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On isomorphism of minimal direct summands

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Abstract

Let G be a quasi-complete p-group and let A be a subgroup of G such that there exists a direct summand L of G containing A which is minimal among the direct summands of G that contain A. Such a direct summand L is said to be a minimal direct summand of G containing A. We prove that all minimal direct summands of G containing A are isomorphic.

Introduction

All groups considered here are p-primary abelian groups for a fixed prime number p. It is well-known that a separable group is isomorphic to a pure and dense subgroup of some torsion-complete group. Therefore it is important to study torsion-complete groups and their subgroups in order to clarify the structure of separable groups.

A subgroup A of a group G is said to be **purifiable** if there exists a pure subgroup H of G containing A which is minimal among the pure subgroups of G that contain A. Such a subgroup H is said to be a **pure hull** of A in G. In a direct sum of cyclic groups, every subsocle is purifiable and all pure hulls of a subsocle are isomorphic. However, in a torsion-complete group, every subsocle is also purifiable, but all pure hulls of the same subsocle are not necessarily isomorphic. (See [7, 66, Exercise 8].) We can raise the following problem:

For which purifiable subgroup A are all pure hulls of A isomorphic?

From [2], [4], [8], and [11], purifiable subgroups A and their pure hulls H have the following properties:

(1) *There exists a non-negative integer m such that $V_n(G, A)=0$ for all $n\geq m$. (i.e. A is eventually vertical in G.)*

(2) *$H=M\oplus N$, where M and N are subgroups of H, $M[p]=A[p]$, $p^{m-1}N\neq 0$, and $p^m N=0$.*

(3) *A is almost-dense in H.*

The subgroup N in (2) is said to be a **residual subgroup** of H determined by A. In [4], it is shown that all residual subgroups determined by a purifiable subgroup are isomorphic.

We extend the concept of purifiable subgroups to the concept of quasi-purifiable subgroups. A subgroup A of a group G is said to be **quasi-purifiable** in G if there exists a pure subgroup K of G such that A is an almost-dense subgroup of K. Namely, A and K satisfy condition (3) above. Such a pure subgroup K is called a **quasi pure hull** of A in G. It is obvious that purifiable subgroups are quasipurifiable. But the converse is not true. For example, the subgroup L constructed in the proof of [8, Proposition 1] is quasi-purifiable but not purifiable. (See Example 2.4) We prove that a quasi-purifiable subgroup A of a group G is purifiable in G if and only if A is eventually vertical in G. Moreover, we show that if A is quasi-purifiable in G, then there exists a maximal quasi pure hull of A in G.

A subgroup A of a group G is said to be **summandable** if there exists a direct summand L of G containing A which is minimal among the direct summands of G that contain A. Such a direct summand L is a **minimal direct summand** of G containing A.

It is obvious that summandable subgroups are quasi-purifiable. Moreover, we show that, in a torsion-complete group, A is summandable if and only if A is quasi-purifiable, and L is a maximal quasi pure hull of A if and only if L is a minimal direct summand of A. In general, every subgroup is not necessarily summandable in a given group. (See Example 3.8.)

We establish another characterization of torsion-complete groups; namely, a reduced group G is torsion-complete if and only if all quasi-purifiable subgroups of G are summandable subgroups. Moreover, we determine when quasi-purifiable subgroups of a quasi-complete but not torsion-complete group are summandable.

Finally, we use these concepts and results to prove our main result: Namely, in a quasi-complete group, all minimal direct summands containing a summandable subgroup are isomorphic.

The terminologies and notations not expressly introduced here follow the usage of [7]. All topological references are to the p-adic topology. Throughout this note, let A be a subgroup of a group G.

1. Purifiable subgroups

We recall some definitions and results that are frequently used in this note, and we make an abstract of the process of studying purifiable subgroups.

DEFINITION 1.1. A is said to be a **purifiable subgroup** of G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a **pure hull** of A in G.

B. Charles was first to consider this notion in [6]. P. Hill and C. Megibben [8] and T. Okuyama [11] determined the structure of pure hulls that is concerned with condition (2) mentioned in the introduction.

On the other hand, in [2], K. Benabdallah and J. Irwin introduced the concept of almost-dense subgroups. This is concerned with the condition (3) mentioned in the introduction.

DEFINITION 1.2. A is said to be **almost-dense** in G if G/K is divisible for every pure subgroup K of G containing A.

PROPOSITION 1.3. ([2], *Theorem* 2) *A is almost-dense in G if and only if, for every non-negative integer n, $A+p^{n+1}G\supset p^nG[p]$.*

In [4], K. Benabdallah and T. Okuyama introduced new invariants, the so-called n-th overhangs of a subgroup in a given group and obtained a necessary condition for a subgroup to be purifiable in a given group. This is concerned with condition (1) mentioned in the introduction. Moreover, they determined when almost-dense subgroups are purifiable in a given group.

DEFINITION 1.4. For every non-negative integer n, the **n-th overhang** of A in G is the vector space

$$V_n(G, A)=((A+p^{n+1}G)\cap p^nG[p])/((A\cap p^nG[p])+p^{n+1}G[p]).$$

It is convenient to use the following notation for the numerator and denominator of $V_n(G, A)$:

$$A_G^n=(A+p^{n+1}G)\cap p^nG[p]=((A\cap p^nG)+p^{n+1}G)[p]$$

and

$$A_n^G=(A\cap p^nG[p])+p^{n+1}G[p]=A[p]_G^n.$$

DEFINITION 1.5. A is said to be a **vertical subgroup** of G if $V_n(G, A)=0$ for all $n\geq 0$. If there exists a non-negative integer m such that $V_n(G, A)=0$ for all $n\geq m$, then A is said to be **eventually vertical**.

PROPOSITION 1.6. ([4], *Theorem* 1.8) *If A is a purifiable subgroup of G, then A is eventually vertical in G.*

PROPOSITION 1.7. ([4], *Theorem* 1.11) *Let A be almost-dense in G. Then A is purifiable if and only if A is eventually vertical.*

PROPOSITION 1.8. ([4], *Theorem* 1.7) *For every pure subgroup K of G containing A, we have $V_n(G, A) \simeq V_n(K, A)$ for all $n \geq 0$.*

Next, in [3], K. Benabdallah, B. Charles, and A. Mader introduced the concept of maximal vertical subgroups. Let S be a subsocle of G. A subgroup M is said to be a maximal vertical subgroup of G supported by S if M is maximal among vertical subgroups of G supported by S. The existence of maximal vertical subgroups supported by any subsocle of G are guaranteed by Zorn's Lemma. If A is vertical in G, then there exists a maximal vertical subgroup B of G supported by $A[p]$ containing A.

PROPOSITION 1.9. ([3], *Theorem* 4.5 *and Theorem* 5.5) *The following properties are equivalent for a group G.*

(1) *All maximal vertical subgroups of G are pure in G.*

(2) *All eventually vertical subgroups of G are purifiable in G.*

(3) *The reduced part of G is a quasi-complete group.*

2. Quasi-purifiable subgroups

We have studied eventually vertical subgroups in [3], [4], [10], and [11]. We are interested in subgroups which are not eventually vertical. Such subgroups have not been studied yet. First, we define the concept of quasi-purifiable subgroups.

DEFINITION 2.1. A is said to be a **quasi-purifiable subgroup** of G if there exists a pure subgroup H of G such that A is almost-dense in H. Such a subgroup H is called a **quasi pure hull** of A.

From the definition, we immediately obtain the following:

PROPOSITION 2.2. *If A is purifiable in G, then A is quasi-purifiable in G.* □

We establish the following useful lemma for almost-dense subgroups. Before we do this, we give a definition concerning certain subsocles.

DEFINITION 2.3. For every non-negative integer n,

$$p^nG[p]=S_n\oplus A_G^n=S_n\oplus P_n\oplus A_n^G=S_n\oplus P_n\oplus A_n\oplus p^{n+1}G[p],$$

where S_n, P_n, and A_n are subgroups of $p^nG[p]$, A_G^n, and A_n^G, respectively. Put $P=\bigoplus_n P_n$, P is said to be an **overhang subsocle** of A in G.

LEMMA 2.4. *Let P be an overhang subsocle of A in G. If A is almost-dense in G, then there exists a quasi pure hull K of A supported by $(A+P)[p]=A[p]\oplus P$.*

PROOF. Since A is almost-dense in G, we have $p^n G[p] \subset (A + p^{n+1}G)[p] = A_G^n$ for every $n \geq 0$. By Definition 2.3, for every $n \geq 0$, we have

$$p^n G[p] = A_G^n = P_n \oplus A_n^G = P_n \oplus A_n \oplus p^{n+1}G[p],$$

where P_n and A_n are subgroups of A_G^n and A_n^G, respectively. Then $(A+P)[p] = A[p] \oplus P$ is dense in $G[p]$. By [7, Theorem 66.3], there exists a pure hull K of $A+P$. Since $K[p] = A[p] \oplus P$, A is almost-dense in K by Proposition 1.3. Hence K is a quasi pure hull of A. □

The next example shows that the converse of Proposition 2.2 is not true. This was constructed in the proof of [8, Proposition 1].

EXAMPLE 2.5. Let $B = \bigoplus_n B_n$ where $B_n \neq 0$ for infinitely many n and is a homogeneous direct sum of cyclic groups of order p^n. Let $n(i)$ be a sequence of positive integers such that $n(i+1) - n(i) \geq 2$ and $B_{n(i)} \neq 0$ for all i. Let $t(i) = n(2i+1) - n(2i) - 1$ and let

$$L = \sum_{i=1}^{\infty} < b_{n(2i)} + p^{t(i)} b_{n(2i+1)} > \text{ and } H = \bigoplus_{i=2}^{\infty} < b_{n(i)} >,$$

where $< b_{n(i)} >$ is a non-zero cyclic summand of $B_{n(i)}$. Since

$$\begin{aligned} p^{n(2i)-1}\, b_{n(2i)} = (&p^{n(2i)-1}\, b_{n(2i)} + p^{n(2i+1)-2}\, b_{n(2i+1)}) \\ &- p^{n(2i+1)-2}\, b_{n(2i+1)} \in L + p^{n(2i)} H, \end{aligned}$$

we have $L + p^{n+1}H \supset p^n H[p]$ for every $n \geq 0$. Hence L is almost-dense in H by Proposition 1.3, and so L is quasi-purifiable in H. However, since L is not eventually vertical in H by Proposition 1.8 and [4], L is not purifiable in H. □

Next, we determine when quasi-purifiable subgroups are purifiable in a given group.

PROPOSITION 2.6. *Let A be a quasi-purifiable subgroup of G. Then A is purifiable in G if and only if A is eventually vertical in G.*

PROOF. The necessity is immediate by Proposition 1.6. Let H be a quasi pure hull of A in G, then A is almost-dense in H. If A is eventually vertical in G, then A is eventually vertical in H by Proposition 1.8. Hence, by Proposition 1.7, A is purifiable in H, and so A is purifiable in G. □

If A is quasi-purifiable in G, there exists a quasi pure hull H of A in G. But such a subgroup H is not necessarily a pure hull. Thus there exists a proper quasi pure hull K of A in H. In general, we obtain the following result.

PROPOSITION 2.7. *Let A be quasi-purifiable and not purifiable in G, and let H be a quasi pure hull of A in G. Then there exists a quasi pure hull K of A in G such that K is a proper subgroup of H.*

PROOF. Since A is not purifiable in G, there exists a proper pure subgroup K of H containing A. Then A is almost-dense in K, K is a quasi pure hull of A in G. $\square$

Proceeding by Proposition 2.7, we obtain an infinite properly decreasing chain

$$H > K > K_2 > \cdots\cdots > K_n > \cdots,$$

where the subgroups K_n are all quasi pure hulls of A in G.

On the other hand, for maximal quasi pure hulls of A, we establish Proposition 2.8. We use the expression "*maximal quasi pure hull of A*" to refer to a quasi pure hull of A which is maximal among the quasi pure hulls of A in G.

PROPOSITION 2.8. *If A is quasi-purifiable in G, there exists a maximal quasi pure hull of A in G.*

PROOF. Let $\mathrm{P}=\{L\leq G | L$ is a quasi-pure-hull of A in $G\}$. By hypothesis, $\mathrm{P}\neq\phi$. Let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a chain of elements in P. We show that $L=\bigcup_{\lambda\in\Lambda} L_\lambda\in\mathrm{P}$. It is immediate that L is pure in G. Let $x\in p^nL[p]$, then $x\in p^nL_\lambda[p]$ for some $\lambda\in\Lambda$. Since A is almost-dense in L_λ, we have $x\in A+p^{n+1}L_\lambda\subset A+p^{n+1}L$. Hence A is almost-dense in L, and L is a quasi-pure-hull of A in G. By Zorn's Lemma, P contains a maximal element. $\square$

3. Minimal direct summands

First, we introduce the concept of summandable subgroups and give a definition of minimal direct summands.

DEFINITION 3.1. A is said to be a **summandable subgroup** of G if, among the direct summands of G containing A, there exists a minimal one. Such a direct summand is called a **minimal direct summand** of G containing A.

From the proof of [2, Lemma 1.5], we immediately obtain the following lemma.

LEMMA 3.2. *Let A be summandable in G and let H be a minimal direct summand of G containing A. Then A is almost-dense in H. Hence if A is summandable in G, then A is quasi-purifiable in G.* □

In the case that G is reduced, we obtain the following useful characterization.

LEMMA 3.3. *Let A be summandable in a reduced group G and let H be a direct summand of G containing A. Then H is a minimal direct summand of G containing A if and only if A is almost-dense in H.*

PROOF. By Lemma 3.2, the necessity is immediate. Conversely, suppose that the condition holds. If there exists a direct summand K of G with $A \subset K \subset H$, then we have $G = K \oplus L$ for some subgroup L of G, and so we have $H = K \oplus (H \cap L)$. Since A is almost-dense in H, $H/K \simeq H \cap L$ is divisible. However, since G is reduced, $H \cap L = 0$. Thus $H = K$ and so H is a minimal direct summand of G containing A. □

We use the concept of summandable subgroups to give a new characterization of a torsion-complete group. Before we do this, we give an interesting property of torsion-complete groups.

PROPOSITION 3.4. *Let G be torsion-complete. Then the following properties hold*:

(1) *A is summandable in G if and only if A is quasi-purifiable in G.*

(2) *Let A be quasi-purifiable in G and let L be a quasi pure hull of A, then $\bar{L}$ is a minimal direct summand of G containing A. Moreover, L is a maximal quasi pure hull of A if and only if L is a minimal direct summand of G containing A.*

(3) *Let M be a minimal direct summand of G containing A and P be an overhang subsocle of A in M. Then there exists a quasi pure hull H of A supported by $(A+P)[p]$ such that $\bar{H} = M$.*

PROOF. The necessity of (1) is immediate by Lemma 3.2. Conversely, suppose that A is quasi-purifiable in G. Let H be a quasi pure hull of A in G. Since A is almost-dense in H and H is pure and dense in $\bar{H}$, A is almost-dense in $\bar{H}$ by [5, Lemma 1.6]. Since $\bar{H}$ is a direct summand of G by [9, Teorem 3], A is summandable in G by Lemma 3.3. Hence (1) and the first part of (2) is proved.

Let L be a maximal quasi pure hull of A, then $\bar{L}$ is a direct summand. By [5, Lemma 1.6] and Lemma 3.3, we have $\bar{L} = L$. Hence $\bar{L}$ is a minimal direct summand of G containing A. Conversely, suppose that L is a minimal direct summand of G containing A. If there exists a quasi pure hull K of A containing L, $\bar{K}$ is a minimal direct summand of G containing A by [5, Lemma 1.6] and Lemma 3.3. Hence we have $L=K=\bar{K}$ and so L is a maximal quasi pure hull of A. Hence (2) is proved.

By Lemma 2.4, it is immediate that there exists a quasi pure hull H of A in M such that $H[p]=(A+P)[p]$. Since M is closed in G, we have $\bar{H}\subset\bar{M}=M$. By (2), we have $\bar{H}=M$. □

THEOREM 3.5. *A reduced group G is torsion-complete if and only if all quasi-purifiable subgroups of G are summandable subgroups.*

PROOF. The necessity is immediate by Proposition 3.4. Conversely, suppose that the conditions hold. Let H be a pure subgroup of G. Since H is quasi-purifiable in G, there exists a minimal direct summand L of G containing A by hypothesis. By Lemma 3.3, H is almost-dense in L, and so L/H is divisible. Then we have $L\subset\bar{H}$. Since G is reduced, we have $\bar{H}\subset\bar{L}=L\subset\bar{H}$. Therefore G is torsion-complete by [9, Theorem 3]. □

Next, we give a necessary condition for a subgroup A of a quasi-complete but not torsion-complete group G to be summandable in G.

LEMMA 3.6. *Let G be a quasi-complete but not torsion-complete group and let A be summandable in G. Then A satisfies either of the following properties:*

(1) *$A[p]$ is discrete.*

(2) *There exists a least non-negative integer m such that $A\cap p^nG$ is almost-dense in p^nG for every $n\geq m$. Let H and K be minimal direct summands of G containing A, then m is the least integer such that $p^mH=p^mK=p^mG$.*

PROOF. Let H be a minimal direct summand of G containing A. Then we have $G=H\oplus M$ for some subgroup M of G. If H is bounded, then $A[p]$ is discrete. Hence we may assume that H is unbounded. By [7, Corollary 74.6], M is bounded and $p^mG=p^mH$ for some integer $m\geq 0$. We have $p^{m+k}G[p]=p^{m+k}H[p]\subset A+p^{m+k+1}H\subset A+p^{m+k+1}H$ for every integer $k\geq 0$ by Lemma 3.2. Hence we have $p^{m+k}G[p]\subset(A\cap p^mG)+p^{m+k+1}G$ for every $k\geq 0$ and $A\cap p^mG$ is almost-dense in p^mG. Let $G=K\oplus L$ for some subgroup L of G, and let t be the least integer such that $p^tK=p^tG$. If $t>m$, then there exists an element $x\in p^mL[p]$ such that $h_G(x)=t-1$ and $x=a+p^tg$ where $a\in A\subset K$ and $g\in G$. Then $h_G(a)=t-1$. Since $G=K\oplus L$, we have $x+(-a)\in p^{t-1}G$. This is a contradiction. Therefore $t\leq m$. If $t<m$, then $A\cap p^tG$ is almost-dense in p^tG. Similarily, this is a contradiction. Hence we have $m=t$. □

These conditions turn out to be also sufficient if A is quasi-purifiable in G. We establish the following result.

THEOREM 3.7. *Let G be a quasi-complete but not torsion-complete group and let A be quasi-purifiable in G. Then A is summandable in G if and only if A saisfies either of the following properties*:

(1) *$A[p]$ is discrete.*

(2) *$A\cap p^mG$ is almost-dense in p^mG for some integer $m\geq 0$.*

PROOF. It suffices to prove the sufficiency. If $A[p]$ is discrete, then there exists a bounded pure hull H of A in G. By [7, Theorem 27.5] H is a minimal direct summand of G containing A. Hence we may assume that $A[p]$ is non-discrete. Since A is quasi-purifiable, there exists a maximal quasi pure hull K of A in G. If we have $p^mG[p]=p^mK[p]\oplus S$ for some subsocle $S(\neq 0)$ of G, then there exists $x\in S\cap(A+p^{m+k}G)$ for some $k>0$. Then $K+\langle x\rangle$ is vertical in G. In fact, let $y\in(K+\langle x\rangle+p^nG)[p]$, then we have $y-ax\in(K+p^nG)[p]=K[p]+p^nG[p]$ for some integer a, by [3, Proposition 2.3], since K is vertical in G.

By Proposition 1.9, there exists a pure subgroup L of G such that $L[p]=K[p]\oplus\langle x\rangle$. Since A is almost-dense in L, L is a quasi pure hull of A in G. This contradicts the maximality of K. Thus $S=0$ and so $p^mG[p]=p^mK[p]\subset K$. By [1, Corollary 3.4], we have $p^mG\subset K$. Since K is pure in G and G/K is bounded, K is a direct summand of G by [7, Theorem 28.4]. Moreover, since A is almost-dense in K, K is a minimal direct summand of G containing A by Lemma 3.3. □

We conclude this section with the following example of a subgroup that is not summandable. This was constructed in the proof of [9, Theorem 2].

EXAMPLE 3.8. Let $G=\bigoplus_{i=1}^{\infty}\langle x_i\rangle$, and let $o(x_i)=p^{n(i)}$ where $n(i)$ is a strictly increasing sequence of positive integers for all $i\geq 1$. Set

$$y_i=x_{2i}+p^{n(2i+1)-n(2i)+1}x_{2i+1}-p^{n(2i+2)-n(2i)}x_{2i+2}.$$

Let $B=\bigoplus_{i=1}^{\infty}\langle y_i\rangle$ and $\overline{B}$ be the closure of B in G. Suppose that $\overline{B}$ is summandable in G. Then there exists a minimal direct summand L of G containing A. By Lemma 3.2, $\overline{B}$ is almost-dense in L. Since B is pure in G, $\overline{B}$ is maximal vertical in L by [3, Proposition 3.4] and Proposition 1.8. Since $\overline{B}$ is purifiable in L by Proposition 1.7, $\overline{B}$ is pure in G. This is a contradiction. Hence $\overline{B}$ is not summandable in G. □

4. Isomorphism of minimal direct summands

The purifiable subgroups of a direct sum of cyclic groups have isomorphic pure hulls by [4, Corollary 3. 3]. But, torsion-complete groups have non-isomorphic pure hulls with the same socle by [7, 66 Exercise 8].

In this section, we first extend the concept of residual subgroups introduced in [4], and we show that all residual subgroups of a quasi-purifiable subgroup are isomorphic. Next, we use this result to prove that all minimal direct summands of a quasi-complete group containing a summandable subgroup are isomorphic.

Let A be a quasi-purifiable subgroup of G and let H be a quasi pure hull of A in G. Let P be an overhang subsocle of A in H. Then there exists a pure subgroup R of H such that $R[p]=P$ and R is a direct sum of cyclic groups. Such a subgroup R is called a **residual subgroup** determined by a subsocle P of a quasi pure hull H.

In [8], it is shown that if K is a pure hull of a purifiable subgroup C of G, then $K=M\oplus N$ where $M[p]=C[p]$, N is a bounded subgroup, and C is almost-dense in K. Hence N is a residual subgroup determined by a subsocle $N[p]$ of a quasi pure hull K.

In [4], K. Benabdallah and T. Okuyama call such a subgroup N a residual subgroup of G determined by the purifiable subgroup C. Hence, if A is purifiable in G, their definition coincides with ours.

LEMMA 4. 1. *All residual subgroups of a quasi-purifiable subgroup A are isomorphic.*

PROOF. Let R and R' be residual subgroups determined by two overhang subsocles P and P' of quasi pure hulls H and K of A, respectively. By Proposition 1. 8, we have

$$\begin{aligned} P\cap p^nG[p]&\simeq F_n(R)\simeq V_n(H, A)\simeq V_n(G, A)\\ &\simeq V_n(K, A)\simeq F(R'_n)\simeq P'\cap p^nG[p]\end{aligned}$$

for all $n\geq 0$, where $F_n(R)$ and $F_n(R')$ are the n-th Ulm-Kaplansky invariants of R and R', respectively. Therefore, R and R' are direct sums of cyclic groups with isomorphic finite Ulm-Kaplansky invariants, and so $R\simeq R'$. □

THEOREM 4. 2. *Let A be summandable in a torsion complete group G. Then all minimal direct summands of G containing A are isomorphic.*

PROOF. Let L and M be minimal direct summands of G containing A, then A is almost-dense in L and M by Lemma 3. 2. Thus, by Proposition 1. 3 for every $n\geq 0$, we have

$$p^n L[p] = P_n \oplus A_n \oplus p^{n+1} L[p] \text{ and } p^n M[p] = Q_n \oplus A'_n \oplus p^{n+1} M[p],$$

where P_n, A_n and Q_n, A'_n are subsocles of $p^n L[p]$ and $p^n M[p]$, respectively. Put $P = \bigoplus_n P_n$ and $Q = \bigoplus_n Q_n$. By Lemma 4.1, we have $P_n \simeq Q_n$ for every $n \geq 0$. By Lemma 2.4, there exist quasi pure hulls H, K of A in L, M, respectively, such that $H[p] = A[p] \oplus P$ and $K[p] = A[p] \oplus Q$. Since $A[p] \cap p^n G = A[p] \cap p^n L = A_n \oplus (A[p] \cap p^{n+1} L) = A_n \oplus (A[p] \cap p^{n+1} G)$ and $A[p] \cap p^n G = A'_n \oplus (A[p] \cap p^{n+1} G)$, we have $A_n \simeq A'_n$ for every $n \geq 0$. On the other hand, there exist basic subgroups B, B' of L, M, respectively, such that $B[p] = P \oplus (\bigoplus_n A_n)$ and $B'[p] = Q \oplus (\bigoplus_n A'_n)$. Therefore we have $B \simeq B'$. Since L and M are torsion-complete groups, it follows that $L \simeq M$. □

THEOREM 4.3. *Let A be summandable in a quasi-complete group G. Then all minimal direct summand of G containing A are isomorphic.*

PROOF. By Theorem 4.2, we may assume that G is a quasi-complete but not torsion-complete group. If $A[p]$ is discrete, then it is immediate by [4, Corollary 3.4]. By Theorem 3.6, we may assume that $A \cap p^m G$ is almost-dense in $p^m G$ for some integer $m \geq 0$. Let H and K be minimal direct summands of G containing A, then we have $p^m H = p^m K = p^m G$ by Lemma 3.6. Since $A_G^n = A_H^n + A_n^G = A_K^n + A_n^G$ for every $n \geq 0$ by [4, Theorem 1.7], it follows that

$$\begin{aligned} p^n G[p] &= S_n \oplus A_G^n = S_n \oplus (A_H^n + A_n^G) = S_n \oplus (A_K^n + A_n^G) \\ &= S_n \oplus P_n \oplus A_n^G = S_n \oplus Q_n \oplus A_n^G \\ &= S_n \oplus P_n \oplus A_n \oplus p^{n+1} G[p] = S_n \oplus Q_n \oplus A'_n \oplus p^{n+1} G[p], \end{aligned}$$

where P_n, A_n and Q_n, A'_n are subsocles of A_H^n, A_K^n, respectively. Then it follows that

$$\begin{aligned} G[p] &= \left(\bigoplus_{i=0}^{m-1} S_i\right) \oplus \left(\bigoplus_{i=0}^{m-1} P_i\right) \oplus \left(\bigoplus_{i=0}^{m-1} A_i\right) \oplus p^m H[p] \\ &= \left(\bigoplus_{i=0}^{m-1} S_i\right) \oplus \left(\bigoplus_{i=0}^{m-1} Q_i\right) \oplus \left(\bigoplus_{i=0}^{m-1} A'_i\right) \oplus p^m K[p]. \end{aligned}$$

We show that there exists a direct summand N of G such that $N[p] = \bigoplus_{i=0}^{m-1} S_i$ and $G = N \oplus H = N \oplus K$.

There exists a direct summand N_0 of G such that $N_0[p] = S_0$. By [2, Lemma 1.5], we have $((N_0 \oplus pG)/pG)[p] = (S_0 \oplus pG)/pG$ and $(N_0 \oplus pG)/pG$ is an absolute direct summand of G/pG. Suppose that $(S_0 \oplus pG)/pG \cap ((H + pG)/pG)[p] \neq 0$. Then there exist $s \in S_0$, $h \in H$, and $pg \in pG$ such that $s = h + pg$ and $h_G(s) = h_G(h) = 0$. Since $s \in (H + pG)[p] = H[p] + pG[p]$ by verticality of H, this is a contradiction. Hence $(N_0 \oplus pG)/pG \cap$

$(H+pG)/pG=0$ and so there exists a subgroup H_0 of G such that $G/pG=(N_0\oplus pG)/pG\oplus H_0/pG$, $H_0\supset H$, and $H_0[p]=\bigoplus_{i=1}^{m-1}S_i\oplus H[p]$. Then we have $G=N_0\oplus H_0$. Applying the same process to H_0, we have $G=N_0\oplus N_1\oplus H_1$, where N_1 and H_1 are direct summands of G with $N_1[p]=S_1$, $H_1[p]=\bigoplus_{i=2}^{m-1}S_i\oplus H[p]$ and $H_1\supset H$. Therefore, by finitely many steps, we have $G=N\oplus H$ where $N=\bigoplus_{i=0}^{m-1}N_i$. Similarly, we have $G=N\oplus K$. Hence it follows that $H\simeq K$. □

Theomem 4.2 leads to the following result:

COROLLARY 4.4. *Let S be a closed subsocle of a torsion-complete group G. Then all pure subgroups supported by S are isomorphic.*

PROOF. Let H and K be pure subgroups supported by S. Then H and K are closed maximal vertical subgroups of G and so H and K are minimal direct summands of G containing S. Thus, by Theorem 4.2, $H\simeq K$. □

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