of

$\mathfrak{z}(\mathfrak{g}_0(0))$ such that ad $h|G_- = \text{ad } e^\vee|G_-$, where $G_- = \bigoplus_{p<0} G_p$. It follows from transitivity of $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G_p$ that ad $e^\vee|\mathfrak{g}_- = \text{ad } h|\mathfrak{g}_-$. Hence by transitivity of $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} \mathscr{G}_p$, we have $e^\vee = h \in \mathfrak{z}(\mathfrak{g}_0(0))$. □

REMARK 1. We may assume that $\bigoplus_{p<0} G_p \subset \bigoplus_{p<0} K_p$. In fact, $\mathfrak{g}$ has a standard gradation $\{G''_p\}_{p\in\mathbf{Z}}$ such that $[\mathfrak{h}, G''_p] \subset G''_p$ and $\bigoplus_{p\geq 0} G''_p \supset \bigoplus_{p\geq 0} K_p \cap \mathfrak{g}$ (Theorem 6). Further $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G_p$ is isomorphic to $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G''_p$ as a GLA and $e^\vee \in \mathfrak{z}(G''_0 \cap \mathfrak{g}_0)$. Hence we may replace $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G_p$ by $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G''_p$.

3.3. Let $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p$ be a GLA of Cartan type of depth μ, E be the defining element of $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p$ and $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} \mathscr{G}_p$ be the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ $(k \geq -1)$. By Lemma 9 and Remark 1, there is a standard gradation $\{G_p\}_{p\in\mathbf{Z}}$ (resp. $\{K_p\}_{p\in\mathbf{Z}}$) on $\mathfrak{g}$ (resp. $\mathscr{G}$) such that each K_p is $(\mathfrak{g}_0 \cap G_0)$-stable, $e^\vee \in \mathfrak{z}(\mathfrak{g}_0 \cap G_0)$ and $\bigoplus_{p<0} G_p \subset \bigoplus_{p<0} K_p \subset \mathfrak{g}_-$, where $e^\vee$ is the defining element of the GLA $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} K_p$. We set $G'_p = \mathfrak{g} \cap K_p$; then $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G'_p$. Further $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G'_p$ is a GLA of depth $\nu = 1$ or 2, and $G'_p \cap \mathrm{Trun}_k(\mathfrak{g}) = K_p \cap \mathrm{Trun}_k(\mathfrak{g})$. In particular, $G'_p = K_p$ for $p<0$, and $G'_{-2} = \mathfrak{g}_{-\mu}$ if $\nu = 2$. By construction, $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G'_p$ is isomorphic to one of $W(n : \mathbf{1})$, $W(n : (\mathbf{1}_{n-1}, 2))$, $CS(n : \mathbf{1})$, $CS(n : (\mathbf{1}_{n-1}, 2))$, $CH(n : \mathbf{1} : \mathbf{2})$, $K(n : \mathbf{1} : \mathbf{2})$.

Then we have

THEOREM 10. *Let* $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p = K(n : \boldsymbol{t} : \mu)$, *Then* $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p$ *is the prolongation of* $\mathrm{Trun}_k(\mathfrak{g})$ *for all* $k \geq -1$.

PROOF. By Lemma 8 (2), $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} \mathscr{G}_p$ is isomorphic to $W(n' : \boldsymbol{t}')$ or $K(n' : \boldsymbol{t}' : \mu')$. If $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} \mathscr{G}_p$ is isomorphic to $W(n' : \boldsymbol{t}')$, then $\nu = 1$. It is impossible. If $\mathscr{G} = \bigoplus_{p\in\mathbf{Z}} \mathscr{G}_p$ is isomorphic to $K(n' : \boldsymbol{t}' : \mu')$, then $\nu = 2$. Thus $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G'_p$ is isomorphic to $K(n' : \mathbf{1} : 2)$. Since $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} G'_p$ is the prolongation of $G'_{-2} \oplus G'_{-1}$, we have $\mathfrak{g} = \mathscr{G}$. □

THEOREM 11. *Let* $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p = W(n : \boldsymbol{t})$. *Then*:

(1) $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p$ *is the prolongation of* $\mathrm{Trun}_k(\mathfrak{g})$ *if one of the following conditions holds*: (α) $n \geq 2$, $t_n = t_{n-1}$; (β) $n \geq 2$, $t_{n-1} < t_n < 2t_{n-1} + (k+1)$; ($\gamma$) $n = 1$.

(2) *If* $n\geq 2$ *and* $t_n\geq 2t_{n-1}+(k+1)$, *then the prolongation of* $Trun_k(\mathfrak{g})$ *is isomorphic to* $K(n-1:\boldsymbol{s}:t_n)$, *where* $s_i=min\{t_i, t_n-t_i\}$.

PROOF. (1) The case $n=1$ is clear. Therefore we assume $n\geq 2$. By Lemma 8, $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(n':\boldsymbol{t}')$ or $K(n':\boldsymbol{t}':\mu)$. If $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(n':\boldsymbol{t}')$, then $\nu=1$, so $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}G'_p$ is isomorphic to $W(n:\boldsymbol{1})$, Since $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}G'_p$ is the prolongation of G'_{-1}, we have $\mathfrak{g}=\mathscr{G}$. Suppose that $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $K(n':\boldsymbol{t}':\mu)$. Then $\nu=2$ and $\dim\mathfrak{g}_{-\mu}=1$. Hence the condition (α) dose not hold. If the condition (β) holds, then $\dim G'_{-2}=1$, and hence $\mathfrak{g}=\bigoplus_{p\geq -2}G'_p$ is isomorphic to $W(n:(\boldsymbol{1}_{n-1},2))$. We set $G'_p(q)=G'_p\cap G_q$. Then we can easily prove that $G'_0=G'_0(0)\oplus G'_0(1)$ and $G'_0(1)$ is a nilpotent ideal of G'_0 isomorphic to $G'_{-2}\otimes S^2(G'_{-1}(-1)^*)$ as a $G'_0(0)$-module, so the maximum of the eigenvalues of ad E on $G'_0(1)$ is $2t_{n-1}-t_n$. Applying Lemma 1 (2) to the case when $\mathfrak{l}=K_0$, $\mathfrak{u}=G'_0$, we have $G'_0(1)\subset\mathrm{Trun}_{-k-1}(\mathfrak{g})$. Hence $2t_{n-1}-t_n\leq -k-1$, which is a contradiction. This proves (1).

(2) We set $F_p=W(n:(\boldsymbol{1}_{n-1},2))_p$ and $F_p(q)=F_p\cap W(n:\boldsymbol{1})_q$. Let $\mathscr{H}=\bigoplus_{p\in\boldsymbol{Z}}H_p$ be the prolongation of $F_{-2}\oplus F_{-1}$. Then $\mathscr{H}=\bigoplus_{p\in\boldsymbol{Z}}H_p$ is isomorphic to $K(n-1:\boldsymbol{1}:2)$. Therefore $\dim F_{-1}=2(n-1)$ and $\dim H_0=\dim\mathfrak{csp}(F_{-1})=2n^2-3n+2$. Applying Lemma 1 (1) to the case when $\mathfrak{n}=F_0(1)$, $\mathfrak{u}=F_0(0)$, $\mathfrak{l}=H_0$, we know that there exists a nilpotent subalgebra $\mathfrak{n}_+$ of H_0 such that $\mathfrak{n}_++F_0(0)+F_0(1)$ is a direct sum and that $\mathfrak{n}_+$ is contragredient to $F_0(1)$ as a $F_0(0)$-module, so $\dim(\mathfrak{n}_+\oplus F_0(0)\oplus F_0(1))=2n^2-3n+2$. This implies $H_0=\mathfrak{n}_+\oplus F_0(0)\oplus F_0(1)$. We set $\mathscr{H}_p=\{x\in\mathscr{H}: [E,x]=px\}$; then $\mathscr{H}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{H}_p$. As in the proof of (1), since $F_0(1)\subset\mathrm{Trun}_{-k-1}(\mathfrak{g})$, we have $\mathfrak{n}_+\subset\bigoplus_{p\geq k+1}\mathscr{H}_p$, whence $H_0\cap\bigoplus_{p\leq k}\mathscr{H}_p=F_0\cap\mathrm{Trun}_k(\mathfrak{g})$. Moreover we can inductively prove that $[H_\ell\cap\bigoplus_{p\leq k}\mathscr{G}_p, F_{-1}]\subset F_{-1}\cap\mathrm{Trun}_k(\mathfrak{g})$ for $\ell\geq 0$. Hence we see that $\bigoplus_{p\geq 1}H_p\cap\mathrm{Trun}_k(\mathfrak{g})$ is contained in the prolongation of $F_{-2}\oplus F_{-1}\oplus F_0$. However since $\mathfrak{g}=\bigoplus_{p\geq -2}F_p$ coincides with the prolongation of $F_{-2}\oplus F_{-1}\oplus F_0$ (by (1)), we have $\mathrm{Trun}_k(\mathfrak{g})=\bigoplus_{p\leq k}\mathscr{H}_p$, so $\mathscr{H}$ is contained in $\mathscr{G}$. On the other hand, since $K(n:\boldsymbol{t}:\mu)$ is the prolongation of $\mathrm{Trun}_k(K(n:\boldsymbol{t}:\mu))$ for any $\boldsymbol{t}$ and μ (Theorem 10), we have $\mathscr{G}=\mathscr{H}$. Finally since $\dim F_{-1}=2(n-1)$ and since F_{-1} is isomorphic to $F_{-1}(-1)\oplus F_{-1}(-1)^*\otimes F_{-2}$ as a $F_0(0)$-module, $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $K(n-1:\boldsymbol{s}:\mu)$, where $s_i=\min\{t_i, t_n-t_i\}$. □

REMARK 2. In the case of $\boldsymbol{t}=\mathbf{1}$ (resp. $\boldsymbol{s}=(\mathbf{1}, \mu\mathbf{1})$), Theorem 10 (resp. Theorem 11) is included in the results of [Mor88].

THEOREM 12. *Let* $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p=CS(n:\boldsymbol{t})$ $(n\geq 2)$. *Then*:

(1) $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ *is the prolongation of* $Trun_k(\mathfrak{g})$ *if* $t_1\leq k$.

(2) *The prolongation of* $Trun_k(\mathfrak{g})$ *is isomorphic to* $W(n:\boldsymbol{t})$ *if one of the following conditions holds*; (β) $t_n=t_{n-1}$, $t_1>k$: (α) $t_{n-1}<t_n<2t_{n-1}+(k+1)$, $t_1>k$.

(3) *If* $t_n\geq 2t_{n-1}+k+1$ *and* $t_1>k$, *then the prolongation of* $Trun_k(\mathfrak{g})$ *is isomorphic to* $K(n-1:\boldsymbol{s}:t_n)$, *where* $s_i=min\ \{t_i, t_n-t_i\}$.

PROOF. If $t_1>k$, then we can easily prove that $W(n:\boldsymbol{t})_\ell=CS(n:\boldsymbol{t})_\ell$ for $\ell\leq k$. Hence the assertions (2) and (3) follow from Theorem 11. Now suppose that $t_1\leq k$. If $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}K_p$ is isomorphic to $CS(n':\mathbf{1})$, then $\nu=1$ and $n=n'$, so $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}G'_p$ is isomorphic to $CS(n:\mathbf{1})$. Hence $\mathfrak{g}=\mathscr{G}$. If $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}K_p$ is isomorphic to $CH(n':\mathbf{1}:2)$ $(n'\geq 2)$, then $\nu=1$ and $2n'=n$, so $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}G'_p$ is isomorphic to $CS(2n':\mathbf{1})$. We have $\dim K_0=\dim \mathfrak{csp}(G'_{-1})=2n'^2+n'+1$ and $\dim G'_0=\dim \mathfrak{gl}(G'_{-1})=4n'^2$. Since $K_0\supset G'_0$, we have $(2n'+1)(n'-1)\leq 0$, which is a contradiction. If $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}K_p$ is isomorphic to $W(n':\mathbf{1})$, then $\nu=1$ and $n'=n$, so $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}G'_p$ is isomorphic to $CS(n:\mathbf{1})$. Moreover $K_0=G'_0$ and there is a K_0-submodule K'_1 of K_1 such that $K_1=K'_1\oplus G'_1$ and that K'_1 is contragredient to G'_{-1} as a G'_0-module (Theorem 3). Since the minimum of the eigenvalues of $\mathrm{ad}(E)$ on K'_1 is t_1 and since $G'_1\cap \mathrm{Trun}_k(\mathfrak{g})=K_1\cap \mathrm{Trun}_k(\mathfrak{g})$, we have $t_1>k$, which is a contradiction. If $\mathscr{H}=\bigoplus_{p\in \boldsymbol{Z}}K_p$ is isomorphic to $K(n':\mathbf{1}:2)$, then $\nu=2$ and $\dim G'_{-2}=1$, so $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}G'_p$ is isomorphic to $CS(n:(\mathbf{1}_{n-1}, 2))$. Let $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}H_p$ be the prolongation of $G'_{-2}\oplus G'_{-1}\oplus G'_0$. By (2), $\mathscr{H}=\bigoplus_{p\in \boldsymbol{Z}}H_p$ is isomorphic to $W(n:(\mathbf{1}_{n-1}, 2))$ and $\mathscr{H}\subset\mathscr{G}$, By Theorem 6, there exists a standard gradation $\{F_p\}_{p\in \boldsymbol{Z}}$ on $\mathscr{H}$ such that $[\mathfrak{h}, F_p]\subset F_p$ and $F_{-1}\subset G'_{-2}\oplus G'_{-1}\subset \mathfrak{g}_-$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0\cap G_0$. We set $C_p=F_p\cap \mathfrak{g}$; then $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}C_p$ and $C_q=F_q$ for $q=0, -1$. Moreover $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}F_p$ is isomorphic to $CS(n:\mathbf{1})$. We set $\mathscr{H}_p=\mathscr{H}\cap\mathscr{G}_p$; then $\mathscr{H}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{H}_p$ and it becomes a GLA isomorphic to $W(n:\boldsymbol{t}')$. As in the proof of Theorem 7, since $\boldsymbol{t}'$ is determined by C_0 and C_{-1}, we have $\boldsymbol{t}=\boldsymbol{t}'$. Since $\mathrm{Trun}_k(\mathfrak{g})=\mathrm{Trun}_k(\mathscr{G})$, we have $\mathrm{Trun}_k(\mathfrak{g})=\oplus_{p\leq k}\mathscr{H}_p$, so $\dim W(n:\boldsymbol{t})_\ell$

$=\dim CS(n: \boldsymbol{t})_\ell$ for $\ell \leq k$. But it is impossible (e. g., $x^{(\varepsilon_1+\varepsilon_2)}D_2 \in W(n: \boldsymbol{t})_{t_1} \backslash CS(n: \boldsymbol{t})_{t_1}$). This proves (1). □

THEOREM 13. *Let* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p=CH(n: \boldsymbol{t}: \mu)$ $(n\geq 2)$. *Then*:

(1) $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *is the prolongation of* $Trun_k(\mathfrak{g})$ *unless* $k=-1$, $t_1=\ldots=t_n$ *and* $\mu=2t_1$.

(2) *If* $t_1=\ldots=t_n$ *and* $\mu=2t_1$, *then the prolongation of* $Trun_{-1}(\mathfrak{g})$ *is* $W(2n: t_1\mathbf{1})$.

PROOF. (1) If $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}K_p$ is isomorphic to $W(n': \mathbf{1})$ or $CS(n': \mathbf{1})$, then $\nu=1$ and $n'=2n$. Thus $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}G'_p$ is isomorphic to $CH(n: \mathbf{1}: 2)$; then $\dim K_0=\dim\mathfrak{gl}(G'_{-1})=4n^2$ and $\dim G'_0=\dim\mathfrak{csp}(G'_{-1})=2n^2+n+1$. If $K_0\cap\mathfrak{g}_-\neq\{0\}$, then $K_0=G'_0$ (Lemma 1 (3)), so $0=2n^2-n-1=(2n+1)(n-1)$, which is a contradiction. If $K_0\cap\mathfrak{g}_-=\{0\}$, then $t_1=\ldots=t_n$ and $\mu=2t_1$, and hence $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is isomorphic to $CH(n: t_1\mathbf{1}: 2t_1)$. But since $k\geq 0$, we have $\mathfrak{g}=\mathscr{G}$, which is a contradiction. If $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}K_p$ is isomorphic to $K(n: \mathbf{1}: 2)$, then $\nu=2$. Hence $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}G'_p$ must be isomorphic to $CH(n: \mathbf{1}: 3)$, which contradicts the fact that $\dim G'_{-2}=1$. If $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}K_p$ is isomorphic to $CH(n': \mathbf{1}: 2)$, then $\nu=1$ and $n'=n$. Hence $\mathfrak{g}=\mathscr{G}$.

(2) This is a well-known fact. □

3. 4. Let $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ be a GLA of finite depth isomorphic to $S(n: \boldsymbol{t})$ or $H(n: \boldsymbol{t}: \mu)$, and $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ be the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. We set $\mathfrak{g}_0^c=\mathfrak{g}_0+KE$ and $\mathfrak{g}_p^c=\mathfrak{g}_p$ for $p\neq 0$, where E is the defining element of $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$. Then $\mathfrak{g}^c=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p^c$ is a GLA isomorphic to $CS(n: \boldsymbol{t})$ or $CS(n: \boldsymbol{t}: \mu)$ respectively. Also we set $\mathscr{G}_0^c=\mathscr{G}_0+KE$ and $\mathscr{G}_p^c=\mathscr{G}_p$ for $p\neq 0$. Then $\mathscr{G}^c=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p^c$ is a GLA of finite depth satisfying (G. 1) and (G. 2). We remark that $\mathscr{G}^c=\mathscr{G}$ if $k=-1$. Then we have

THEOREM 14. *Let* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p=S(n: \boldsymbol{t})$ $(n\geq 2)$. *Then*:

(1) *If* $k\geq 0$, *then* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *is the prolongation of* $Trun_k(\mathfrak{g})$.

(2) *The prolongation of* $Trun_k(\mathfrak{g})$ *is isomorphic to* $W(n: \boldsymbol{t})$ *if one of the following conditions holds*: $(\alpha)'$ $k=-1$, $t_n=t_{n-1}$; $(\beta)'$ $k=-1$, $t_{n-1}<t_n<2t_{n-1}$.

(3) *If* $t_n\geq 2t_{n-1}$, *then the prolongation of* $Trun_{-1}(\mathfrak{g})$ *is* $K(n-1: \boldsymbol{s}: t_n)$, *where* $s_i=\min\{t_i, t_n-t_i\}$.

PROOF. If $k\geq 0$, then $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $S(n' : \boldsymbol{t}')$ or $H(n' : \boldsymbol{t}' : \mu')$ (Lemma 8 (3)). Then as in the proof of Theorem 11, we have $\mathfrak{g}^c=\mathscr{G}^c$, so $\mathfrak{g}=\mathscr{G}$, Since the prolongation of $\mathrm{Trun}_{-1}(\mathfrak{g})$ coincides with that of $\mathrm{Trun}_{-1}(\mathfrak{g}^c)$, our assertion follows from Theorem 12. □

Similarly we have

THEOREM 15. *Let* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p=H(n : \boldsymbol{t} : \mu)$ $(n\geq 2)$. *Then*:

(1) *If* $k\geq 0$, *then* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *is the prolongation of* $Trun_k(\mathfrak{g})$.

(2) *The prolongation of* $Trun_{-1}(\mathfrak{g})$ *is* $CH(n : \boldsymbol{t} : \mu)$ *unless* $t_1=\ldots=t_n$ *and* $\mu=2t_1$.

(3) *If* $t_1=\ldots=t_n$ *and* $\mu=2t_1$, *then the prolongation of* $Trun_{-1}(\mathfrak{g})$ *is* $W(2n : t_1\mathbf{1})$.

Let $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ be a finite dimensional simple GLA. Then $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ if and only if $\bigoplus_{p\geq 1}\mathfrak{g}_p$ is generated by $\mathfrak{g}_1$. By contrast, in the cases of GLAs of Cartan type, it is not true. Indeed, as a corollary to Theorems 10–15, we have

COROLLARY. *Let* $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ *be one of* $W(m : \boldsymbol{s})$ $(m\geq 2)$, $S(m : \boldsymbol{s})$ $(m\geq 2)$, $CS(m : \boldsymbol{s})$ $(m\geq 2)$, $H(n : \boldsymbol{t} : \mu)$ $(n\geq 2)$, $CH(n : \boldsymbol{t} : \mu)$ $(n\geq 2)$, $K(n : \boldsymbol{t} : \mu)$. *Suppose that* $\mathfrak{g}_-$ *is generated by* $\mathfrak{g}_{-1}$. *The following conditions are equivalent*:

(*a*) $\bigoplus_{p\geq 1}\mathfrak{g}_p$ *is generated by* $\mathfrak{g}_1$.

(*b*) (i) $s_{i+1}-s_i\leq 1$ *for* $\mathfrak{g}=W(m : \boldsymbol{s})$, $S(m : \boldsymbol{s})$ *or* $CS(m : \boldsymbol{s})$; (ii) $t_{i+1}-t_i\leq 1$, $[\mu/2]=t_n$ *for* $\mathfrak{g}=H(n : \boldsymbol{t} : \mu)$, $CH(n : \boldsymbol{t} : \mu)$ *or* $K(n : \boldsymbol{t} : \mu)$ $(n\geq 2)$; (iii) $t_1=[\mu/2]$ *for* $\mathfrak{g}=K(1 : \boldsymbol{t} : \mu)$.

PROOF. By assumption, $s_1=1$ (resp. $t_1=1$) for $\mathfrak{g}=W(m : \boldsymbol{s})$, $S(m : \boldsymbol{s})$ or $CS(m : \boldsymbol{s})$ (resp. $\mathfrak{g}=H(n : \boldsymbol{t} : \mu)$, $CH(n : \boldsymbol{t} : \mu)$ or $K(n : \boldsymbol{t} : \mu)$). Suppose that (a) holds and (b) does not hold. Then there exists a positive integer p_0 such that $p_0<m$, $s_{p+1}-s_p\leq 1$ for $p>p_0$ and $s_{p_0}+1<s_{p_0+1}$ (resp. $p_0\leq n$, $t_p-t_{p-1}\leq 1$ for $p\leq p_0$ and $t_{p_0}+1<t_{p_0+1}$, where we consider $t_{p_0+1}=\mu-t_n$ if $p_0=n$). We set $\mathfrak{a}=(\sum_{p\geq p_0}\mathfrak{A}(m : \boldsymbol{s})_{-1+s_p}D_p)\cap\mathfrak{g}_{-1}$ (resp. $(\sum_{p\geq p_0}\mathfrak{A}(m : \boldsymbol{s})_{-1+\mu-t_p}D_{n+p})\cap\mathfrak{g}_{-1}$). Then it is not difficult to show that $\mathfrak{a}$ is a nontrivial $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$ such that $[[\mathfrak{a},\mathfrak{g}_1],\mathfrak{g}_{-1}]\subset\mathfrak{a}$. Let $\mathfrak{b}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{b}_p$ be the ideal of $\mathfrak{g}$ generated by $\mathfrak{a}$; then $\mathfrak{b}_{-1}=\mathfrak{a}$ (cf. [Kan70, Lemma 16]), which contradicts (G. 2); thus (a) implies (b). Conversely we suppose that (b) holds. Then

we can easily prove that $[\mathfrak{g}_p, \mathfrak{g}_1]=\mathfrak{g}_{p+1}$ for $p<0$. Let $\tilde{\mathfrak{g}}=\bigoplus_{p\in \boldsymbol{Z}}\tilde{\mathfrak{g}}_p$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$. By the above-mentioned fact, $\tilde{\mathfrak{g}}=\bigoplus_{p\in \boldsymbol{Z}}\tilde{\mathfrak{g}}_p$ satisfes (G. 2), so it is isomorphic to one of GLAs of Cartan type (Corollary to Theorem 7). Since the condition (a) is clearly satisfied for the GLA $\tilde{\mathfrak{g}}=\bigoplus_{p\in \boldsymbol{Z}}\tilde{\mathfrak{g}}_p$ the condition (b) also satified by virtue of the implication (a)$\Longrightarrow$(b). Applying Theorems 10-15 to $\tilde{\mathfrak{g}}$, we see that $\tilde{\mathfrak{g}}$ is the prolongation of $\mathrm{Trun}_1(\mathfrak{g})$, which implies $\mathfrak{g}=\tilde{\mathfrak{g}}$. Hence the condition (a) holds for $\mathfrak{g}$. $\square$

3. 5. Let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be a finite dimensional simple GLA of depth μ and $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ be the prolongation of $\mathrm{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. We can easily prove that if dim $\mathscr{G}<\infty$, then $\mathscr{G}=\mathfrak{g}$. Hence we investigate only the case that dim $\mathscr{G}=\infty$, Then by Lemma 8 (2), $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(n: \boldsymbol{t})$ or $K(n: \boldsymbol{t}: \mu)$. Just as in the proof of Lemma 9, there exists a standard gradation $\{K_p\}_{p\in \boldsymbol{Z}}$ on $\mathscr{G}$ whose defining element $e^{\vee}$ is contained in $\mathfrak{z}(\mathfrak{g}_0)$. The following theorem is due to K. Yamaguchi ([Yam-pre]). Here we prove the theorem by a different approach to his proof.

THEOREM 16. *Let* $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ *be a finite dimensional simple GLA of depth* μ *with the gradation of type* $(s_0,\ldots,s_\ell)$ *and* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ *be the prolongation of* $\mathrm{Trun}_k(\mathfrak{g})$ $(k\geq -1)$. *If* $\mathscr{G}$ *is infinite dimensional, then only the following cases occur*:

(α) $\mathfrak{g}$ *is of type* $A_{\ell+1}(\ell\geq 1)$ *and* $k+1\leq s_0-\sum_{i=1}^{\ell}s_i$, *where we except the cases* (i) $k=-1$, $s_0=s_\ell>0$ *and* $s_i=0$ *for* $1\leq i\leq \ell-1$, *and* (ii) $k=-1$, $s_0>0$ *and* $s_i=0$ *for* $1\leq i\leq \ell$. *In this case,* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ *is isomorphic to* $W(\ell+1: \boldsymbol{t})$, *where* $t_i=\sum_{j=0}^{i-1}s_j$.

(β) $\mathfrak{g}$ *is of type* $C_{\ell+1}$ *and* $k+1\leq s_0-2\sum_{i=1}^{\ell}s_i-s_\ell$, *where we except the case* $k=-1$, $s_0>0$ *and* $s_i=0$ *for* $1\leq i\leq \ell$. *In this case,* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ *is isomorphic to* $K(\ell: \boldsymbol{t}: \mu)$, *where* $t_i=\sum_{j=0}^{i-1}s_j$ *and* $\mu=2\sum_{j=0}^{\ell-1}s_j+s_\ell$.

(γ) $\mathfrak{g}$ *is of contact type of order* m *and* $k=-1$. *In this case,* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ *is isomorphic to* $K(n: m\mathbf{1}: 2m)$, *where dim* $\mathfrak{g}_{-m}=2n$.

(δ) $\mathfrak{g}$ *is of general type of order* m *such that dim* $\mathfrak{g}_{-m}=n>1$ *and* $k=-1$. *In this case,* $\mathscr{G}=\bigoplus_{p\in \boldsymbol{Z}}\mathscr{G}_p$ *is isomorphic to* $W(n: m\mathbf{1})$.

(ε) *$\mathfrak{g}$ is of type A_1 and $-1\leq k\leq s_0$. In this case, $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(1:s_0)$.*

PROOF. We use the same notation as in 1.2. We set $\mathscr{G}_p(q)=\mathscr{G}_p\cap K_q$. Since $[\mathfrak{h}, \mathscr{G}_p(q)]\subset\mathscr{G}_p(q)$, we know that $\mathscr{G}_p(q)(p<0)$ is spanned by root vectors of $(\mathfrak{g}, \mathfrak{h})$. We first prove that $\mathscr{G}_p(q)=\{0\}$ for $q>0$, $p<0$. If $\mathscr{G}_p(q)\neq\{0\}$ for some $q>0$, $p<0$, there exists a root α with $e_\alpha\in\mathscr{G}_p(q)$. Then $[E, e_{-\alpha}]=-pe_{-\alpha}$ and $[e^\vee, e_{-\alpha}]=-qe_{-\alpha}$. Thus $e_{-\alpha}\in\mathfrak{g}_{-p}(-q)=\{0\}$, which is a contradiction. Hence $\mathscr{G}_p(q)=\{0\}$ for $q>0$, $p<0$. Clearly, if $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(1:t_1)$, then $\mathfrak{g}$ is of type A_1, $s_0=t_1$ and $-1\leq k\leq t_1$. Therefore we assume that $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $K(n:\boldsymbol{t}:\mu)$ or $W(n:\boldsymbol{t})$ $(n\geq 2)$. Then there exist a K_0-submodule $K_1^{(1)}$ of K_1 not contragredient to K_{-1} and a K_0-submodule $K_1^{(2)}$ of K_1 contragredient to K_{-1}. We set $G_q=\mathfrak{g}\cap K_q$ and $\mathfrak{g}_p(q)=\mathfrak{g}_p\cap G_q$. We first suppose that $K_0\cap\mathfrak{g}_-\neq\{0\}$. Then $\mathfrak{g}_p(0)=\mathscr{G}_p(0)$ for $p<0$ and $\mathfrak{g}_p(0)\subset\mathscr{G}_p(0)$ for $p\geq 0$, so by Lemma 1 (3), we have $K_0=G_0$. Since G_1 is contragredient G_{-1} as a G_0-module and $K_1=K_1^{(1)}\oplus K_1^{(2)}$, we have $G_1=K_1^{(2)}$, Thus $\mathfrak{g}$ coincides with the subalgebra of $\mathscr{G}$ generated by $K_{-1}\oplus K_0\oplus K_1^{(2)}$, and therefore the gradation of type $(s_0,\ldots,s_\ell)$ must be as in the proof of Theorem 7. Thus, when $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(n:\boldsymbol{t})$ (resp. $K(n:\boldsymbol{t}:\mu)$), since $K_1^{(1)}\subset\bigoplus_{p\geq k+1}\mathscr{G}_p(1)$ and since the minimum of the eigenvalues of ad E on $K_1^{(1)}$ is $2t_1-\mu$ (resp. $3t_1-\mu$), we obtain that $\mathfrak{g}$ is of type $A_{\ell+1}(\ell\geq 1)$ (resp. $C_{\ell+1}$) and $k+1\leq s_0-\sum_{i=1}^{\ell}s_i$ (resp. $k+1\leq s_0-2\sum_{i=1}^{\ell-1}s_i-s_\ell$). Next we suppose that $K_0\cap\mathfrak{g}_-=\{0\}$. Then $t_i=t_1$ (resp. $t_i=t_1$ and $\mu=2t_1$) for all $1\leq i\leq n$ when $\mathscr{G}=\bigoplus_{p\in\boldsymbol{Z}}\mathscr{G}_p$ is isomorphic to $W(n:\boldsymbol{t})$ (resp. $K(n:\boldsymbol{t}:\mu)$), so $\mathfrak{g}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is of general type (resp. of contact type). The remaining statements are obvious from the above proof, Theorem 9 and Theorem 10. □

References

[Bou60] N. BOURBAKI, "Groupes et algebres de Lie", Chap. 1. Hermann Paris (1960).

[Bou68] N. BOURBAKI, "Groupes et algebres de Lie", Chaps 4, 5, 6, Hermann Paris (1968).

[Bou75] N. BOURBAKI, "Groupes et algebres de Lie", Chaps 7, 8, Diffusion C. C. L. S Paris (1975).

[But67] C. BUTTIN, Étude d'un cas d'isomorphisme d'une algēbra de Lie filtrēe avec son algebra graduēe associēe, C. R. Acad. Sc. Paris, t. 246 (1967), 496–498.

[GQS66] V. GUILLEMIN, D. QUILLEN, and S. STERNBERG, On the classification of the complex primitive infinite pseudogroups, Proc. Nat. Acad. Sci. U. S. A. 55 (1966) 687–690.

[Kac68] V. G. KAC, Simple irreducible graded Lie algebras of finite growth, Math. USSR-Izvestija (1968), 1271-1311.

[Kac70] V. G. KAC, The classification of the simple Lie algebras over a field with nonzero characteristic, Math. USSR-Izvestija 4 (1970), 391-413.

[Kan70] I. L. KANTOR, Graded Lie algebras, Trudy sem. Vect. Tens. Anal. 15, (1970), 227-266 (in Russian).

[KN65] S. KOBAYASHI and T. NAGANO, On filtered Lie algebras and geometric structure III, J. of Math. Mech. 14 (1965) 679-706.

[KN66] S. KOBAYASHI and T. NAGANO, On filtered Lie algebras and geometric structure IV, J. of Math. Mech. 15 (1966) 163-171.

[Mor88] T. MORIMOTO, Transitive Lie algebras admitting differential systems, Hokkaido Math. J. 17 (1988) 45-81.

[MT70] T. MORIMOTO and N. TANAKA, The classification of the real primitive infinite Lie algebras, J. Math. Kyoto Univ. 10-2 (1970) 207-243.

[SS65] I. M. SINGER and S. STERNBERG, On the infinite groups of Lie and Cartan, J. Analyse Math. 15 (1965) 1-114.

[Tan70] N. TANAKA, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1970) 1-82.

[Tan79] N. TANAKA, On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J., 8 (1979) 23-84.

[Tan85] N. TANAKA, On affine symmetric spaces and the automorphism groups of product manifolds, Hokkaido Math. J. (1985) 207-351.

[Wei68] B. Ju. WEISFEILER, Infinite dimensional filtered Lie algebras and their connection with graded Lie algebras, Funktional anal. i Prilozhen (1968) 94-94.

[Wei78] B. Ju. WEISFEILER, On the structure of the minimal ideal of some graded Lie algebras in characteristic $p>0$, J. of Alg. 53 (1978) 344-361.

[Yam82] K. YAMAGUCHI, Contact geometry of higher order, Japan J. Math. 8 (1982) 108-176.

[Yam83] K. YAMAGUCHI, Geometrization of jet bundles, Hokkaido Math. J. 12 (1983) 27-40.

[Yam-pre] K. YAMAGUCHI, Differential systems associated with simple graded Lie algebras, preprint.

[Yat88] T. YATSUI, On pseudo-product graded Lie algebras, Hokkaido Math. J. 17 (1988) 333-343.

[Yat89] T. YATSUI, On simple graded Lie algebras of finite depth, In: Geometry of manifolds pp. 239-243: Academic Press 1989.

Department of mathematics,
Hokkaido University
Sapporo 060
Japan