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Bourgain algebras on $M(H^{\infty})$

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Abstract

In this paper we consider closed subalgebras of $C(M)$ and study the structure of algebras $\mathscr{A}$ satisfying $\mathscr{A}_b=C(M)$. We show that the Bourgain algebra of A is contained in $H^{\infty}(D)+C(\bar{D})$ if A is between the disk algebra $A(D)$ and $H^{\infty}(D)$ or between $H^{\infty}(D)$ and $H^{\infty}(D)+C(\bar{D})$, and the Bourgain algebra of $H^{\infty}\circ L_m$ is contained in $H^{\infty}(D)+C(\bar{D})$ if m is a nontrivial point.

1. Introduction.

Let D be the open unit disk, and let $H^{\infty}(D)$ be the algebra of all bounded analytic functions on D. $L^{\infty}(\partial D)$ denotes the usual space of bounded measurable functions on ∂D, and let $H^{\infty}(\partial D)$ (or simply H^{∞}) be the subalgebra of $L^{\infty}(\partial D)$ consisting of boundary values of functions in $H^{\infty}(D)$.

Let $M=M(H^{\infty})$ denote the maximal ideal space of H^{∞}. The open unit disk D can be identified as an open set in M. By using the Gelfand transform we think of $H^{\infty}(D)$ as a closed subalgebra of $C(M)$, the space of continuous functions on M.

For φ, $\tau\in M$, the pseudohyperbolic distance between φ and τ, denoted by $\rho(\varphi, \tau)$, is defined by

$$\rho(\varphi, \tau)=\sup\{|\varphi(f)|: f\in H^{\infty}, \|f\|<1, \textit{ and } \tau(f)=0\}.$$

The Gleason part of φ is denoted by $P(\varphi)$, and is defined by

$$P(\varphi)=\{\tau\in M: \rho(\varphi, \tau)<1\}.$$

For each $\varphi\in M$, Hoffman [Ho2] constructed a fundamental canonical map L_{φ} of the unit disk D onto the part $P(\varphi)$. This map is defined by taking a net $\{w_{\alpha}\}$ in D such that $w_{\alpha}\rightarrow\varphi$ and defining

$$f\circ L_{\varphi}(z)=\lim_{\alpha} f\left(\frac{w_{\alpha}+z}{1+\overline{w_{\alpha}}z}\right)$$

for $z\in D$ and $f\in H^{\infty}$, the above limit exists and is independent of the net

$\{w_\alpha\}$ provided that $w_\alpha \to \varphi$. Budde [Bu] extended the map L_φ from the maximal ideal space M onto the closure of the part $P(\varphi)$ in M. We shall use the symbol L_φ for this extension.

If f is in $C(M)$ (or L^∞), then the closed subalgebra of $C(M)$ (respectively L^∞) generated by $H^\infty(D)$ (respectively H^∞) and f is denoted by $H^\infty[f]$.

Let A be a subalgebra of $C(X)$ where X is a compact Hausdorff space. Cima and Timoney [CT] introduced the notation of the Bourgain algebra. The Bourgain algebra A_b consists of those f in $C(X)$ such that if $f_n \to 0$ weakly in A, then $dist(ff_n, A) \to 0$. The distance, $dist(ff_n, A)$ is the quotient norm of $ff_n + A$ in the space $C(X)/A$. The proof in [CT] shows that A_b is a closed subalgebra of $C(X)$ and contains A. Several authors have studied Bourgain algebras, [Bi], [CJY], [CSY 1], [CSY 2], [GSZ], [GIM], [I], [MY], [Y], [Z]. In this paper we consider close subalgebras of $C(M)$.

In Section 2, we present one lemma that will be used frequently in this paper. In Section 3, we consider closed subalgebras $\mathscr{A}$ of $C(M)$ which contain $H^\infty(D)$, and study the structure of $\mathscr{A}$ satisfying $\mathscr{A}_b = C(M)$. In Section 4, we consider algebras A between the disk algebra $A(D)$ and $H^\infty(D)$ or between $H^\infty(D)$ and $H^\infty(D) + C(\bar{D})$ and obtain that A_b is still contained in $H^\infty(D) + C(\bar{D})$. Also we show that $(H^\infty \circ L_m)_b$ is contained in $H^\infty(D) + C(\bar{D})$ if m is a nontrivial point although $H^\infty \circ L_m$ does not contain the disk algebra $A(D)$ in case m is a nonhomeomorphic point.

2. Preliminaries and notations.

A sequence $\{z_n\}$ in D is called an interpolating sequence if for every bounded sequence of complex numbers $\{w_n\}$ there exists a function f in H^∞ such that $f(z_n\} = w_n$ for all n. A Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

is said to be interpolating if its zero sequence $\{a_n\}$ in D is an interpolating sequence. Let $Z(B) = \{x \in M(H^\infty) : B(x) = 0\}$, then by [G, p. 379], $Z(B) = closure\{a_n\}$. The Blaschke product B is called thin if

$$\lim_{k \to \infty} \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z_n} z_k} \right| = 1.$$

In this paper we use $\tilde{f}$ to denote the harmonic extension of f to the unit disk D if f is a function on the unit circle ∂D. f^* denotes the nontangential limit of f if f is defined on D and its nontangential limit exists.

We use H^∞ to denote $H^\infty(\partial D)$ or $H^\infty(D)$ for simplicity. Since $C(M)$ is an algebra generated by $H^\infty(D)$ and $\overline{H^\infty(D)}$, the nontagential limit f^* always exists for each function f in $C(M)$.

Throughout this paper, the following lemma will be used several times.

LEMMA 2.1. *Let m be a nontrivial point and $\{z_n\}$ be a sequence in D such that $|z_n|\to 1$, as $n\to\infty$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a weakly null sequence $\{f_n\}$ in H^∞ such that $\|f_n\|\leq 2$ and $|f_k\circ L_m(z_{n_k})|>\frac{3}{4}$, whenever m is a nontrivial point. Moreover $\{f_k\circ L_m\}$ is also a weakly null sequence in $H^\infty\circ L_m$.*

PROOF. Let m be a nontrivial point, and let $\phi_n=L_m(z_n)$, and $\phi\in\bigcap_{n=1}^\infty$ *closure* $\{\phi_n, \phi_{n+1}, \ldots\}$. If ϕ is in $P(m)$, then there is a point z_o in D such that $L_m(z_0)=\phi$. Since m is nontrivial, it follows from [Ho2, p.105] that

$$\rho(\phi, \phi_n)=\rho(L_m(z_0),\ L_m(z_n))=\rho(z_0, z_n)\to 1,$$

as $n\to\infty$. In case that ϕ is not in $P(m)$, $\rho(\phi, \phi_n)=1$ for all n. Thus the proof of Theorem 3 in [AG2] still works, and from its proof we can see that there are functions F_n and G_n in H^∞ such that for some subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$,

$$\sum_{n=1}^{\infty}|F_n(z)\prod_{j=1}^{n-1}G_j(z)|<2$$

and $F_k(\phi_{n_k})=1$, and $|1-(\prod_{j=1}^{k-1}G_j)(\phi_{n_k})|<1/4$. Let $f_k=F_k\prod_{j=1}^{k-1}G_j$. Then

$$\sum_{k=1}^{\infty}|f_k(z)|<2$$

and $|1-f_k(\phi_{n_k})|<1/4$. Thus $|f_k\circ L_m(z_{n_k})|>3/4$, and f_k is a weakly null sequence in H^∞.

Since m is nontrivial, by [Ho2] there exists a net z_α in D such that $z_\alpha\to m$ and

$$f_k\circ L_m(z)=\lim_{z_\alpha\to m}f_k\left(\frac{z+z_\alpha}{1+\overline{z_\alpha}z}\right).$$

Thus $\sum_{k=1}^{\infty}|f_k\circ L_m(z)|<2$. Thus $\{f_k\circ L_m\}$ is a weakly null sequence in $H^\infty\circ L_m$, as desired.

From the above proof we can easily get the following lemma.

LEMMA 2.2. *Let b be an interpolating Blaschke product. There are points $\{m_n\}$ in $Z(b)-D$ and a weakly null sequence $\{f_n\}$ such that $\|f_n\| \leq 2$ and $|f_n(m_n)| > 3/4$.*

3. Subalgebras $\mathscr{A}$ of $C(M)$ satisfying $\mathscr{A}_b = C(M)$.

In this section we consider closed subalgebras $\mathscr{A}$ of $C(M)$ having the property $\mathscr{A}_b = C(M)$. Here we present some properties that shed some lights on the structure of these algebras.

THEOREM 3.1. *Let $\mathscr{A}$ be a closed subalgebra of $C(M)$ which contains $H^\infty(D)$. If $\mathscr{A}_b = C(M)$, then $\mathscr{A} \circ L_m \neq H^\infty \circ L_m$ whenever m is a nontrivial point.*

The proof of Theorem 3.1 will be given at the end of this section. Now let us mention the following consequences of Theorem 3.1. If $\mathscr{B}$ be a subset of $C(M)$, we use $H^\infty[\mathscr{B}]$ to denote the algebra generated by $\mathscr{B}$ over H^∞.

COROLLARY 3.2. *Let $\mathscr{B}$ be a subset of the complex conjugates of H^∞, and let $\mathscr{A} = H^\infty[\mathscr{B}]$. If the Bourgain algebra $\mathscr{A}_b$ is $C(M)$, then any nontrivial point is a maximal antisymmetric set for $\mathscr{A}$.*

P. Gorkin and R. Mortini found examples of proper subalgebras $\mathscr{A}$ of $C(M)$ such that $\mathscr{A}_b = C(M)$ (private communication).

COROLLARY 3.3. *Let $\mathscr{B}$ be a subset of the complex conjugates of H^∞, and let $\mathscr{A} = H^\infty[\mathscr{B}]$. If the Bourgain algebra $\mathscr{A}_b$ is $C(M)$, then for any inner function b with $|b| = 1$ on trivial points, the conjugate of b is in $\mathscr{A}$. In particular, $\mathscr{A}$ contains the conjugate of every thin Blaschke product.*

The following lemma will be used in the proof of Corollary 3.2, which first appeared in [Z]. Let $\mathscr{B}$ be a subset of the complex conjugates of H^∞. We define

$$E(\mathscr{B}) = \{m \in M : f \circ L_m \text{ is not constant for some } f \in \mathscr{B}\}.$$

LEMMA 3.4. *If S is a maximal antisymmetric set for $H^\infty[\mathscr{B}]$, and $S \cap E(B)$ is not empty, then S contains only one point.*

PROOF. Since $S \cap E(B)$ is not empty, for some f in $\mathscr{B}$, $S \cap E(f)$ is not empty. From the proof of Theorem 1 in [AG2] it follows that there is an interpolating Blaschke product b such that S is a subset of $Z(b)$. Now observe that $Z(b)$ is totally disconnected because by [Ho 1, p. 205], it is homeomorphic to the Stone-Cech compactification βN of N. Since S is connected, this forces S to be just one point.

PROOF of COROLLARY 3.2. Let S be a maximal antisymmetric set for $\mathscr{A}$, and suppose that S contains a nontrivial point m. If m is not in $E(\mathscr{B})$, then $\mathscr{A}\circ L_m = H^\infty\circ L_m$, contradicting Theorem 3.1. If m is in $E(\mathscr{B})$, then by Lemma 3.4, S is just one point.

PROOF of COROLLARY 3.3. Let b be an inner function such that $|b| = 1$ on any trivial point, and let S be a maximal antisymmetric set for $\mathscr{A}$. If S does not contain any nontrivial point, then $\bar{b}|_S = \frac{1}{b}|_S$ is in $\mathscr{A}|_S$. On the other hand, if S contains any nontrivial point, then by Lemma 3.4, S is just one point. So $\bar{b}|_S$ is in $\mathscr{A}|_S$. By Bishop antisymmetric decomposition theorem we have $\bar{b}$ is in $\mathscr{A}$.

In particular, if b is thin, the zero set of b does not contain any trivial point. So $|b| = 1$ on trivial points [He]. From above we get $\bar{b}$ is in $\mathscr{A}$.

Now we return to the proof of Theorem 3.1.

PROOF of THEOREM 3.1. Suppose that $\mathscr{A}\circ L_m = H^\infty\circ L_m$ for some nontrivial point m. Let

$E = \{g\in C(M) : dist_D(gf_n\circ L_m, H^\infty)\to 0$, *for any weakly null sequence* $f_n\in\mathscr{A}\}$.

It is easy to see that E is a closed vector space. Because $\mathscr{A}\circ L_m = H^\infty\circ L_m$, we see that E contains $H^\infty(D)$.

The rest of the proof will be divided into several steps.

STEP 1. The set E contains $C(M)\circ L_m$.

Since $\mathscr{A}_b = C(M)$, it suffices to show that E contains $\mathscr{A}_b\circ L_m$. Let f be in $\mathscr{A}_b$. Then for any weakly null sequence $\{f_n\}$ in $\mathscr{A}$, we have

$$\begin{aligned} &dist_D(f\circ L_m f_n\circ L_m, H^\infty)\le dist_D(f\circ L_m f_n\circ L_m, H^\infty\circ L_m)\\ &= dist_D(f\circ L_m f_n\circ L_m, \mathscr{A}\circ L_m)\le dist_D(ff_n, \mathscr{A})\to 0, \end{aligned}$$

as $n\to\infty$. Thus $f\circ L_m$ is in E, completing the proof of Step 1.

STEP 2. $H^\infty[f]\subset E$ whenever f is in $C(M)\circ L_m$.

Let g be in H^∞ and f in $C(M)\circ L_m$. By Step 1, f is in E. Let $\{f_n\}$ be any weakly null sequence in $\mathscr{A}$. Then

$$\begin{aligned} dist_D(gff_n\circ L_m, H^\infty) &\le dist_D(gff_n\circ L_m, gH^\infty)\\ &\le \|g\|_\infty dist_D(ff_n\circ L_m, H^\infty)\to 0, \end{aligned}$$

as $n\rightarrow\infty$ because f is in E. Thus gf is in E.

Since $C(M)\circ L_m$ is an algebra, and E is a closed vector space containing H^∞, we get that $\sum_{j=0}^{n} g_j f^j$ is in E for any g_j in H^∞, and that E contains $H^\infty[f]$, as required.

STEP 3. There exists an interpolating Blaschke product b such that $\bar{b}$ is in E.

First we claim that there is a function f in $H^\infty\circ L_m$ which is not constant on some nontrivial point. To prove the claim we consider two cases.

(i) $L_m(z)$ is homeomorphic. Hence $L_m(z)$ has an injective extension on M. Let τ be a nontrivial point in M/D and $z\neq 0$ in D. Since $L_m(z)$ is homeomorphic, by Theorem 1.4 in [GLM] we have $L_m\circ L_\tau(z)\neq L_m(\tau)$. Because H^∞ separates points of M, then there is a function g in H^∞ such that $g\circ L_m\circ L_\tau(z)\neq g\circ L_m(\tau)$. Let $f=g\circ L_m$. Thus f is not constant on the Gleason part $P(\tau)$.

(ii) $L_m(z)$ is not homeomorphic. There are a nontrivial point τ in M/D and a point w in D such that $L_m(\tau)=L_m(w)$, we may assume that w is 0. By Theorem 2.5 of Chapter X in [G], then we have $L_m\circ L_\tau(z)=L_m(\alpha z)$ for some constant α with $|\alpha|=1$. Since H^∞ separates its maximal ideal space M, there is a function g in H^∞ such that $g\circ L_m(z)$ is not constant. Let $f=g\circ L_m$. Thus $f\circ L_\tau(z)=g\circ L_m\circ L_\tau(z)=g\circ L_m(\alpha z)$ is not constant. This finishes the proof of our claim.

To finish the proof of Step 3, we let f be a function in $H^\infty\circ L_m$ as above. By Theorem 2 in [AG], there is an interpolating Blaschke product b such that $\bar{b}$ is in $H^\infty[\bar{f}]$. Since $\overline{H^\infty\circ L_m}$ is a subset of $C(M)\circ L_m$, by Step 2 we have $H^\infty[\bar{f}]\subset E$. Thus $\bar{b}$ is in E, as promised.

Now we are ready to finish the proof of Theorem 3.1. Let b the interpolating Blaschke product b in Step 3 and let $\{z_n\}$ be its zero set. By Lemma 2.1, we can choose a weakly null sequence $\{f_k\}$ in $\mathscr{A}$ such that $|f_k\circ L_m(z_{n_k})|>\frac{3}{4}$. Since $\bar{b}$ is in E, for such weakly null sequence $\{f_k\}$, we have

$$0\leftarrow dist_D(\bar{b}f_k\circ L_m, H^\infty)\geq dist_{\partial D}(\overline{b^* f_k\circ L_m^*}, H^\infty).$$
$$=dist_{\partial D}(f_k\circ L_m^*, b^*H^\infty)=dist_D(f_k\circ L_m, bH^\infty)\geq|f_k\circ L_m(z_{n_k})|\geq\frac{3}{4}.$$

This contradication shows that $\mathscr{A}\circ L_m\neq H^\infty\circ L_m$, and this completes the proof of Theorem 3.1.

4. Bourgain algebras of some subalgebras of $H^\infty(D)+C(\bar{D})$.

K. Hoffman [Ho2] showed that if m is a nontrivial point, then for any f in $H^\infty(D)$, $f\circ L_m$ is in $H^\infty(D)$. Thus $H^\infty\circ L_m$ is a subalgebra of $H^\infty(D)$. In many cases $H^\infty\circ L_m$ does not contain $A(D)$. In case that m is locally thin, then [GLM], $H^\infty\circ L_m=H^\infty(D)$. Nevertheless, we have the following theorem.

THEOREM 4.1. *If m is a nontrivial point, then*

$$(H^\infty\circ L_m)_b\subset H^\infty(D)+C(\bar{D}).$$

PROOF. Let f be in $(H^\infty\circ L_m)_b$. First we claim that $H^\infty[f^*]\subset H^\infty+C$. To prove the claim, by the Chang-Marshall theorem ([C] [G], [M]), it suffices to show that $H^\infty[f^*]$ doesn't contain any conjugate of infinite Blaschke products.

Let ψ be an infinite Blaschke product with zeros $\{z_n\}$ and assume that $\bar{\psi}$ is in $H^\infty[f^*]$. By Lemma 2.1, there is a weakly null sequence $\{g_k\}$ in $H^\infty\circ L_m$ such that $|g_k(z_{n_k})|>3/4$. Pick some h_i in H^∞ for $i=0,\dots,n$ so that $\|\bar{\psi}-\sum_{i=0}^{n}h_i(f^*)^i\|_{\partial D}<1/8$. Let $C=\max_{j=0,\dots,n}\|h_j\|_{\partial D}$. Since f^i is in $(H^\infty\circ L_m)_b$ for $i=0,\dots,n$, we choose f_i in $H^\infty\circ L_m$ such that $\|g_kf^i-f_i\|_D<\frac{1}{8nC}$ for $i=0,\dots,n$ as k is sufficiently large. For such k, we have $\|g_k^*(f^*)^i-f_i^*\|_{\partial D}\le\frac{1}{8nC}$ for $i=0,\dots,n$. Now we obtain

$$\begin{aligned}\|\bar{\psi}g_k^*-\sum_{i=0}^{n}f_i^*h_i\|_{\partial D}&\le\|(\bar{\psi}-\sum_{i=0}^{n}h_i(f^*)^i)g_k^*\|_{\partial D}+\|\sum_{i=0}^{n}(h_i(f^*)^ig_k^*-f_i^*h_i)\|_{\partial D}\\&\le\|g_k^*\|_{\partial D}\|\bar{\psi}-\sum_{i=0}^{n}h_i(f^*)^i\|_{\partial D}+\sum_{i=0}^{n}\|h_i\|_{\partial D}\|(f^*)^ig_k^*-f_i^*\|_{\partial D}\\&\le\frac{2}{8}+\sum_{i=0}^{n}\|h_i\|_{\partial D}\frac{1}{8nC}\le\frac{1}{4}+\frac{1}{8}=\frac{3}{8}.\end{aligned}$$

On the other hand, because $|\bar{\psi}|=1$ a.e. on ∂D, we have

$$\|\bar{\psi}g_k^*-\sum_{i=0}^{n}f_i^*h_i\|_{\partial D}=\|g_k^*-\psi\sum_{i=0}^{n}f_i^*h_i\|_{\partial D}.$$

As $\sum_{i=0}^{n}f_i^*h_i$, and g_k^* are in H^∞, the maximum modulus principle yields

$$\begin{aligned}\|g_k^*-\psi\sum_{i=0}^{n}f_i^*h_i\|_{\partial D}&=\|g_k-\psi\sum_{i=0}^{n}f_ih_i\|_D\\&\ge|g_k(z_{n_k})-\psi(z_{n_k})\sum_{i=0}^{n}f_i(z_{n_k})h_i(z_{n_k})|=|g_k(z_{n_k})|.\end{aligned}$$

Thus $\frac{3}{8} \geq |g_k(z_{n_k})|$. This contradicts the choice of g_k, which satisfy $|g_k(z_{n_k})| > \frac{3}{4}$ for any k. Thus $H^\infty[f^*]$ doesn't contain $\bar{\psi}$.

Now we have that f^* is in $H^\infty + C$. Thus the harmonic extension $\tilde{f}^*$ is in $H^\infty + C(\bar{D})$. Let $g = f - \tilde{f}^*$. Clearly $g|_{\partial D} \equiv 0$. Now we are going to show that $g(z) \to 0$ as $z \to \partial D$.

If not, then there is an interpolating sequence $\{w_n\}$ in D such that $|g(w_n)| > \delta$, for some $\delta > 0$. By Lemma 2.1, there exists a weakly null sequence $\{g_k \circ L_m\}$ in $H^\infty \circ L_m$ such that $|g_k \circ L_m(w_{n_k})| > 3/4$.

Since f is in $(H^\infty \circ L_m)_b$, there is a sequence $\{l_k\}$ in $H^\infty \circ L_m$ such that $\|fg_k \circ L_m - l_k\|_D \to 0$. Because $H^\infty(D) + C(\bar{D}) = (H^\infty(D))_b$ [GSZ], $\tilde{f}^*$ is in $(H^\infty(D))_b$. Thus there exists a sequence $\{u_k\}$ in $H^\infty(D)$ such that $\|\tilde{f}^* g_k \circ L_m - u_k\|_D \to 0$. Consequently $\|gg_k \circ L_m - (l_k - u_k)\|_D \to 0$.

Set $h_k = l_k - u_k$. Then we get $\|h_k^*\|_{\partial D} \to 0$. But g vanishes on ∂D, and hence $\|h_k\|_D \to 0$. Thus $\|gg_k \circ L_m\|_D \to 0$. But $|g(w_{n_k}) g_k \circ L_m(w_{n_k})| > \frac{3\delta}{4}$. This contradiction shows that $g(z) \to 0$ as $z \to \partial D$, completing the proof that g is in $C(\bar{D})$. Thus $f = \tilde{f}^* + g$ is in $H^\infty(D) + C(\bar{D})$, as required.

K. Izuchi [I] proved that the Bourgain algebra of a closed subalgebra between $A(\partial D)$ and H^∞ on $L^\infty(\partial D)$ is always contained in $H^\infty + C$. Since on the disk there are many closed subalgebras of $C(M)$ which are between $H^\infty(D)$ and $H^\infty(D) + C(\bar{D})$, the following theorem says that the Bourgain algebra of a closed subalgebra between $A(D)$ and $H^\infty(D)$ or between $H^\infty(D)$ and $H^\infty(D) + C(\bar{D})$ is always contained in $H^\infty(D) + C(\bar{D})$.

THEOREM 4.2. *Let $\mathscr{A}$ be a closed subalgebra of $C(M)$. If $A(D) \subset \mathscr{A} \subset H^\infty(D)$ or $H^\infty(D) \subset \mathscr{A} \subset H^\infty(D) + C(\bar{D})$, then $(\mathscr{A})_b$ is contained in $H^\infty(D) + C(\bar{D})$.*

PROOF. Case 1. $\mathscr{A}$ is between $A(D)$ and $H^\infty(D)$.

In this case the proof that $(\mathscr{A})_b$ is contained in $H^\infty + C$ is exactly the same as the proof of Theorem 4.1, but here we use the fact [I] that for an interpolating sequence $\{z_n\}$ in D, there is a weakly null sequence $\{g_k\}$ in the disk algebra $A(D)$ such that $|g_k(z_{n_k})| > \frac{3}{4}$ for a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ instead of Lemma 2.1.

Case 2. $\mathscr{A}$ is between $H^\infty(D)$ and $H^\infty(D) + C(\bar{D})$.

Let f be in $(\mathscr{A})_b$. The proof will be divided into two steps.

Step 1. The non-tangential limit f^* is in $H^\infty + C$.

If not, then by Chang-Marshall theorem there is an interpolating Blaschke product b such that $\bar{b}$ is in the Douglas algebra $H^\infty[f^*]$.

Define

$$\mathscr{A}_2=\{f\in C(M): dist_D(ff_n, H^\infty(D)+C(\bar{D}))\to 0,\ \forall\ f_n\to 0 \text{ weakly in } \mathscr{A}\}.$$

It is easy to check that $\mathscr{A}_b\subset\mathscr{A}_2$, and that $H^\infty[f^*]\subset\mathscr{A}_b|_X\subset\mathscr{A}_2|_X$, where $X=M(L^\infty)$. So for any weakly null sequence $\{f_n\}$ in $\mathscr{A}$,

$$(*)\qquad dist_X(\bar{b}f_n^*, H^\infty+C)\to 0.$$

On the other hand,

$$dist_X(\bar{b}f_n^*, H^\infty+C)=dist_X(f_n^*, b(H^\infty+C))\geq\|f_n\|_{Z(b)-D}.$$

But by Lemma 2.2, there is a weakly null sequence $\{f_n\}$ in $H^\infty(D)$ such that $\|f_n\|_{Z(b)-D}>3/4$. Thus $dist_X(\bar{b}f_n^*, H^\infty+C)>3/4$, which contradicts (*). Thus the proof of Step 1 is now completed.

Step 2. Let $g=f-\tilde{f}^*$. We are going to show that g is in $C(\bar{D})$.

By Step 1, f^* belongs to $H^\infty+C$, and so $\tilde{f}^*$ is in $H^\infty(D)+C(\bar{D})$. Thus g vanishes on ∂D, and g is continuous on D. In order to prove that g is in $C(\bar{D})$, we need only to show that $g(z)\to 0$ as $z\to\partial D$. If this is not true, then we may assume that there is an interpolating sequence $\{z_k\}$ in D such that for some constant $\delta>0$, $|g(z_k)|>\delta$.

Let b be a Blaschke product with zeros $\{z_k\}$. By Lemma 2.2, there is a weakly null sequence $\{f_n\}$ in $H^\infty(D)$ such that $|f_n(m_n)|\geq 3/4$ for some $m_n\in Z(b)-D$. Since $\tilde{f}^*$ is in $H^\infty(D)+C(\bar{D})$, and f in $\mathscr{A}_b$, there is a sequence $\{g_n\}\subset H^\infty(D)+C(\bar{D})$ such that

$$(**)\qquad \|gf_n-g_n\|_D\to 0.$$

Since g vanishes on ∂D, $\|g_n^*\|_{\partial D}\to 0$. So $\|\tilde{g}_n^*\|_D\to 0$. Since g_n is in $H^\infty(D)+C(\bar{D})$, $\tilde{g}_n^*|_{M-D}=g_n|_{M-D}$, and so $\|g_n\|_{M-D}\to 0$. Thus it follows from (**) that

$$(***)\qquad \|gf_n\|_{M-D}\to 0.$$

But from above we have $|g(m_n)f_n(m_n)|>\delta/2$. Thus the proof of Step 2 is completed.

Since $f=g+\tilde{f}^*$, from Steps 1 and 2 we get that f is in $H^\infty(D)+C(\bar{D})$, and this completes the proof of Theorem 4.2.

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