Hokkaido Mathematical Journal Vol. 23 (1994) p. 143-183

Substitution of open subhypergroups

Michael VOIT
(Received March 22, 1993)

Summary: We generalize the join of hypergroups as follows: If H is an open subhypergroup of a hypergroup K and W a compact subhypergroup of a hypergroup L such that $L/W=H$, then there is a natural hypergroup structure on the disjoint union $M:=(K-H)\cup L$. Properties of this hypergroup M are discussed, and its Haar measure and its dual space are determined. As an application we determine the conjugacy class hypergroups G^{G} as well as the dual hypergroups $\hat{G}$ of some compact groups G which are close to the commutative case.

1. Introduction

Hypergroups generalize locally compact groups. They appear when the Banach space of all bounded Radon measures on a locally compact space carries a convolution having all properties of a group convolution apart from the fact that the convolution of two point measures is a probability measure with compact support and not necessarily a point measure. Hypergroups were introduced by Dunkl [4, 5], Jewett [14], and Spector [23] to unify harmonic analysis on duals of compact groups, double coset spaces $G/\!/H$ (H a compact subgroup of a locally compact group G), and commutative convolution algebras associated with linearization formulas of special functions.

There exist several methods to construct hypergroups from given ones which are unknown for groups. These methods lead sometimes to hypergroups with particular strange properties. One method is the join as introduced by Jewett [14] and studied in [6, 7, 25, 30, 32]. The join $L\vee K$ of a compact hypergroup L and a discrete hypergroup K is formed by replacing the neutral element in K by the hypergroup L. The purpose of this paper is to generalize this method as follows: If H is an open subhypergroup of a hypergroup K, and if π is an open and proper hypergroup homomorphism from a further hypergroup L onto H, then the disjoint union of $K-H$ and L carries a natural hypergroup structure. We shall denote this hypergroup by $S(K, H\xrightarrow{\pi} L)$, and we shall say that this hypergroup is formed by substituting H by L in K. It will turn out that hypergroups formed by substitution can be characterized by a universal property and that many properties of hypergroups are preserved under substitution. In particular, duals of hypergroups formed by substitution can be described in a satisfying way.

We give some motivation for introducting a construction like substitution. First of all, there exist some compact groups G whose conjugacy class hypergroups G^G and whose dual hypergroups $\widehat{G}$ may be described by substitution; see Vrem [30] for applications of the join in this field, and Ch. VII in [11] as well as [9, 13, 16] for a general account of dual hypegroups of compact groups. A second motivation comes from the study of hypergroup structures on the one point compactification $\boldsymbol{N}\cup\{\infty\}$ of $\boldsymbol{N}$. Examples are given by orbit hypergroups which occur when the compact groups of p-adic units act on the compact groups of p-adic integers; see Dunkl and Ramirez [6]. Vrem noticed in Section 4.5 of [30] that repeated joins of finite hypergroups and then taking a projective limit lead to hypergroups on $\boldsymbol{N}\cup\{\infty\}$. We shall prove in a forthcoming paper that repeated substitutions of finite hypergroups and then taking a projective limit also lead to hypergroup structures on $\boldsymbol{N}\cup\{\infty\}$, and that in fact all hypergroup structures on $\boldsymbol{N}\cup\{\infty\}$ can be obtained in this way. Finally, substitution also appears as a natural tool when describing all hypergroups that contain a given subgroup of index 2. These remarks indicate that substitution is a suitable frame for some structural results for hypergroups.

This paper is organized as follows: In the end of Section 1 we recapitulate some basic facts about hypergroups. Section 2 then contains the construction of hypergroups formed by substitution. We also collect some basic properties preserved by substitution there. In Section 3 we shall determine the set of all irreducible representations and study positive definite functions on hypergroups formed by substitution. In particular, if the hypergroups K and L are commutative and admit dual hypergroups, it will turn out that $S(K, H \xrightarrow{\pi} L)$ also admits a dual hypergroup which may be described by substitution. In Section 4 we shall study some connections between substitution and commutative diagrams formed by hypergroups and their homomorphisms. In Section 5 we shall describe the dual hypergroups $\widehat{G}$ and conjugacy class hypergroups G^G of some compact groups via substitution. In Section 6, substitution is used to determine all hypergroups having a given subgroup of index 2. As an application, we determine all hypergroup structures on $\boldsymbol{R}\times\{0, 1\}$ and $T\times\{0, 1\}$. This classification is based on the fact that $\boldsymbol{R}$ and the complex torus T admit only the usual group structures by Zeuner [32]. Further classification results for hypergroups can be found in [3, 20, 31, 32].

We next recapitulate some basic facts; for details see Jewett [14].

1. 1. Some notation. Let K be a locally compact (Hausdorff) space. By $M(K)$, $M_b(K)$, $M_b^+(K)$, and $M^1(K)$ we denote all Radon measures, the bounded ones, those that are bounded and nonnegative, and the probability measures on K respectively. The spaces $C_b(K)$, $C_0(K)$ and $C_c(K)$ consist of the $\boldsymbol{C}$-valued continuous bounded functions on K, those that are continuous and zero at infinity, and those that are continuous and compactly supported respectively. $\delta_x \in M^1(K)$ is the point measure at $x \in K$. We consider two topologies on subspaces of $M(K)$, namely the vague topology $\sigma(M(K), C_c(K))$ as well as the weak topology $\sigma(M_b(K), C_b(K))$. Both topologies agree on $M^1(K)$.

We also mention the Michael topology on the space $\mathscr{C}(K)$ of all nonvoid compact subsets of K. This topology is generated by the subbasis of sets $U_{V,W} := \{L \in \mathscr{C}(K) : L \cap V \neq \emptyset,\ L \subset W\}$ where V and W run through the open subsets of K; see Michael [17] and Jewett [14]. We always assume that $\mathscr{C}(K)$ carries this topology.

Let K and L be locally compact spaces and $p : K \to L$ a continuous mapping. The associated mapping from $M_b(K)$ to $M_b(L)$ is denoted by p again. If we take $\mu \in M_b(L)$ and a weakly continuous mapping $L \to M_b^+(K)$, $x \mapsto q_x$, then $\int_L q_x \, d\mu(x)$ stands for the unique $\rho \in M_b(K)$ such that $\int_K f \, d\rho = \int_L \int_K f(y) \, dq_x(y) d\mu(x)$ for all $f \in C_b(K)$.

1.2. Hypergroups. Let K be a locally compact Hausdorff space and $*$ a bilinear, associative mapping on $M_b(K)$. $(K, *)$ (or, for short, K) is called a hypergroup, if the following conditions are satisfied:

(1) $\delta_x * \delta_y$ is a probability measure with compact support for all $x, y \in K$.

(2) The mapping $K \times K \longrightarrow M^1(K)$, $(x, y) \mapsto \delta_x * \delta_y$, is (weakly) continuous.

(3) The mapping $K \times K \longrightarrow \mathscr{C}(K)$, $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$, is continuous.

(4) There exists an identity element $e \in K$ satisfying $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$, and there is a continuous involution $x \mapsto \bar{x}$ on K such that $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$ and that $e \in \operatorname{supp}(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$ for all $x, y \in K$.

K is called commutative if the convolution $*$ is commutative and symmetric if the hypergroup involution is the identity mapping.

Obviously, each symmetric hypergroup is commutative.

Assume from now on that K is a hypergroup.

1.3. Subhypergroups and convolution of subsets. For $A, B\in K$ we define $A*B:=\bigcup_{x\in A,y\in B}\operatorname{supp}(\delta_x*\delta_y)$. If $x\in K$, then we write $x*A$ or $A*x$ instead of $\{x\}*A$ or $A*\{x\}$ respectively. A closed nonvoid subset H of K is called a subhypergroup if $H*H=H=\bar{H}$ where $\bar{H}:=\{x\in K:\bar{x}\in H\}$. A subhypergroup is said to be normal if $x*H=H*x$ for all $x\in K$.

1.4. The Haar measure. For $f\in C(K)$ and $x\in K$ we use the notation $f(x*y):={}_xf(y):=\int_K f(z)\,d(\delta_x*\delta_y)(z)$. In this case, ${}_xf\in C(K)$ holds. A nontrivial positive Radon measure ω on K is said to be a left Haar measure if $\omega(f)=\omega({}_xf)$ for all $x\in K$ and $f\in C_c(K)$. A left Haar measure is essentially unique (Jewett [14]), but it is unknown whether each hypergroup admits a left Haar measure. The existence of a left Haar measure is proved only for the cases of commutative, compact, and discrete hypergroups (Spector [23] and Jewett [14]). Analogous results hold for right Haar measures. If K is commutative, compact or discrete, then a left Haar measure is also a right Haar measure. In the latter case, ω is called a Haar measure and K unimodular. If K is compact, then ω is assumed to be normalized by $\omega(K)=1$.

1.5. Hypergroup homomorphisms. Let $(K,*)$ and $(J,\bullet)$ be hypergroups. A continuous mapping $p:K\to J$ is said to be a hypergroup homomorphism if

$$p(e_K)=e_J \quad\text{and}\quad \delta_{p(x)}\bullet\delta_{p(y)}=p(\delta_x*\delta_y) \quad \text{for all } x, y\in K \tag{1.1}$$

(e_J and e_K being the identity elements of J and K). For each $x\in K$, we then have $e_J\in\{p(x)\}\bullet\{p(\bar{x})\}$ and thus $p(\bar{x})=\overline{p(x)}$. p is said to be hypergroup isomorphism if it is also a homeomorphism. We next recall the obvious relation between homomorphisms and quotients; cf. Theorem 1.6 in [26].

1.6. THEOREM. *Let $p:K\to J$ be an open and surjective hypergroup homomorphism. Then $H:=p^{-1}(e)$ is a normal subhypergroup in K, $K/H:=\{x*H:x\in K\}$ is a locally compact space with respect to the quotient topology, and*

$$\delta_{x*H}*\delta_{y*H}:=\int_K \delta_{z*H}\,d(\delta_x*\delta_y)(z) \quad (x, y\in K) \tag{1.2}$$

defines a hypergroup on K/H being isomorphic with J. Conversely, if H is a normal subhypergroup of K such that Eq. (1.2) *defines a hypergroup on*

K/H, then $\pi : K \to K/H$, $x \mapsto x*H$, is an open hypergroup homomorphism. This is, in particular, true if H is a compact normal subhypergroup.

There exist subhypergroups H of (commutative) hypergroups K such that K/H fails to bear a well-defined quotient convolution (1. 2); see [26, 33]. The following fact will be needed below (see Section 14.2 in [14] and Lemma 1.7 in [28]):

1.7 Let H be a compact normal subhypergroup of K with normalized Haar measure ω_H. If $\pi : K \to K/H$ is the natural homomorphism, then

$$\pi : M_b(K|H) := \{\mu \in M_b(K) : \omega_H * \mu = \mu\} \to M_b(K/H),\ \mu \mapsto \pi(\mu)$$

is an isometric isomorphism of Banach algebras, and the mapping $\tilde{\pi} : M_b(K|H) \cap M_b^+(K) \to M_b^+(K/H)$ is a homeomorphism with respect to the weak topology.

1.8. Orbital morphisms (see Jewett [14] and Voit [27]). Let J and K be hypergroups with identities e_J and e_K. A continuous, proper, surjective, and open mapping $\Phi : J \to K$ is called an orbital mapping. Φ is said to be unary if $\Phi^{-1}(e_K) = \{e_J\}$.

A recomposition of Φ is a weakly continuous mapping $x \mapsto q_x$ from K to $M^1(J)$ with $\operatorname{supp} q_x = \Phi^{-1}(x)$ for all $x \in K$. Φ is a generalized orbital morphism associated with the recomposition $(q_x)_{x\in K}$ if $q_{\bar{x}} = q_x^{-}$ and $\Phi(q_x * q_y) = \delta_x * \delta_y$ for all $x, y \in K$.

If there exists a measure $l \in M^+(J)$ such that $l = \int_J q_{\Phi(y)}\, dl(y)$, then this recomposition is said to be consistent with l. If $(q_x)_{x\in K}$ is consistent with the Haar measure ω on J, then the generalized orbital morphism Φ is called an orbital morphism.

Let Φ be a generalized orbital morphism associated with the recomposition $(q_x)_{x\in K}$. If $M := \{\mu \in M_b(J) : \mu = \int_K q_y\, d\nu(y),\ \nu \in M_b(K)\}$ is closed under convolution (i.e., M is a Banach-$*$-subalgebra of $M_b(J)$), then Φ is called consistent. Obviously, each injective consistent generalized orbital morphism is a hypergroup isomorphism.

2. Substitution of open subhypergroups.

We here generalize the join of hypergroups (Section 10.5 of Jewett [14]) as follows: We replace an open subhypergroup H of a hypergroup K by a hypergroup L which is related to H via a proper surjective hypergroup homomorphism $\pi : L \to H$. We show that the resulting space becomes a hypergroup in a natural way, and we characterize this hypergroup by a universal property.

2.1. THEOREM AND DEFINITION. *Let $(K, *)$ and $(L, \bullet)$ be hypergroups. Let $\pi : L \to K$ be a proper and open hypergroup homomorphism (which is not necessarily surjective) and $W := \operatorname{kern} \pi$ the associated compact normal subhypergroup of L. Then there exists a hypergroup $(M, \diamond)$ having following properties:*

(i) *There exists an injective and open hypergroup homomorphism $\tau : L \to M$ and a proper, surjective, and open hypergroup homomorphism $p : M \to K$ such that $p \circ \tau = \pi$ and $\operatorname{kern} p = \tau(W)$.*

(ii) *If there exists a further hypergroup $\tilde{M}$, an injective and open hypergroup homomorphism $\tilde{\tau} : L \to \tilde{M}$, and a proper, surjective hypergroup homomorphism $\tilde{p} : \tilde{M} \to K$ such that $\tilde{p} \circ \tilde{\tau} = \pi$ and $\operatorname{kern} \tilde{p} = \tilde{\tau}(W)$, then there exists a unary consistent generalized orbital morphism $\varphi : \tilde{M} \to M$ with $\tilde{p} = p \circ \varphi$.*

M is determined uniquely by (i) and (ii) up to isomorphism. We say that M is obtained from K by substituting the open subhypergroup $H := \pi(L)$ of K by L via π. M will be denoted by $S(K, H \xrightarrow{\pi} L)$ where the π is omitted if there is no possible confusion. $(M, \diamond)$ can be realized as follows: Take $M := (K-H) \cup L$ as the disjoint union of $K-H$ and L such that both sets are embedded into M as open sets. Then, $\diamond$ is given by

$$\delta_x \diamond \delta_y := \begin{cases} \delta_x \bullet \delta_y & \text{for } x, y \in L \subset M \\ \delta_{\pi(x)} * \delta_y & \text{for } x \in L \text{ and } y \in M-L=K-H \\ \delta_x * \delta_{\pi(y)} & \text{for } y \in L \text{ and } x \in M-L=K-H \\ ((\delta_x * \delta_y)|_{M-L} + \tilde{\pi}^{-1}((\delta_x * \delta_y)|_H) & \text{for } x, y \in M-L \end{cases} \tag{2.1}$$

where $\tilde{\pi} : M_b(L|W) \to M_b(L)$ is the isometric Banach-$*$-algebra isomorphism associated with π as in Section 1.7. The identity of M agrees with the identity of L, and the involution $^-$ on M is inherited from the involutions on $M-L=K-H$ and L.

PROOF. The proof of the theorem will be divided into two major parts.

STEP 1: $(M, \diamond)$ as given above is a hypergroup. For this, we first note that the hypergroup axiom concerning the identity element and the involution is obviously true for $\diamond$. Moreover, $\delta_x \diamond \delta_y$ is a probability measure with compact support for all $x, y \in M$.

We next check that

$$\sigma:\ M_b^+(M)\times M_b^+(M)\longrightarrow M_b^+(M),\ (\mu,\nu)\mapsto \mu\diamond\nu$$

is weakly continuous. As L and $M-L$ are open in M, the mappings $\mu\mapsto \mu|_L$ and $\mu\mapsto\mu|_{M-L}$ from $M_b^+(M)$ onto $M_b^+(L)$ and $M_b^+(M-L)$ respectively are weakly continuous. Consequently, using this decomposition of positive measures and the weak continuity of the addition of measures in $M_b^+(M)$, we can restrict our attention to the 4 possible Cartesian products of the spaces $M_b^+(L)$ and $M_b^+(M-L)$. As the canonical projection $\pi: M_b^+(L)\to M_b^+(L/W)=M_b^+(H)$ is weakly continuous, the continuity of σ restricted to $M_b^+(L)\times M_b^+(L)$, $M_b^+(L)\times M_b^+(M-L)$ and $M_b^+(M-L)\times M_b^+(L)$ is a consequence of the continuity properties of $*$ and $\bullet$. It remains to consider σ on $M_b^+(M-L)\times M_b^+(M-L)$. As $M-L$ is open and closed in M and as H is open and closed in K, the mappings $M_b^+(M-L)\times M_b^+(M-L)\to M_b^+(M)$, $(\mu,\nu)\mapsto(\mu*\nu)|_{M-L}$ and $M_b^+(M-L)\times M_b^+(M-L)\to M_b^+(H)$, $(\mu,\nu)\mapsto(\mu*\nu)|_H$ are weakly continuous. Hence, by Lemma 1.7 of Voit [28],

$$\begin{aligned}&M_b^+(M-L)\times M_b^+(M-L)\to\\ &\qquad M_b^+(L)\subset M_b^+(M),\ (\mu,\nu)\mapsto\omega_W\bullet\pi^{-1}((\mu*\nu)|_H)\end{aligned}$$

is weakly continuous which completes the proof of the continuity of σ.

We next prove the continuity of

$$\tau: M\times M\longrightarrow \mathscr{C}(M),\ (x,y)\mapsto \operatorname{supp}(\delta_x\diamond\delta_y).$$

It suffices to check this on the sets $L\times L$, $(M-L)\times L$, $L\times(M-L)$ and $(M-L)\times(M-L)$ separately. This check is trivial for the first 3 cases. Moreover, as the cosets $x*H$ $(x\in K-H)$ are open ([14], Lemma 4.1D) and cover $M-L=K-H$, we restrict our attention to subsets of the form $(x*H)\times(y*H)$ $(x,y\in K-H)$. If $x*H\neq\bar{y}*H$, then $(x*H)\cap(\bar{y}*H)=\emptyset$ ([14], Lemma 10.3A) and $(\{y\}*\{x\})\cap H=\emptyset$ ([14], Lemma 4.1B). Thus, for $u\in x*H$, $v\in\bar{y}*H$, and $x*H\neq y*H$, we have $\delta_u\diamond\delta_v=\delta_u*\delta_v$ which yields that τ is continuous on $(x*H)\times(y*H)$ for $x*H\neq\bar{y}*H$. We have still to study τ on $P_x:=(x*H)\times(\bar{x}*H)$ for an arbitrary $x\in K-H$. To do this, fix $x\in K-H$. As H is open and closed in K, the sets $W_1:=\{(u,v)\in P_x: \{u\}*\{v\}\subset H\}$ and $W_2:=\{(u,v)\in P_x:(\{u\}*\{v\})\cap(K-H)\neq\emptyset\}$ are open, disjoint and satisfy $W_1\cup W_2=P_x$. Thus it suffices to prove the continuity on W_1 and W_2 separately. As τ is defined on W_1 by $\tau(u,v)=\pi^{-1}(\{u\}*\{v\})\in\mathscr{C}(L)$ and as $\pi^{-1}:\mathscr{C}(H)\to\mathscr{C}(L)$ is continuous (Michael [17], 5.10.2 and 5.10.3; note that π is open and closed), τ is continuous on W_1. Moreover, as the operations $\cup:\mathscr{C}(M)\times\mathscr{C}(M)\to\mathscr{C}(M)$, $(R,S)\mapsto R\cup S$ and intersection of a compact set with a fixed closed set are continuous, the methods used for W_1 above ensure that τ is continuous on W_2. This completes the proof of τ being continuous.

To complete the proof of $(M, \diamond)$ being a hypergroup, we still have to check that $\diamond$ is associative. Restricted to L, $\diamond$ is obviously associative. We next take $x,y,z \in M-L$. As $\operatorname{supp}(\delta_u * \delta_v) \cap H = \emptyset$ for all $u \in K-H = M-L$ and $v \in H$, we see that

$$\begin{aligned}
\delta_x \diamond (\delta_y \diamond \delta_z) &= \delta_x \diamond ((\delta_y * \delta_z)|_{M-L}) + \delta_x \diamond (\omega_W \bullet \pi^{-1}((\delta_y * \delta_z)|_H)) \\
&= \int \Big[(\delta_x * \delta_u)|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_u)|_H) \Big] d((\delta_y * \delta_z)|_{M-L})(u) + \delta_x * ((\delta_y * \delta_z)|_H) \\
&= (\delta_x * ((\delta_y * \delta_z)|_{M-L}))|_{M-1} + \omega_W \bullet \pi^{-1}((\delta_x * ((\delta_y * \delta_z)|_{M-L}))|_H) \\
&\qquad + (\delta_x * ((\delta_y * \delta_z)|_H))|_{M-L} \\
&= (\delta_x * \delta_y * \delta_z)|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_y * \delta_z)|_H) = \ldots = (\delta_x \diamond \delta_y) \diamond \delta_z. \qquad (2.2)
\end{aligned}$$

Taking $x, y \in L$ and $z \in M-L$, we have

$$\delta_x \diamond (\delta_y \diamond \delta_z) = \delta_{\pi(x)} * \delta_{\pi(y)} * \delta_z = \pi(\delta_x \bullet \delta_y) * \delta_z = (\delta_x \bullet \delta_y) \diamond \delta_z = (\delta_x \diamond \delta_y) \diamond \delta_z. \qquad (2.3)$$

Next, if $x, z \in L$ and $y \in M-L$, then we obtain

$$\delta_x \diamond (\delta_y \diamond \delta_z) = \delta_x \diamond (\delta_y * \delta_{\pi(z)}) = \delta_{\pi(x)} * \delta_y * \delta_{\pi(z)} = \ldots = (\delta_x \diamond \delta_y) \diamond \delta_z. \qquad (2.4)$$

Taking $x, y \in M-L$ and $z \in L$, we observe that

$$\begin{aligned}
\delta_x \diamond (\delta_y \diamond \delta_z) &= \delta_x \diamond (\delta_y * \delta_{\pi(z)}) \\
&= \int \Big[(\delta_x * \delta_u)|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_u)|_H) \Big] d((\delta_y * \delta_{\pi(z)})(u) \\
&= (\delta_x * \delta_y * \delta_{\pi(z)})|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_y * \delta_{\pi(z)})|_H) \\
&= ((\delta_x * \delta_y)|_{M-L}) * \delta_{\pi(z)} + \omega_W * \pi^{-1}((\delta_x * \delta_y)|_H * \delta_{\pi(z)}) \\
&= \int \delta_u * \delta_{\pi(z)} \, d((\delta_x * \delta_y)|_{M-L})(u) + \omega_W \bullet \pi^{-1}((\delta_x * \delta_y)|_H) \bullet \delta_z) \\
&= ((\delta_x * \delta_y)|_{M-L}) \diamond \delta_z + (\omega_W \bullet \pi^{-1}((\delta_x * \delta_y)|_H)) \diamond \delta_z) = (\delta_x \diamond \delta_y) \diamond \delta_z. \qquad (2.5)
\end{aligned}$$

Finally, taking $x, z \in M-L$ and $y \in L$, we see that

$$\begin{aligned}
\delta_x \diamond (\delta_y \diamond \delta_z) &= \delta_x \diamond (\delta_{\pi(y)} * \delta_z) \\
&= \int \Big[(\delta_x * \delta_u)|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_u)|_H \Big] d((\delta_{\pi(y)} * \delta_z)(u) \\
&= (\delta_x * \delta_{\pi(y)} * \delta_z)|_{M-L} + \omega_W \bullet \pi^{-1}((\delta_x * \delta_{\pi(y)} * \delta_z)|_H) = \ldots = (\delta_x \diamond \delta_y) \diamond \delta_z. \qquad (2.6)
\end{aligned}$$

The remaining two cases are symmetric to the cases (2.3) and (2.5) and will be omitted. Therefore the proof of $\diamond$ being associative is finished.

STEP 2: $(M, \diamond)$ has the properties (i) and (ii). To check (i) we take $\tau : L \to M$ as the identity mapping. $W \subset L \subset M$ is a compact normal subhypergroup of M by the construction of M, and M/W can be identified with K. We define $p: M \to M/W=K$ as the canonical homomorphism, i.e.

$$p(x) := \begin{cases} \pi(x) & \text{if } x \in L \\ x & \text{if } x \in M-L. \end{cases} \tag{2.7}$$

p is proper, surjective, and open by Theorem 1.6.

To check (ii), we take $\tilde{M}$, $\tilde{p}$ and $\tilde{\tau}$ an assumed in (ii). We define $\varphi: \tilde{M} \to M$ by

$$\varphi(x) = \begin{cases} \tilde{\tau}^{-1}(x) \in L \subset M & \text{if } x \in \tilde{\tau}(L) \\ \tilde{p}(x) \in K-H \subset M & \text{if } x \in \tilde{M} - \tilde{\tau}(L) \end{cases} \tag{2.8}$$

where we have used in the second case that $\tilde{p}^{-1}(H) = \tilde{\tau}(L)$ and thus $\tilde{p}(\tilde{M} - \tilde{\tau}(L)) \subset K-H \subset M$. $\tilde{p}$ is a proper surjective homomorphism and hence open (Proposition 1.7 of [26]). Moreover, $\tilde{\tau}(L)$ is an open (and closed) subhypergroup of $\tilde{M}$. These facts and the assumptions imply that φ is continuous and open. Obviously, φ is unary, surjective, and proper. To verify that φ is a consistent generalized orbital morphism, we define an associated recomposition $(q_z)_{z \in M} \subset M^1(\tilde{M})$. If $\omega_{\text{kern}\, \tilde{p}}$ is the normalized Haar measure of kern $\tilde{p} \subset \tilde{M}$, and if $\star$ is the convolution on $\tilde{M}$, then we define

$$q_z := \begin{cases} \delta_{\tilde{\tau}^{-1}(z)} & \text{if } z \in L \subset M \\ \delta_{\tilde{z}} \star \omega_{\text{kern}\, \tilde{p}} & \text{if } z \in M-L \text{ and } \varphi(\tilde{z}) = z \end{cases} \tag{2.9}$$

where in the second case the definition is independent of the choice of $\tilde{z}$ (cf. Section 1.7). Taking $z_1, z_2 \in M-L$ and $\tilde{z}_1, \tilde{z}_2 \in \tilde{M} - \tilde{\tau}^{-1}(L)$ such that $\varphi(\tilde{z}_i) = z_i$, we have

$$\begin{aligned} \varphi(q_{z_1} \star q_{z_2}) &= \varphi(\delta_{\tilde{z}_1} \star \omega_{\text{kern}\, \tilde{p}} \star \delta_{\tilde{z}_2} \star \omega_{\text{kern}\, \tilde{p}}) \\ &= \varphi((\delta_{\tilde{z}_1} \star \delta_{\tilde{z}_2} \star \omega_{\text{kern}\, \tilde{p}})|_{\tilde{\tau}(L)}) + \varphi((\delta_{\tilde{z}_1} \star \delta_{\tilde{z}_2} \star \omega_{\text{kern}\, \tilde{p}})|_{\tilde{M} - \tilde{\tau}(L)}) = \delta_{z_1} \diamond \delta_{z_2}. \end{aligned} \tag{2.10}$$

The same methods yield $\varphi(q_{z_1} \star q_{z_2}) = \delta_{z_1} \diamond \delta_{z_2}$ for $z_1, z_2 \in M$. As $\{\mu \in M_b(M) : \mu = \int_{\tilde{M}} q_z \, d\nu(z),\ \nu \in M_b(\tilde{M})\}$ is closed under convolution by (2.9), $(q_z)_{z \in M}$ is a recomposition of the consistent generalized orbital morphism φ. This completes the proof of (ii)

To show that M is determined uniquely by (i) and (ii), we assume that $\tilde{M}$, $\tilde{\tau}$ and $\tilde{p}$ satisfy (i) and (ii). By (ii), we find a generalized orbital morphism $\tilde{\varphi}: M \to \tilde{M}$ with $p = \tilde{p} \circ \varphi$. As $p|_{M-L}$ is injective, the commutative diagram

$$\begin{tikzcd} M \arrow[rr, bend left, "\tilde{\varphi}"] \arrow[dr, "p"'] & L \arrow[l, hook'] \arrow[r, hook, "\tilde{\tau}"] \arrow[d, "\pi"] & \tilde{M} \arrow[dl, "\tilde{p}"] \\ & K & \end{tikzcd} \tag{2.11}$$

shows that $\tilde{\varphi}|_L$ and $\tilde{\varphi}|_{M-L}$ are injective. As $\tilde{\varphi}(L)\cap\tilde{\varphi}(M-L)=\emptyset$, and as an injective generalized orbital morphism is an isomorphism, the proof of Theorem 2.1 is complete.

2.2. The hypergroup join. The join $L\vee K$ of a compact hypergroup L and a discrete hypergroup K appears when the identity element of K is replaced by L. Hence, if $\pi: L\to\{e\}$ is the trivial homomorphism, then we have $L\vee K=S(K,\{e\}\stackrel{\pi}{\to}L)$.

We next present a typical example of a hypergroup constructed by substitution.

2.3. EXAMPLE. For $n\in\boldsymbol{N}$ let $\boldsymbol{Z}_n$ be the cyclic group of order n. The commutative hypergroup $K=S(\boldsymbol{Z}_4, \boldsymbol{Z}_2\to\boldsymbol{Z}_2\times\boldsymbol{Z}_2)$ consists of six elements which we name e, x, y, z, a, b where e is the neutral element and $\{e, x, y, z\}$ is the subgroup isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. In this case, the convolution on K is given by

$*$	e	x	y	z	a	b
e	δ_e	δ_x	δ_y	δ_z	δ_a	δ_b
x	δ_x	δ_e	δ_z	δ_y	δ_a	δ_b
y	δ_y	δ_z	δ_e	δ_x	δ_b	δ_a
z	δ_z	δ_y	δ_x	δ_e	δ_b	δ_a
a	δ_a	δ_a	δ_b	δ_b	$(\delta_z+\delta_y)/2$	$(\delta_e+\delta_x)/2$
b	δ_b	δ_b	δ_a	δ_a	$(\delta_e+\delta_x)/2$	$(\delta_z+\delta_y)/2$

It is natural to ask whether a given hypergroup is isomorphic to a hypergroup constructed via a non-trivial substitution. We here give the following criterion:

2.4. PROPOSITION. *Let $W\neq\{e\}$ be a compact normal subhypergroup of a hypergroup M. Then $T:=\{x\in M: x*W\neq(x)\}$ is open in M. Moreover, if L is a subhypergroup of K satisfying $L\supset T$, then L is open in M, and M is isomorphic to $S(M/W, L/W\to L)$. In particular, if L can be* *chosen to be equal to* W, *then* M *is isomorphic to* $W \vee (M/W)$.

PROOF. Let $(y_\alpha)_{\alpha\in A} \subset M-T$ be a net converging to $y\in M$. Then the net of compact subsets $(y_\alpha * W)_{\alpha\in A}=(\{y_\alpha\})_{\alpha\in A}$ tends in $\mathscr{C}(K)$ to $y*W$ and $\{y\}$ at the same time. Hence, $y*W=\{y\}$ and $y\in M-T$. This proves that T is open in M.

It follows from $L\supset T$ that L is open in M. Hence, $\tilde{M}:=S(M/W, L/W\rightarrow L)$ is a well-defined hypergroup. Applying part (ii) of Theorem 2.1 and Eq. (2.8), we find a consistent generalized orbital morphism $\varphi: M\rightarrow \tilde{M}$ which is given by

$$\varphi(x)=\begin{cases} x*W=\{x\} & \text{for } x\in M-L \\ x & \text{for } x\in L \end{cases}$$

As φ is injective, φ is a hypergroup isomorphism. This completes the proof.

2.5. REMARKS. All assumptions of Theorem 2.1 are in fact necessary in order to construct $(M, \diamond)$. For instance, the continuity properties of $\diamond$ follow from H being open in K. Furthermore, the construction of $\diamond$ depends on the fact that π is open and proper. In particular, the assumption of $\operatorname{kern}\pi$ being compact is necessary since otherwise each coset $x*\operatorname{kern}\pi$ would be non-compact in M. Finally, the assumption of π being a hypergroup homomorphism was needed to ensure that $\diamond$ is associative.

On the other hand, the property of π being a homomorphism has been used mainly to verify Eq. (2.3) and its symmetric counterpart. This observation has the following consequence: Let $\pi: L\rightarrow H$ be an orbital mapping and T a closed $*$-subalgebra of $M_b(L)$ such that $\pi: T\rightarrow M_b(H)$ is a Banach-$*$-algebra isomorphism. If

$$\delta_{\pi(x)}*\delta_{\pi(y)}*\delta_z=\pi(\delta_x\bullet\delta_y)*\delta_z \quad\text{and}\quad \delta_z*\delta_{\pi(x)}*\delta_{\pi(y)}=\delta_z*\pi(\delta_x\bullet\delta_y) \qquad (2.12)$$

for all $x, y\in L$ and $z\in M-L$, then we can construct a hypergroup $(M, \diamond)$ as in Theorem 2.1 by using the isomorphism between $M_b(H)$ and T (instead of $M_b(L|\operatorname{kern}\pi)$).

Using double coset hypergroups, we give a simple example for this setting: Let J and L be hypergroups. Let R and W be compact subhypergroups of L such that R is normal in L, that W is non-normal in L, and that $W\subset R$ holds. Then $R/\!/W$ is a normal subhypergroup of $L/\!/W$, and $L/R\simeq(L/\!/W)/(R/\!/W)$ holds (Theorem 14.3A of Jewett [14]). Assume that L/R is an open subhypergroup of J. Then, by Theorem 2.1, the hypergroups $K:=S(J, L/R\rightarrow L/\!/W)$ and $M:=\mathrm{S}(J, L/R\rightarrow L)$ exist. M can be regarded as the hypergroup that appears if $L/\!/W$ is replaced by

L in K.

We do not know whether every substitution not being associated with a hypergroup homomorphism can be reduced to the usual substitution in this obvious way. However, if L is a group, then we have the following result.

2.6. LEMMA. *Let W be a compact subgroup of a locally compact group L such that the hypergroup $L /\!/ W$ is an open subhypergroup of a further hypergroup K. If the hypergroup $M := S(K, L/\!/W \to L)$ exists in the above way, then there exists a compact normal subgroup R of L with $R \supset W$ such that the hypergroup $J := K/(R/\!/W)$ contains L/R as an open subgroup in the obvious way, and such that we have $K = S(J, L/R \to L/\!/W)$ and $M = S(J, L/R \to L)$.*

PROOF. Let R be the smallest closed normal subgroup of L containing W. The convolution on $S(K, L/\!/W \to L)$ yields that for $c \in K-(L/\!/W) \subset M$ and $x \in L \subset M$ the relation $\{c\} = (x * x^{-1}) * c = (WxW) * (Wx^{-1}) * c$ holds. As the subhypergroup generated by the elements $(WxW) * (Wx^{-1}W)$ $(x \in L)$ of $L/\!/W$ is equal to $R/\!/W$, we conclude that $R * \{c\} = \{c\}$ for all $c \in K-(L/\!/W) \subset M$, and that, in particular, R is compact. As L/R is isomorphic to the subgroup $(L/\!/W)/(R/\!/W)$ of $K/(R/\!/W)$, the remaining assertions of the lemma follow from Proposition 2.4.

We next collect some properties which are preserved under substitution.

2.7. PROPOSITION. *In the setting of Section 2.1 the following statements hold for $M = S(K, H \xrightarrow{\pi} L)$:*

(1) *M admits a left Haar measure if and only if K does. In particular, for every left Haar measure ω_K on K there exists a left Haar measure ω_L on L such that $\pi(\omega_L) = \omega_K|_H$. Then $\omega_M := \omega_K|_{K-H} + \omega_L$ is a left Haar measure on M. The corresponding results are true for right Haar measures.*
(2) *M is unimodular or compact if and only if so is K.*
(3) *M is commutative, symmetric, discrete or totally disconnected if and only if K and L have the same property.*
(4) *M is amenable (i.e., M admits an invariant mean; see Skantharajah [22]) if and only if K is amenable.*
(5) *If K admits a left Haar measure ω_K, then M as well as the hypergroup $\tilde{M}$ of part (ii) of Theorem 2.1 have Haar measures, and the mapping $\varphi : \tilde{M} \to M$ is a consistent orbital morphism.*

PROOF. Theorem 1.8 of Voit [26] states that if N is a normal compact subhypergroup of a hypergroup R, then R admits a (left or right) Haar measure if and only if the hypergroup R/N does. This theorem also gives a natural relation beetween both Haar measures. Therefore, the first statement of part (1) is obvious. Moreover, if ω_K is a left Haar measure on K and ω_M is the associated Haar measure on M according to Theorem 1.8 of [26], then $\omega_L := \omega_M|_L$ is a left Haar measure on the open subhypergroup L of M and satisfies $\pi(\omega_L) = \omega_K|_H$. The equation $\omega_M := \omega_K|_{K-H} + \omega_L$ now follows from the structure of M and the properties of ω_M.

The remaining assertions of part (1) as well as parts (2) and (3) are obvious.

As $\operatorname{kern}\pi \subset L \subset M$ is compact and $M/\operatorname{kern}\pi \simeq K$, part (4) follows from Proposition 3.6 of Skantharajah [22].

To verify part (5), we first notice that $\tilde{M}$ as well as M admit Haar measures $\omega_{\tilde{M}}$ and ω_M by Theorem 1.8 of Voit [26]. We have to check that the recomposition $(q_z)_{z\in M} \subset M^1(\tilde{M})$ (see Eq. (2.9)) of the generalized orbital morphism $\varphi : \tilde{M} \to M$ is consistent with $\omega_{\tilde{M}}$, i.e. that we have $\omega_{\tilde{M}} = \int_{\tilde{M}} q_{\varphi(x)}\, d\omega_{\tilde{M}}(x)$. By the definition of q_z, this equation is obviously true on the set $\tilde{\tau}(L) \subset \tilde{M}$. Moreover, it follows from (2.8), (2.9), and Theorem 1.8 of Voit [26] that

$$\left(\int_{\tilde{M}} q_{\varphi(x)}\, d\omega_{\tilde{M}}(x)\right)\Big|_{\tilde{M}-\tilde{\tau}(L)} = \int_{\tilde{M}-\tilde{\tau}(L)} q_{\varphi(x)} d\omega_{\tilde{M}}(x)$$
$$= \int_{\tilde{M}-\tilde{\tau}(L)} \delta_x * \omega_{\operatorname{kern}\tilde{p}}\, d\omega_{\tilde{M}}(x) = \omega_{\tilde{M}}|_{\tilde{M}-\tilde{\tau}(L)} \tag{2.13}$$

where $\tilde{p} : \tilde{M} \to K$ is given as in part (ii) of Theorem 2.1. Thus the proof is complete.

2.8. Next we list some obvious isomorphisms for hypergroups constructed by substitution. The proofs are straightforward and will be omitted.

(1) $S(K\times J, H\times J \xrightarrow{\tilde{\pi}} L\times J) \simeq S(K, H \xrightarrow{\pi} L)\times J$, if H is an open subhypergroup of K, J is compact, and the proper homorphisms $\pi : L \to H$ and $\tilde{\pi} : L\times J \to H\times J$ are connected by $\tilde{\pi}(x, y) := (\pi(x), y)$ $(x\in L, y\in J)$.
(2) $S(K, H \to H\vee W) \simeq W\vee K$, if K is discrete and W compact.
(3) $S(K, H \to S(H, W \to L)) \simeq S(K, W \to L)$, if W (and thus H) is open in K.
(4) $S(H\vee K, H \to L) \simeq L\vee K$, if K is discrete and H and L are compact.

3. Duals of hypergroups constructed by substitution

The purpose of this section is to discuss dual spaces of hypergroups constructed via substitution. For this we first recapitulate some basic facts about irreducible representations of hypergroups and in particular duals of commutative hypergroups. For details we refer to Bloom and Heyer [1, 2], Jewett [14] and Vrem [29].

3. 1. Duals of hypergroups. Let K be a hypergroup. Let H be a Hilbert space, $B(H)$ the algebra of all bounded linear operators on H, and $T: M_b(K) \to B(H)$ a $*$-representation of the Banach-$*$-algebra $M_b(K)$ such that
(i) $T(\delta_e)$ is equal to the identity operator I,
(ii) $\|T(\mu)\| \leq \|\mu\|$ for all $\mu \in M_b(K)$, and
(iii) the mapping $\mu \mapsto \langle T(\mu)u, v\rangle$ is weakly continuous on $M_b^+(K)$ for all $u, v \in H$. Then the mapping

$$K \to B(H),\ x \mapsto T(\delta_x)$$

is denoted by T again. T is said to be a representation of K. A representation of K is called irreducible if the associated representation of $M_b(K)$ is irreducible. Using the usual concept of unitary equivalence, we denote the set of all equivalence classes of irreducible representations of K by $\hat{K}$.

The annihilator $A(\hat{K}, L)$ of a subset L of K is given by
$A(\hat{K},\ L) := \{T \in \hat{K}: \quad T(x) = I$ for all $x \in L\}$, I being the identity operator.

3. 1. 1. The following facts about annihilators are needed below (see Lemma 1. 9 in Voit [28]): Let L be a compact normal subhypergroup of a hypergroup K. If ω_L is the normalized Haar measure of L, then $T(\omega_L) = 0$ for all $T \in \hat{K} - A(\hat{K},\ L)$. Moreover, if $p: K \to K/L$ is the natural projection, then $T \mapsto T \circ p$ is a bijective mapping from $(K/L)^\wedge$ onto $A(\hat{K}, L)$.

3. 2. Duals of commutative hypergroups. Let K be a commutative hypergroup. Then all irreducible representations are one-dimensional, and it is convenient to consider characters instead of such representations. As usual, the dual $\hat{K}$ is given by

$$\hat{K} := \{\alpha \in C_b(K):\ \alpha(x * \bar{y}) = \alpha(x) \cdot \overline{\alpha(y)} \text{ for all } x, y \in K, \quad \alpha \not\equiv 0\}.$$

Equipped with the topology of uniform convergence on compacta, $\hat{K}$ is a locally compact space. If ω_K is a Haar measure on K, then there is a corresponding Plancherel measure $\pi\in M^+(\hat{K})$ such that the Fourier transformation is an L^2-isometry (Jewett [14]). We say $\hat{K}$ has a dual hypergroup structure whenever $\hat{K}$ carries a hypergroup structure such that the mappings $\hat{K}\mapsto \boldsymbol{C}$, $\alpha\to\alpha(x)$, are characters for all $x\in K$. The dual of the dual hypergroup $\hat{K}$ is written as $K^{\wedge\wedge}$. K is called a Pontryagin hypergroup if $\hat{K}$ is a dual hypergroup and $K^{\wedge\wedge}$ agrees with K in the obvious way. In particular, every locally compact abelian group is a Pontryagin hypergroup. There exist, however, commutative hypergroups having dual hypergroups and not being Pontryagin; see Zeuner [33, 34].

For $L\subset K$ and $W\subset\hat{K}$ we define the annihilators $A(\hat{K}, L):=\{\alpha\in\hat{K}: \alpha|_L=1\}$ and $A(K, W):=\{x\in K: \alpha(x)=1 \text{ for all } \alpha\in W\}$.

For further details on duals of commutative hypergroups we refer to Bloom and Heyer [1, 2], Dunkl [4, 5], Jewett [14], Voit [24, 26], and Zeuner [33].

We next determine duals of hypergroups constructed by substitution.

3.3. THEOREM. *Let $M=S(K, H\xrightarrow{\pi}L)$, $W=\mathrm{kern}\,\pi\subset L\subset M$ and $p: M\to M/W=K$ be given as in Section* 2.1.

(1) *If $T\in\hat{L}-A(\hat{L}, W)$, then $T^\bullet(\mu):=T(\mu|_L)$ $(\mu\in M_b(M))$ yields an irreducible representation $T^\bullet\in\hat{M}$.*
(2) *If $T\in\hat{K}$, then $T^*:=T\circ p\in\hat{M}$.*
(3) *$\hat{M}=\{T^*: T\in\hat{K}\}\cup\{T^\bullet: T\in\hat{L}-A(\hat{L}, W)\}$.*
(4) *Every irreducible representation of M has finite dimension if and only if K and L have the same property.*

PROOF.

(1) As L and $M-L$ are open in M, the continuity of the mapping $\mu\mapsto T^\bullet(\mu)$ from $M_b^+(M)$ into $B(H)$ is obvious. It is also clear that $T^\bullet(\mu^*)=T^\bullet(\mu)^*$. To show that $T^\bullet$ is a $*$-representation, it suffices to check that

$$T^\bullet(\delta_x\diamond\delta_y)=T^\bullet(\delta_x)\ T^\bullet(\delta_y) \quad \text{for all } x, y\in M. \tag{3.1}$$

If $x,y\in L$, then (3.1) is trivial. Moreover, if $x\in L$ and $y\in M-L$, then

$$T^\bullet(\delta_x\diamond\delta_y)=T^\bullet(\delta_{\pi(x)}*\delta_y)=T((\delta_x\diamond\delta_y)|_{M-L})=$$
$$T(0)=0=T^\bullet(\delta_x)*T^\bullet(\delta_y).$$

For $x\in M-L$ and $y\in L$, Eq. (3.1) can be verified in the same way. Finally, if $x, y\in M-L$, then $T(\omega_W)=0$ yields

$$T^{\bullet}(\delta_x \diamond \delta_y) = T((\delta_x \diamond \delta_y)|_L) = T(\omega_W \bullet \pi^{-1}((\delta_x * \delta_y)|_H))$$
$$= T(\omega_W) T(\pi^{-1}((\delta_x * \delta_y)|_H)) = 0 = T^{\bullet}(\delta_x) * T^{\bullet}(\delta_y).$$

This completes the proof of $T^{\bullet}$ being a $*$-representation. It is clear that this representation is irreducible and that the operation $^{\bullet}$ transforms equivalent representations into equivalent representations.

(2) This is a consequence of $M/W=K$ and the statement 3.1.1 above.

(3) $\{T^*: T \in \widehat{K}\} = A(\widehat{M}, W)$ follows from 3.1.1. Moreover, if $\tilde{T} \in \widehat{M} - A(\widehat{M}, W)$, then $T^*(\omega_W)=0$ by 3.1.1. Hence, for all $y \in M-L$,

$$\tilde{T}(\delta_y) = \tilde{T}(\delta_e * \delta_y) = \tilde{T}(\pi(\omega_W) * \delta_y) = \tilde{T}(\omega_W \diamond \delta_y) = \tilde{T}(\omega_W)\tilde{T}(\delta_y) = 0.$$

This proves that $\tilde{T} = (\tilde{T}|_L)^{\bullet}$ which completes the proof.

(4) This is a consequence of (3).

We next study commutative hypergroups, in which case all irreducible representations are one-dimensional. Thus, it is convenient here to consider characters instead of one-dimensional irreducible representations. To translate the operations $^{\bullet}$ and * of Theorem 3.1 into mappings on duals of commutative hypergroups, we use the following conventions.

3.4. DEFINITION. Retaining the setting of Section 2.1, we define the mappings

$$\bullet : C(L) \to C(M),\ f \mapsto f^{\bullet}, \text{ where } f^{\bullet}(x) := \begin{cases} f(x) & \text{if } x \in L \\ 0 & \text{if } x \in M-L, \end{cases}$$

and
$$* : C(K) \to C(M),\ f \mapsto f^* := f \circ p.$$

Both mappings are obviously open and continuous when the space $C(L)$, $C(K)$ and $C(M)$ are equipped with the topology of uniform convergence on compact subsets.

3.5. THEOREM. *If K, L and $M=S(K,H \xrightarrow{\pi} L)$ are commutative, then the following statements hold:*

(1) *$(\widehat{L}-A(\widehat{L}, W))^{\bullet}$ and $(\widehat{K})^*$ are disjoint and open subsets of $\widehat{M}$, and*

$$(\widehat{L}-A(\widehat{L}, W))^{\bullet} \cup (\widehat{K})^* = \widehat{M}.$$

(2) *$A(\widehat{M}, W) = \widehat{K}^*$.*

(3) *Let ω_K, ω_L and ω_M be the Haar measures of K, L and M respectively such that these measures are related as in part (1) of Proposition 2.7. Let π_K, π_L and π_M be the associated Plancherel measures on $\widehat{K}$, $\widehat{L}$ and $\widehat{M}$ respectively. Then*

$$\pi_M=(\pi_K)^*+(\pi_L|_{\hat{L}-A(\hat{L},W)})^\bullet \tag{3, 2}$$

where the symbols $*$ *and* $\bullet$ *stand for taking the image of a measure with respect to the mappings defined in Section* 3.4.

PROOF.

(1) $(\hat{L}-A(\hat{L}, W))^\bullet \cup (\hat{K})^*=\hat{M}$ is an immediate consequence of Theorem 3.3. It is also clear that the sets $(\hat{L}-A(\hat{L}, W))^\bullet$ and $(\hat{K})^*$ are disjoint.
In order to prove that both sets are open in $\hat{M}$, we take $\alpha\in\hat{L}-A(\hat{L}, W)$ and $\beta\subset\hat{K}$. Then $\alpha^\bullet|_W\in\hat{W}$, $\alpha^\bullet|_W\not\equiv 1$ and $\beta^*|_W\equiv 1$. Therefore, $\int_W \alpha^\bullet\beta^*\, d\omega_W=0$ and thus $\sup_{x\in W}|\alpha^\bullet(x)-\beta^*(x)|>1$. Since W is compact, it follows that $(\hat{L}-A(\hat{L}, W))^\bullet$ and $(\hat{K})^*$ are open in $\hat{M}$ which finishes the proof.

(2) This is obvious.

(3) As $K=M/W$, we obtain $\pi_M|_{(\hat{K})^*}=\pi_M|_{A(\hat{M},W)}=(\pi_{\hat{K}})^*$ from Theorem 2.5(3) of [26]. To check that

$$\pi_M|_{\hat{M}-A(\hat{M},W)}=(\pi_L|_{\hat{L}-A(\hat{L},W)})^\bullet, \tag{3. 3}$$

we observe that L is an open subhypergroup of M. Thus, the mapping $r:\hat{M}\to\hat{L}$, $\alpha\mapsto\alpha|_L$ satisfies $r(\pi_M)=\pi_L$ (see Theorem 2.7 of Voit [26]). Using parts (1) and (2), we conclude that the restriction mapping $r:\hat{M}-A(\hat{M}, W)\to\hat{L}-A(\hat{L}, W)$ is the inverse mapping of $\bullet$ and a homeomorphism. This proves Eq. (3.3).

3.6. THEOREM. *Assume that* K *and* L *and hence* $M=S(K, H\xrightarrow{\pi} L)$ *are commutative. Again, we put* $W:=\operatorname{kern}\pi$. *If* $\hat{K}$ *and* $\hat{L}$ *are hypergroups, then* $\hat{M}$ *is a hypergroup isomorphic to* $S(\hat{L}, A(\hat{L}, W)\to\hat{K})$. *This substitution is admissible as* $A(\hat{L}, W)$ *is an open subhypergroup of* $\hat{L}$ *isomorphic to* $\hat{K}/A(\hat{K}, H)$, $A(\hat{K}, H)$ *being a compact subhypergroup of* $\hat{K}$.

Conversely, if $\hat{M}$ *is a hypergroup, then* $\hat{K}$ *is a hypergroup, and* $\hat{L}$ *is a hypergroup if and only if* $\operatorname{supp}\pi_L=\hat{L}$ (π_L *is the Plancherel measure on* $\hat{L}$).

PROOF. Assume first that $\hat{M}$ is a hypergroup. Then $A(\hat{M}, W)$ is a subhypergroup of $\hat{M}$ isomorphic to $(M/W)^\wedge=\hat{K}$ (Theorem 2.5 of Voit [26]). Moreover, if $\operatorname{supp}\pi_L=\hat{L}$, then $\hat{L}$ is a commutative hypergroup by Theorem 3.5. The converse conclusion is clear.

Now assume that $\hat{L}$ and $\hat{K}$ are hypergroups. Then, using $L/W=H$, we observe that $\hat{H}$ is a hypergroup isomorphic to $A(\hat{L}, W)$ (Theorem 2.5 of Voit [26]) where $A(\hat{L}, W)$ is an open subhypergroup of $\hat{L}$ (Proposition

3.1 of Bloom and Heyer [2]). Moreover, since H is open in K, $A(\hat{K}, H)$ is a compact subhypergroup of $\hat{K}$ (Proposition 3.1 of [2]) and $\hat{K}/A(\hat{K}, H)$ is isomorphic to $\hat{H}$ (Theorem 2.7 of Voit [26] and use that $\hat{H}$ is a hypergroup). If we identify the isomorphic hypergroups $\hat{K}/A(\hat{K}, H)$, $\hat{H}$ and $A(\hat{L}, W)$ in the obvious way, then we may form the hypergroup $S(\hat{L}, A(\hat{L}, W) \to \hat{K})$ according to Theorem 2.1 on the locally compact space $(\hat{L}-A(\hat{L}, W)) \cup \hat{K}$. If we identify this space with $\hat{M}=(\hat{L}-A(\hat{L}, W))^\bullet \cup (\hat{K})^*$ by using the mappings $^\bullet$ and *, then we still have to check that the resulting hypergroup structure is consistent with the multiplication of characters on M. To do this, we have to consider three cases:

If $\alpha, \beta \in \hat{K}$, then our assumptions imply that there exists a unique probability measure $\delta_\alpha * \delta_\beta$ on $\hat{K}$ satisfying $\alpha(x)\beta(x)=\int_{\hat{K}} \gamma(x)\, d(\delta_\alpha * \delta_\beta)(\gamma)$ for all $x \in K$. Then, for $\alpha^*, \beta^* \in (\hat{K})^* \subset \hat{M}$, we obtain that

$$\alpha^*(x)\beta^*(x)=\int_{(\hat{K})^*} \gamma^*(x)\, d(\delta_\alpha * \delta_\beta)^*(\gamma^*) \quad \text{for all } x \in M, \tag{3.4}$$

μ^* being the image of a measure μ with respect to the mapping *.

Now take $\alpha \in \hat{K}$ and $\beta \in \hat{L}-A(\hat{L}, W)$. Denote the canonical projection from $\hat{K}$ onto $\hat{K}/A(\hat{K}, H) \simeq A(\hat{L}, W)$ by ρ. Using the construction of $S(\hat{L}, A(\hat{L}, W) \to \hat{K})$, we conclude that there is a unique probability measure $\delta_{\rho(\alpha)} \bullet \delta_\beta$ on $\hat{L}-A(\hat{L}, W)$ such that

$$\alpha(x)\beta(x)=\int_{\hat{L}} \gamma(x)\, d(\delta_{\rho(\alpha)} \bullet \delta_\beta)(\gamma) \quad \text{for all } x \in L. \tag{3.5}$$

Since $\gamma^\bullet(x)=0$ for all $x \in M-L=K-H$ and $\gamma \in \hat{L}-A(\hat{L}, W)$, we obtain

$$\alpha^*(x)\beta^\bullet(x)=0=\int_{\hat{L}-A(\hat{L},W)} \gamma^\bullet(x)\, d(\delta_{\rho(\alpha)} \bullet \delta_\beta)(\gamma) \text{ for all } x \in M-L, \tag{3.6}$$

where $\mu^\bullet$ denotes the image of the measure $\mu \in M^1(\hat{L}-A(\hat{L}, W))$ with respect to the mapping $^\bullet$: $\hat{L}-A(\hat{L}, W) \to (\hat{L}-A(\hat{L}, W))^\bullet \subset \hat{M}$. (3.5) and (3.6) together ensure the consistency of the dual convolution in the second case.

Now take $\alpha, \beta \in \hat{L}-A(\hat{L}, W)$ and let ρ be given as before. We have to prove that

$$\alpha^\bullet(x)\beta^\bullet(x)=\int_{(\hat{L}-A(\hat{L},W))^\bullet} \gamma^\bullet(x)\, d(\delta_\alpha \bullet \delta_\beta)^\bullet(\gamma^\bullet) + \int_{(\hat{K})^*} \gamma^*(x)\, d\Big(\omega_{A(\hat{K},H)} * \rho^{-1}((\delta_\alpha * \delta_\beta)|_{A(\hat{L},W)})\Big)^*(\gamma^*) \tag{3.7}$$

for all $x \in M$. First take $x \in M-L=K-H$. Then $\alpha^\bullet(x)\beta^\bullet(x)=0$ and

$$\int_{(\hat{L}-A(\hat{L},W))\bullet}\gamma^{\bullet}(x)\,d(\delta_{\alpha}\bullet\delta_{\beta})^{\bullet}(\gamma^{\bullet})=0.$$

Furthermore, Lemma 2.10 of Voit [26] shows that $A(\hat{K},H)\to \boldsymbol{C}$, $\gamma\mapsto\gamma(x)$ is a nontrivial character on the compact hypergroup $A(\hat{K},H)$. Thus

$$\int_{A(\hat{K},H)}\gamma(x)\,d\omega_{A(\hat{K},H)}(\gamma)=0.$$

As $\hat{K}\to\boldsymbol{C}$, $\gamma\mapsto\gamma(x)$ is a character on $\hat{K}$, and as $\gamma=\gamma^*$ on $K-H$, we get

$$\int_{\hat{K}}\gamma^*(x)\,d\Big(\omega_{A(\hat{K},H)}*\rho^{-1}((\delta_{\alpha}\bullet\delta_{\beta})|_{A(\hat{L},W)})\Big)(\gamma)$$
$$=\int_{A(\hat{K},H)}\gamma(x)\,d\omega_{A(\hat{K},H)}\cdot\int_{\hat{K}}\gamma(x)\,d\rho^{-1}((\delta_{\alpha}\bullet\delta_{\beta})|_{A(\hat{L},W)})(\gamma)=0$$

which establishes Eq. (3.7) for $x\in M-L$.

Now take $x\in L$. Then the definition of the convolution on $\hat{L}$ yields that $\alpha^{\bullet}(x)\beta^{\bullet}(x)=\int_{\hat{L}}\gamma^{\bullet}(x)\,d(\delta_{\alpha}\bullet\delta_{\beta})(\gamma)$. Thus, Eq. (3.7) follows from

$$\int_{\hat{K}}\gamma^*(x)\,d(\omega_{A(\hat{K},H)}*\rho^{-1}((\delta_{\alpha}\bullet\delta_{\beta})|_{A(\hat{L},W)}))(\gamma)=\int_{A(\hat{L},H)}\gamma(x)\,d(\delta_{\alpha}\bullet\delta_{\beta})(\gamma).\qquad(3.8)$$

Let $\rho:\hat{K}\to A(\hat{L},\mathrm{W})$ be the canonical projection. Then $\gamma^*(x)=\gamma(\pi(x))=\rho(\gamma)(x)$ for all $x\in L$ and $\gamma\in\hat{K}$. Consequently, Eq. (3.8) follows from

$$\rho\Big(\omega_{A(\hat{K},H)}*\rho^{-1}((\delta_{\alpha}\bullet\delta_{\beta})|_{A(\hat{L},W)})\Big)=(\delta_{\alpha}\bullet\delta_{\beta})|_{A(\hat{L},W)}$$

and the definition of the image of a measure. Thus the proof of the theorem is complete.

3.7. COROLLARY. *If K and L are Pontryagin hypergroups, then $S(K,H\to L)$ is a Pontryagin hypergroup.*

PROOF. Apply Theorem 3.6 two times and use the fact that the bidual of a commutative hypergroup R is a hypergroup if and only if this bidual is isomorphic with R (Jewett [14], Theorem 12.4).

We next investigate positive definite functions on arbitrary hypergroups constructed by substitution. We recapitulate that a function $f\in C(K)$ on a hypergroup K is called positive definite if $\int_K f(\mu*\mu^*)\geq 0$ for all $\mu\in M_b(H)$ with compact support. It is well-known that products of bounded positive definite functions can fail to be positive definite (see Example 9.1C in Jewett [14]). We say that K has property (P) if fg is positive definite for all bounded positive definite functions $f,g\in C_b(K)$.

3.8. THEOREM. *Let K and L be hypergroups having property (P). If H is an open subhypergroup of K such that each bounded positive definite function on H can be extended to such a function on K, then the hypergroup $M:=S(K, H\xrightarrow{\pi} L)$ has property (P), and each bounded positive definite function on L can be extended to a positive definite function on M.*

For the proof of Theorem 3.8 we copy the linearization results for characters in Theorem 3.6 and decompose positive definite functions into two parts as follows:

3.9. LEMMA. *Let W be a compact normal subhypergroup of a hypergroup K. If $f\in C(K)$ is positive definite, then $f-\omega_W*f$ and ω_W*f are positive definite.*

PROOF. Fix $\mu\in M_b(K)$ with compact support. As $\omega_W=\omega_W^*$ and $\omega_W*f=\omega_W*f*\omega_W=f*\omega_W$ (Lemma 1.5 in Voit [26]), Lemma 4.2H of Jewett [14] yields

$$\begin{aligned}\int_K (f-\omega_W*f)\,d(\mu*\mu^*)&=\int_K (\delta_e-\omega_W)*f*(\delta_e-\omega_W)^*\,d(\mu*\mu^*)\\&=\int_K fd((\delta_e-\omega_W)*\mu*\mu^**(\delta_e-\omega_W)^*)\geq 0.\end{aligned}$$

In a similar way we obtain $\int_K f*\omega_W\,d(\mu*\mu^*)\geq 0$.

3.10. LEMMA. *Let K, L, M and $W=$*kern *$\pi\subset L$ be given as in Section 2.1. Then $f\in C(M)$ is positive definite on M if and only if $f*\omega_W\in C(M)$ is positive definite on M and $(f-\omega_W*f)|_L\in C(L)$ is positive definite on L.*

PROOF. The only-if-part follows from 3.9. To check the if-part, it suffices to show that $h:=f-\omega_W*f$ is positive definite on M. For this we decompose a given $\rho\in M_b(M)$ having compact support into $\rho=\rho_1+\rho_2$ with supp $\rho_1\subset M-L$ and supp $\rho_2\subset L$. As supp $(\rho_1*\rho_2^*+\rho_2*\rho_1^*)\subset M-L$ and $h|_{M-L}=0$, we have $\int_M h\,d(\rho_1*\rho_2^*+\rho_2*\rho_1^*)=0$. Moreover, 4.2H of Jewett [14] and the definition of the convolution on M lead to

$$\int_M h\,d(\rho_1*\rho_1^*)=\int_M h\,d((\rho_1*\rho_1^*)|_L)=\int_L (\omega_W*h)\,d((\rho_1*\rho_1^*)|_L)=0.$$

Thus, $\int_M h\,d(\rho*\rho^*)=\int_M h\,d(\rho_2*\rho_2^*)\geq 0$ which completes the proof.

PROOF OF THEOREM 3.8. Let $f, g \in C_b(M)$ be positive definite. We prove that $(f*\omega_W)\cdot g$ and $(f-f*\omega_W)\cdot g$ are positive definite on M. We first observe that $f*\omega_W$ and $g*\omega_W$ may be regarded as positive definite functions on $M/W=K$ by 1.7. Hence, $(f*\omega_W)\cdot(g*\omega_W)$ is positive definite on M. As $f*\omega_W$ is constant on W-cosets,

$$(((f*\omega_W)\cdot g)*\omega_W)(x)=\int_K ((f*\omega_W)\cdot g)(x*\bar{y})d\omega_W(y)=f*\omega_W(x)\cdot g*\omega_W(x)$$

for all $x \in K$.
As $(f*\omega_W)|_L$ and $(g-g*\omega_W)|_L$ are positive definite by Lemma 3.9, we obtain that $((f*\omega_W)(g-g*\omega_W))|_L=((f*\omega_W)g-((f*\omega_W)g)*\omega_W)|_L$ is positive definite on L. As $((f*\omega_W)g)*\omega_W=(f*\omega_W)(g*\omega_W)$ is positive definite on M, it follows from Lemma 3.10 that $(f*\omega_W)g$ is positive definite on M. As the function $f-f*\omega_W$ is positive definite on M by Lemma 3.9, the function $h:=((f-f*\omega_W)g)|_L \in C(L)$ is positive definite on L. Hence, by Lemma 3.9, $(h-h*\omega_W)|_L$ is positive definite. Thus, in order to check that $(f-f*\omega_W)g$ is positive definite on M, it suffices to prove by Lemma 3.10 that $r:=((f-f*\omega_W)g)*\omega_W$ is positive definite on M. However, $r|_L$ is obviously positive definite on L and can be regarded as a bounded positive definite function on H. This function can be extended to a positive definite function $\tilde{r}$ on M such that $\tilde{r}*\omega_W=\tilde{r}$. Thus, $\tilde{r}|_{M-L}=0=r|_{M-L}$ and $\tilde{r}=r$. This completes the proof.

3.11. REMARK. We do not know whether we can omit the condition in Theorem 3.8 that each bounded positive definite function on H can be extended to a bounded positive definite function on K. This extension problem did not appear in Theorem 3.6 as it was ensured implicitely there by the assumptions. Applications of induced representations on hypergroups to the extension problem can be found in Hermann [10].

4. Substitution and commutative diagrams

In this section we establish some relations between substitution and drawing commutative diagrams of hypergroups. These results will be useful when applying substitution repeatedly; see, for instance, Section 5 below.

4.1. LEMMA. *Let K_1, K_2, L_1, L_2 be hypergroups. Let $p_i: L_i \to K_i$ be open and proper homomorphisms. Consider the open subhypergroups $H_i:=p_i(L_i)$ of K_i for $i \in \{1, 2\}$. Assume that φ_K and φ_L are hypergroup homomorphisms such that*

$$\begin{array}{ccc} L_1 & \xrightarrow{\varphi_L} & L_2 \\ \downarrow p_1 & & \downarrow p_2 \\ K_1 & \xrightarrow{\varphi_K} & K_2 \end{array} \tag{4.1}$$

commutes. Then the hypergroups $M_i := S(K_i, H_i \xrightarrow{p_i} L_i)$ *exist, and the mapping*

$$\varphi : M_1 \to M_2, \qquad x \mapsto \begin{cases} \varphi_L(x) & \text{if } x \in L_1 \\ \varphi_K(x) & \text{if } x \in M_1 - L_1 \end{cases}$$

is a hypergroup homomorphism. Moreover, if φ_K *and* φ_L *are open, proper, or surjective, then* φ *has the same property.*

Finally, (4.1) *leads to the following extended commutative diagram*

$$\begin{array}{ccccc} L_1 & \hookrightarrow & M_1 & \longrightarrow & K_1 \\ \downarrow \varphi_L & & \downarrow \varphi & & \downarrow \varphi_K \\ L_2 & \hookrightarrow & M_2 & \longrightarrow & K_2 \end{array} \tag{4.2}$$

PROOF. It suffices to check that $\varphi(\delta_x \circ \delta_y) = \varphi(\delta_x) \circ \psi(\delta_y)$ for all $x, y \in M_1$ such that $x \in M_1 - L_1$ or $y \in M_1 - L_1$ holds. If $x \in M_1 - L_1$ and $y \in L_1$, then

$$\varphi(\delta_x \circ \delta_y) = \varphi_K(\delta_x * \delta_{p_1(y)}) = \varphi_K(\delta_x) * \delta_{\varphi_K(p_1(y))} = \varphi(\delta_x) \circ \varphi(\delta_y)$$

and, in a similar way, $\varphi(\delta_y \circ \delta_x) = \varphi(\delta_y) \circ \varphi(\delta_x)$.

It remains to study the case $x, y \in M_1 - L_1$. Let ω_1 and ω_2 be the normalized Haar measures of $\ker p_1$ and $\ker p_2$. As $\varphi_L(\ker p_1) \supset \ker p_2$, for each $\rho \in M_b(L_1)$ satisfying $\rho * \omega_1 = \rho$ we have $\varphi_L(\rho) * \omega_2 = \varphi_L(\rho * \omega_1) * \omega_2 = \varphi_L(\rho) * \varphi_L(\omega_1) * \omega_2 = \varphi_L(\rho) * \varphi_L(\omega_1) = \varphi_L(\rho)$. If $\tilde{p}_i^{-1} : M_b(H_i) \to M_b(L_i)$ is defined as in Theorem 2.1 ($i \in \{1, 2\}$), it follows that $\varphi_L(\tilde{p}_1^{-1}(\mu)) = \tilde{p}_2^{-1}(\varphi_K(\mu))$ for each $\mu \in M_b(H_1)$. Using this fact, $p(H_1) = H_2 \cap \varphi_K(K_1)$, as well as $\varphi_K(K_1 - H_1) = \varphi_K(K_1) - H_2$, we conclude that

$$\begin{aligned} \varphi(\delta_x \circ \delta_y) &= \varphi_K((\delta_x * \delta_y)|_{K_1 - H_1}) + \varphi_L(\tilde{p}_1^{-1}((\delta_x * \delta_y)|_{H_1})) \\ &= (\varphi_K(\delta_x * \delta_y))|_{K_2 - H_2} + \tilde{p}_1^{-1}(\varphi_K((\delta_x * \delta_y)|_{H_1})) \\ &= (\varphi_K(\delta_x) * \varphi_K(\delta_y))|_{K_2 - H_2} + \tilde{p}_1^{-1}((\varphi_K(\delta_x) * \varphi_K(\delta_y))|_{H_2}) \\ &= \varphi(\delta_x) \circ \varphi(\delta_y) \qquad \text{for } x, y \in M_1 - L_1 \end{aligned}$$

This completes the proof.

4.2. Let K and L be commutative hypergroups having dual hypergroups. If $p : K \to L$ is an open and proper hypergroup homomorphism, then we may introduce the dual homomorphism $\hat{p} : \hat{L} \to \hat{K}$ with $\hat{p}(\alpha)(x) =$

$\alpha(p(x))$ for $\alpha \in \hat{L}$, $x \in K$. $\hat{p}$ is again open and proper by Theorems 2.5 and 2.7 in Voit [26].

Assume now that the hypergroups K_1, K_2, L_1, L_2 of Lemma 4.1 are commutative and have dual hypergroups. Then we may draw the associated dual commutative diagram

$$\begin{array}{ccc} \hat{L}_1 & \xleftarrow{\hat{\varphi}_L} & \hat{L}_2 \\ \uparrow \hat{p}_1 & & \uparrow \hat{p}_2 \\ \hat{K}_1 & \xleftarrow{\hat{\varphi}_K} & \hat{K}_2 \end{array} \tag{4.3}$$

Moreover, we have $\hat{M}_i = S(\hat{L}_i, \hat{H}_i \xrightarrow{\hat{p}_i} \hat{K}_i)$ for $i=1,2$ by Theorem 3.6. Therefore, we have a natural dual homomorphism $\hat{\varphi}: \hat{M}_2 \rightarrow \hat{M}_1$ such that the diagram

$$\begin{array}{ccccc} \hat{L}_1 & \leftarrow & \hat{M}_1 & \hookleftarrow & \hat{K}_1 \\ \uparrow \hat{\varphi}_L & & \uparrow \hat{\varphi} & & \uparrow \hat{\varphi}_K \\ \hat{L}_2 & \leftarrow & \hat{M}_2 & \hookleftarrow & \hat{K}_2 \end{array} \tag{4.4}$$

commutes. This diagram is dual to the diagram (4.2).

4.3. Chains of hypergroups and substitution. Let $(H_i)_{1 \leq i \leq n}$ be a chain of hypergroups together with open and proper hypergroup homomorphisms $p_i: H_i \rightarrow H_{i+1}$ $(1 \leq i \leq n-1)$. We inductively construct new hypergroups $(K_i)_{1 \leq i \leq n}$ together with open and injective homomorphisms $\pi_i: K_i \rightarrow K_{i+1}$ $(1 \leq i \leq n-1)$ and surjective, open and proper homomorphisms $q_i: K_i \rightarrow H_i$ $(1 \leq i \leq n)$ as follows:

(1) Put $K_1 := H_1$ and $q_1 := id$.

(2) If K_i and q_i are constructed, then we define

$$K_{i+1} := S(H_{i+1}, p_i \circ q_i(K_i) \xrightarrow{p_i \circ q_i} K_i)$$

and take π_i as the canonical embedding and q_{i+1} as the canonical projection associated with this substitution.

$$\begin{array}{ccccccccc} H_1 & \xrightarrow{p_1} & H_2 & \xrightarrow{p_2} & H_3 & \cdots & \xrightarrow{p_{n-1}} & H_n \\ \uparrow q_1 & & \uparrow q_2 & & \uparrow q_3 & & & \uparrow q_n \\ K_1 & \overset{\pi_1}{\hookrightarrow} & K_2 & \overset{\pi_2}{\hookrightarrow} & K_3 & \cdots & \overset{\pi_{n-1}}{\hookrightarrow} & K_n \end{array} \tag{4.5}$$

Using the definition of substitution as well as induction, we may realize the largest hypergroup $K := K_n$ of this diagram as follows: If $W_1 := H_1$

and $W_k := H_k - p_{k-1}(H_{k-1})$ $(k=2, \dots, n)$, then $K := \bigcup_{k=1}^{n} W_k$ is the disjoint union of the sets W_k which are embedded into K as open subsets.

Let e_k and $*_k$ be the identity element and the convolution on H_k respectively. Then the convolution $*$ on K is given by

$$\delta_x * \delta_y = \begin{cases} \delta_x *_k \delta_{p_{k-1}\circ\dots\circ p_l(y)} & \text{if } x \in W_k,\ y \in W_l,\ l<k \\ \delta_{p_{l-1}\circ\dots\circ p_k(x)} *_l \delta_y & \text{if } x \in W_k,\ y \in W_l,\ l>k \\ (\delta_x *_k \delta_y)|_{W_k} + \\ \sum_{j=1}^{k-1} \tilde{p}_j^{-1}\Big(\dots(\tilde{p}_{k-1}^{-1}((\delta_x *_k \delta_y)|_{p_{k-1}(W_{k-1})})|_{p_{k-2}(W_{k-2})}\dots)|_{p_j(W_j)}\Big) & \\ & \text{if } x, y \in W_k \end{cases} \tag{4.6}$$

where $\tilde{p}_j^{-1}$ is the inverse mapping of $\tilde{p}_j : M_b(H_j|\mathrm{kern}\, p_j) \to M_b(p_1(H_j))$; cf. Section 1.7.

4.4. Dual chains and substitution. Let $(H^i)_{1\le i\le n}$ be a chain of hypergroups together with open and proper hypergroup homomorphisms $p^i : H^{i+1} \to H^i$ $(1 \le i \le n-1)$. We inductively construct new hypergroups $V(K^i)_{1\le i\le n}$ together with open and injective homomorphisms $q^i : H^i \to K^i$ $(1\le i\le n)$ and surjective, open and proper homomorphisms $\pi^i : K^{i+1} \to K^i$ $(1 \le i \le n-1)$ as follows:

(1) Put $K^1 := H^1$ and $q^1 := id$.

(2) If K^i and q^i are constructed, we set

$$K^{i+1} := S(K^i, q^i \circ p^i(H^{i+1}) \xrightarrow{q^i\circ p^i} H^{i+1})$$

and take π^i as the canonical projection and q_{i+1} as the canonical embedding associated with this substitution.

$$\begin{array}{ccccccccc} H^1 & \xleftarrow{p^1} & H^2 & \xleftarrow{p^2} & H^3 & \cdots & \xleftarrow{p^{n-1}} & H^n & \\ \downarrow q^1 & & \downarrow q^2 & & \downarrow q^3 & & & \downarrow q^n & \quad (4.7) \\ K^1 & \xleftarrow{\pi^1} & K^2 & \xleftarrow{\pi^2} & K^3 & \cdots & \xleftarrow{\pi^{n-1}} & K^n & \end{array}$$

Using the definition of substitution as well as induction, we may realize the largest hypergroup K^n of this diagram as follows: If $W^n := H^n$ and $W^k := H^k - p^k(H^{k-1})$ $(k=1, \dots, n-1)$, then $K^n := \bigcup_{k=1}^{n} W^k$ is the disjoint union of the sets W^k which are embedded into K^n as open subsets. The convolution on K^n can be computed explicitly as in Eq. (4.6). It turns out that the hypergroup K^n is isomorphic to the hypergroup K_n of Section 4.3 if we set $H_i := H^{n-i}$ and $p^i := p_{n-i}$. We therefore do not write down the convolution on K^n explicitly.

The constructions 4.3 and 4.4 are dual in the following way: If the hypergroups H_i of Section 4.3 are commutative and admit dual hypergroups $H^i := \hat{H}_i$, then the associated dual homomorphisms $p^i := \hat{p}_i : \hat{H}_{i+1} \to \hat{H}_i$ $(i=1, \dots, n-1)$ are again open and proper (cf. Theorems 2.5 and 2.7 in Voit [26]). Moreover, the hypergroups K_i are also commutative and they admit dual hypergroups $K^i := \hat{K}_i$. Then the hypergroups K^i are constructed (up to isomorphism) as described in Section 4.4. This follows inductively from Theorem 3.6.

It is clear that a corresponding result holds if we consider the hypergroups H^i of Section 4.4, and if these hypergroups are commutative and admit dual hypergroups.

5. Conjugacy class hypergroups and duals of some compact groups

Let G be a compact group. If G acts on itself by conjugation, then the space G^G of all orbits becomes a commutative hypergroup in a canonical way (see Jewett [14], Section 8) which admits a discrete dual hypergroup. This dual may be identified with the set $\hat{G}$ of all equivalence classes of irreducible representations of G with the convolution

$$\delta_\pi * \delta_\rho = \sum_{\tau \in \pi \otimes \rho} \frac{\dim \tau}{\dim \pi \cdot \dim \rho} m_{\tau,\pi,\rho} \cdot \delta_\tau \qquad (\pi, \rho \in \hat{G})$$

where $m_{\tau,\pi,\rho} \in \boldsymbol{N}$ is the multiplicity of τ in $\pi \otimes \rho$ (see, for instance, [8, 9, 14]).

The purpose of this section is to show how substitution of open subhypergroups can be used to describe the structure of the hypergroups G^G and $\hat{G}$ for compact groups which are sufficiently close to the abelian case. It is clear that our method works for a very particular kind of compact groups only. Moreover, it does not lead to any explicit irreducible representation as mothods like induced representations do.

Assume from now on that G is a compact group having a commutative normal subgroup L such that G/L is a finite cyclic group of order $n \subset \boldsymbol{N}$. For sake of convenience we identify G/L with $\boldsymbol{Z}(n) = \{0, 1, \dots, n-1\}$. For $a \in G$ we consider the automorphism $t_a : x \mapsto axa^{-1}$ on L. Let S_a be the subgroup of G/L generated by aL. Moreover, we introduce the closed subgroup $J(a) := \{t_a(x) \cdot x^{-1} : x \in L\}$ of L. We next determine the structure of G^G:

5.1. Lemma. *The following statements hold for $a, b, c \in G$:*
(1) *If $S_b \subset S_a$, then $J(b) \subset J(a)$.*

(2) *$t_b(J(a))=J(a)$.*
(3) *If cL generates the cyclic group S_aS_b, then $J(a)J(b)=J(c)$.*
(4) *If cL generates G/L, then all conjugacy classes of G are given by the sets $R(x, k) :=\{t_d(x)\cdot J(c^k)\cdot c^k : d\in G\}$ where $x\in L$ and $k=0, 1, \dots, n-1$.*
(5) *The convolution on the conjugacy class hypergroup $G^G :=\{R(x, k) : x\in L, k=0, \dots, n-1\}$ is defined by*

$$\delta_{R(x,k)}*\delta_{R(y,l)}=\frac{1}{n}\sum_{u=0}^{n-1}\int_{J(c^l)}\delta_{R(x\cdot t_{c^u}(y)\cdot w, k+l)}\,d\omega_{J(c^l)}(w)$$

where $\omega_{J(c^l)}$ is the normalized Haar measure on $J(c^l)$.

PROOF.
(1) If $i\in\boldsymbol{N}$ and $x\in L$, then $xJ(a)=t_a(x)J(a)=\dots=t_{a^i}(x)J(a)$. As $S_b\subset S_a$ yields some $i\in\boldsymbol{N}$ with $t_b=t_{a^i}$, it follows that $t_b(x)x^{-1}\in J(a)$. Hence, $J(b)\subset J(a)$.
(2) Take $x\in L$. Then $t_b(t_a(x)x^{-1})=t_b(t_a(x))t_b(x^{-1})=t_a(t_b(x))t_b(x)^{-1}\in J(a)$, and thus $t_b(J(a))\in J(a)$. Taking b^{-1} instead of b, we obtain the converse inclusion.
(3) Part (1) yields $J(a)J(b)\subset J(c)$. As $S_c\subset S_aS_b$, we find $p, q\in\boldsymbol{N}$ such that $a^pb^qL=cL$. Thus, $t_c(x)x^{-1}=t_{a^p}(t_{b^p}(x))\cdot(t_{b^p}(x))^{-1}\cdot t_{b^p}(x)x^{-1}\in J(a)J(b)$ for all $x\in L$. Hence, $J(c)\subset J(a)J(b)$.
(4) By the assumption, each element of G can be written in a unique way as xc^k where $x\in L$ and $k\in\{0, 1, \dots, n-1\}$. As $yc^l\cdot xc^k\cdot(yc^l)^{-1}=t_{c^l}(x)\cdot(yc^ky^{-1}c^{-k})\cdot c^k$ and $\{yc^ky^{-1}c^{-k} : y\in L\}=J(c^k)$, it follows that $\{t_d(x)\cdot J(c^k)\cdot c^k : d\in G\}$ is the conjugacy class of xc^k.
(5) The normalized Haar measure of any compact group W will be denoted by ω_W. Let $p: G\to G^G$ be the canonical projection. Take representatives xc^k and yc^l of $R(x, k)$ and $R(y, l)$ repectively. For each $a\in L$, the mapping $\varphi_a : L\to J(a)$, $w\mapsto w^{-1}t_a(w)$, is a homomorphism. Hence, by part (2), $t_{c^k}\circ p_{c^l}(\omega_L)=\omega_{J(c^l)}$. The definition of the convolution on G^G (c.f. Section 8.4 of Jewett [14]) now yields

$$\begin{aligned}\delta_{R(x,k)}*\delta_{R(y,l)}&=\int_G p(\delta_{xc^kzyc^lz^{-1}})\,d\omega_G(z)=\frac{1}{n}\sum_{u=0}^{n-1}\int_L p(\delta_{xc^kvc^uyc^{l-u}v^{-1}})\,d\omega_L(v)\\&=\frac{1}{n}\sum_{u=0}^{n-1}\int_L\delta_{R(xt_{c^{u+k}}(y)t_{c^k}(vt_{c^l}(v^{-1})), k+l)}\,d\omega_L(v)\\&=\frac{1}{n}\sum_{u=0}^{n-1}\int_{J(c^l)}\delta_{R(xt_{c^u}(y)w, k+l)}\,d\omega_{J(c^l)}(w).\end{aligned}$$

5.2. Our next aim is to find another description of G^G in terms of substitution of subhypergroups. For this, we need some notations and facts:

(1) Fix $a,b\in G$. Assume that $S_b\subset S_a\subset G/L$. Then $J(b)\subset J(a)\subset L$ by 5.1(1). The group G acts on $J(b)$, $J(a)$ and L by conjugation (see 5.1(2)). Thus we may form the associated commutative orbit hypergroups $J(b)^G\subset J(a)^G\subset L^G$. Moreover, we may form the coset hypergroups $L^G/J(a)^G$, $L^G/J(b)^G$ and $J(a)^G/J(b)^G$. As $(L^G/J(b)^G)/(J(a)^G/J(b)^G)\simeq L^G/J(a)^G$ by Theorem 14.3A in [14], we obtain a natural surjective hypergroup horomorphism $\pi(b,a):L^G/J(b)^G\to L^G/J(a)^G$. We define the hypergroups $K(a):=L^G/J(a)^G\times S_a$ and $K(b):=L^G/J(b)^G\times S_b$. Then $\varphi(b,a):K(b)\to K(a)$, $(v,w)\mapsto(\pi(b,a)(v),w)$, is a hypergroup homomorphism from $K(b)$ onto the open subhypergroup $L^G/J(a)^G\times S_b$ of $K(a)$.

(2) Take $c\in G$ such that cL generates G/L. Then $L^G/J(c)^G$ is a group isomorphic with $L/J(c)$. In fact, $\tau:L\to L^G/J(c)^G$, $x\mapsto x^G*J(c)^G$, is a consistent orbital morphism. Using the factorization theorem 14.3B of [14] and $\tau^{-1}(e)=J(c)$, we see that $\pi:L/J(c)\to L^G/J(c)^G$, $xJ(c)\mapsto x^G*J(c)^G$ is a unary consistent orbital morphism. As $xJ(c)=\{axa^{-1}\cdot byb^{-1}: a,b\in G,\ y\in J(c)\}$ for $x\in L$, π is also injective which proves that π is a hypergroup isomorphism as claimed.

(3) If $a\in L$, then $J(a)$ is trivial, and $L^G/J(a)^G$ can be identified with L^G.

5.3. Assume that $|G/L|=n$ has the form

$$n=p_1^{k_1}\cdots p_l^{k_l}\quad\text{with}\quad l,k_1,\dots,k_l\in\boldsymbol{N},\ p_1,\dots,p_l \text{ different primes.}$$

The hypergroups $K(a)$ of Section 5.2(1) depend on the subgroups S_a of G/L only and not on a itself by Lemma 5.1(1). As the subgroups of $G/L=\boldsymbol{Z}(p)$ are generated by the elements $p_1^{i_1}\cdots p_l^{i_l}\in\boldsymbol{Z}(p)$ $(0\le i_j\le k_j)$, we find a unique associated hypergroup $K(a)$ which we shall call $K(i_1,\dots,i_l)$ from now on. Section 5.2 shows that we have a natural homomorphism $\pi(j;i_1,\dots,i_l)$ from $K(i_1,\dots,i_l)$ onto an open subhypergroup of $K(i_1,\dots,i_j-1,\dots,i_l)$ for all $j=1,\dots,l$ and $i_j=1,\dots,k_j$. It is clear from 5.2 that the hypergroups $K(i_1,\dots,i_l)$ and the homomorphisms $\pi(j;i_1,\dots,i_l)$ form a commutative diagram which has the form of an l-dimensional lattice.

$$\begin{array}{ccccc}
K(k_1,0) & \longleftarrow & K(k_1,1) & \longleftarrow & \dots\ K(k_1,k_2) \\
\vdots & & \vdots & & \vdots \\
K(1,0) & \longleftarrow & K(1,1) & \longleftarrow & \dots\ K(1,k_2) \\
\downarrow & & \downarrow & & \downarrow \\
K(0,0) & \longleftarrow & K(0,1) & \longleftarrow & \dots\ K(0,k_2)
\end{array}\quad (l=2) \tag{5.1}$$

The homomorphisms in this lattice have the following properties for $h\neq j$, $1\leq i_j\leq k_j$ and $0\leq i_h\leq k_h-1$:

$$\operatorname{kern}\pi(j\,;i_1,\ldots,i_l)=\pi(h\,;i_1,\ldots,i_h+1,\ldots,i_l)(\operatorname{kern}\pi(j\,;i_1,\ldots,i_h+1,\ldots,i_l)) \tag{5.2}$$

and

$$\begin{aligned}&\pi(j\,;i_1\ldots i_l)(K(i_1\ldots i_l)-\pi(h\,;i_1\ldots,i_h+1,\ldots i_l)(K(i_1\ldots,i_h+1,\ldots i_l)))=\\&\quad K(i_1\ldots i_j-1\ldots i_l)-\pi(h\,;i_1\ldots i_j-1\ldots i_h+1\ldots i_l)(K(i_i\ldots i_j-1\ldots i_h+1\ldots i_l))\end{aligned} \tag{5.3}$$

We put $K^{(1)}(i_1,\ldots,i_{l-1},k_l):=K(i_1,\ldots,i_{l-1},k_l)$ and

$$\begin{aligned}&K^{(1)}(i_1,\ldots,i_{l-1},k_l-1):=\\&\quad S(K(i_1,\ldots,i_{l-1},k_l-1),\ \pi(l\,;i_1\ldots i_{l-1},k_l)(K(i_1,\ldots,i_l))\to K(i_1,\ldots,i_l)).\end{aligned}$$

We then obtain a natural embedding $\pi^{(1)}(l\,;i_1,\ldots,i_{l-1},k_l)$ from $K^{(1)}(i_1,\ldots,i_{l-1},k_l)$ into $K^{(1)}(i_1,\ldots,i_{l-1},k_l-1)$ as well as a natural homomorphism $\tilde{\pi}(l\,;i_1,\ldots,i_{1-1},k_l-1)$ from $K^{(1)}(i_1,\ldots,i_{l-1},k_l-1)$ into $K(i_1,\ldots,i_{l-1},k_l-2)$. Moreover, (5.2), (5.3) and Lemma 4.1 ensure that there exist unique homomorphisms $\pi^{(1)}(h\,;i_1,\ldots,i_{l-1},k_l-1)$ from $K^{(1)}(i_1,\ldots,i_{l-1},k_l-1)$ into $K^{(1)}(i_1,\ldots,i_h-1,\ldots,i_{l-1},k_l-1)$ such that the complete diagram remains commutative after replacing the hypergroups $K(i_1,\ldots,i_{l-1},k_l-1)$ (and their homomorphisms) by $K^{(1)}(i_1,\ldots,i_{l-1},k_i-1)$. As the properties (5.2) and (5.3) remain valid for the modified lattice diagram by Lemma 4.1, we may recapitulate the procedure above with k_l-2 instead of k_l-1 and so on. This yields a new lattice consisting of hypergroups $K^{(1)}(i_1,\ldots,i_l)$ and homomorphisms $\pi^{(1)}(h\,;i_1,\ldots,i_l)$ where now the homomorphisms $\pi^{(1)}(l\,;i_1,\ldots,i_l)$ are injective.

We now recapitulate the complete procedure with the index $l-1$ instead of l and so on. After l steps we arrive at a lattice diagram consisting of hypergroups $K^{(l)}(i_1,\ldots,i_l)$ and homomorphisms $\pi^{(l)}(h\,;i_1,\ldots,i_l)$ where now all homomorphisms are injective.

We claim that the hypergroup $K^{(l)}(0,\ldots,0)$ is isomorphic with the conjugacy class hypergroup G^G. In fact, the hypergroups $K^{(1)}(i_1,\ldots,i_{l-1},0)$ above are constructed just as the hypergroups resulting in Section 4.3. This also holds for $K^{(2)}(i_1,\ldots,i_{l-2},0,0)$ and so on. Thus, applying the results of Section 4.3 l-times, we readily obtain that the hypergroup $K^{(l)}(0,\ldots,0)$ consists of the sets $R(x,k)$ of Lemma 5.1(4) where the convolution is given as in Lemma 5.1(5).

G^G can easily be written down explicitely for special cases like $|G/L|$ being a prime power. For the case that $|G/L|$ is a prime, we have the following result:

5.4. PROPOSITION. *Let G be a compact group having a commutative normal subgroup L such that $|G/L|=:p$ is a prime. If we fix $c\in G-L$, then $J:=\{cxc^{-1}x^{-1}:x\in L\}$ is a subgroup of L, and the group $\boldsymbol{Z}(p)$ acts continuously on L via $(k, x)\mapsto c^{k}xc^{-k}$. The mapping $\pi:L^{\boldsymbol{Z}(p)}\to L/J$, $\{c^{k}xc^{-k}:k\in\boldsymbol{Z}(p)\}\mapsto xJ$ is a hypergroup homomorphism. The conjugacy class hypergroup G^{G} is isomorphic to*

$$S(\boldsymbol{Z}(p)\times L/J, \{0\}\times L/J\xrightarrow{\pi}\{0\}\times L^{\boldsymbol{Z}(P)}). \tag{5.4}$$

Moreover, the dual hypergroup $\widehat{G}$ is given - up to isomorphism - by

$$S(\widehat{L}^{\boldsymbol{Z}(p)},\ A(\widehat{L}, J)\to\boldsymbol{Z}(p)\times A(\widehat{L}, J)) \tag{5.5}$$

where $\boldsymbol{Z}(p)$ acts on $\widehat{L}$ via $(k, \alpha)\mapsto\alpha^{k}$, $\alpha^{k}(x):=\alpha(c^{k}xc^{-k})$.

PROOF. (5.4) follows from Lemma 5.1 and Section 5.2. To check (5.5), we first notice that the action of $\boldsymbol{Z}(p)$ on L leads to a dual action on $\widehat{L}$ such that the hypergroups $(L^{\boldsymbol{Z}(p)})^{\wedge}$ and $\widehat{L}^{\boldsymbol{Z}(p)}$ are isomorphic (see, for instance, [9]). Moreover, as $\boldsymbol{Z}(p)$ acts trivially on the subgroup $A(\widehat{L}, J)$ of $\widehat{L}$, $A(\widehat{L}, J)$ is a subgroup of $\widehat{L}^{\boldsymbol{Z}(p)}$. Using $(\boldsymbol{Z}(p)\times L/J)^{\wedge}\simeq\widehat{\boldsymbol{Z}(p)}\times(L/J)^{\wedge}\simeq\boldsymbol{Z}(p)\times A(\widehat{L}, J)$ (see (3.3) and (5.3) in Zeuner [33]), (5.5) is an immediate consequence of Eq. (5.4) and Theorem 3.6.

Let H be a finite group acting on the finite abelian group L. Let G be the semidirect product of A and H. Then the method of Mackey and Wigner leads to a description of $\widehat{G}$ in terms of induced representations from the normal subgroup L of G (see Ch. 8 in Serre [21]). In the situation of Proposition 5.4, this description and the description of $\widehat{G}$ in (5.5) are well matched.

We next return to the general case:

5.5. The dual hypergroup $\widehat{G}$. The method of the proof of Proposition 5.4 can be used to constuct $\widehat{G}$ in the general case. For this we fix $c\in G$ such that cL generates $G/L\simeq\boldsymbol{Z}(n)$. We form the dual lattice of the hypergroups $K(i_{1}, \ldots, i_{l})$ of Section 5.3 which consists of the hypergroups

$$K(i_{1}, \ldots, i_{l})^{\wedge}\simeq A((L^{G})^{\wedge},\ J(c^{p_{1}^{i_{1}}\cdots p_{l}^{i_{l}}}))\times\widehat{S}_{c^{p_{1}^{i_{1}}\cdots p_{l}^{i_{l}}}}\qquad(0\leq i_{j}\leq k_{j}).$$

If we regard the groups $\widehat{S}_{c^{p_{1}^{i_{1}}\cdots p_{l}^{i_{l}}}}\simeq\boldsymbol{Z}(p_{1}^{k_{1}-i_{1}}\ldots p_{l}^{k_{l}-i_{l}})$ as subgroups of $\boldsymbol{Z}(n)$ of index $p_{1}^{i_{1}}\ldots p_{l}^{i_{l}}$, the dual homomorphisms of the dual lattice are given by

$$\hat{\pi}(j\,;i_1,\dots,i_l):\ K(i_1,\dots,i_j-1,\dots,i_l)^\wedge\to K(i_1,\dots,i_l)^\wedge,\ (w,s)\mapsto(w,s^{p_j}).$$

If we apply the construction of Section 5. 3 to this dual lattice, i.e., if we apply the construction of Section 4. 3 l-times, then we obtain the following description of the resulting hypergroup $\hat{G}$ (cf. Section 4. 4).

For any $w\in(L^G)^\wedge$ we choose $i_1(w),\dots,i_l(w)\geq 0$ as small as possible such that $w\in A((L^G)^\wedge, J(c^{p_1^{i_1(w)}\cdots p_l^{i_l(w)}}))$. We then define $H(w)$ as the subgroup of the cyclic group $\boldsymbol{Z}(n)$ generated by $p_1^{i_1(w)}\cdots p_l^{i_l(w)}\in\boldsymbol{Z}(n)$. We then have

$$\hat{G}\simeq\Big\{(w,s):\ w\in(L^G)^\wedge,\ s\in H(w)\Big\}. \tag{5.6}$$

Using $(L^G)^\wedge\simeq\hat{L}^G$, the description (5. 6) of $\hat{G}$ is again well matched with that of Section 8 of [21]. The convolution on $\hat{G}$ is given as follows: Let the convolution on $(L^G)^\wedge$ be given by

$$\delta_u*\delta_v=\sum_{w\in(L^G)^\wedge}g(u,v,w)\,\delta_w\qquad(u,v\in(L^G)^\wedge).$$

Then $g(u,v,w)>0$ yieds that $H(u)\subset H(w)$ and $H(v)\subset H(w)$, and that projections from $H(w)$ onto $H(u)$ and $H(v)$ are given by $x\mapsto x\cdot r(u,w)$ and $x\mapsto x\cdot r(v,w)$ respectively with

$$r(u,w):=p_1^{i_1(u)-i_1(w)}\cdots p_l^{i_l(u)-i_l(w)},\quad r(v,w):=p_1^{i_1(v)-i_1(w)}\cdots p_l^{i_l(v)-i_l(w)}.$$

For $u,v\in(L^G)^\wedge$, $s\in H(u)$ and $t\in H(v)$ we then have

$$\delta_{(u,s)}*\delta_{(v,t)}=\sum_{w\in(L^G)^\wedge}\sum_{a=1}^{r(u,w)}\sum_{b=1}^{r(v,w)}\frac{g(u,v,w)}{r(u,w)r(v,w)}\delta_{(w,s+t+a\cdot h(u,w)+b\cdot h(v,w))} \tag{5.7}$$

where

$$h(u,w):=p_1^{k_1+i_1(w)-i_1(u)}\cdots p_l^{k_l+i_l(w)-i_l(u)},\ h(v,w):=p_1^{k_1+i_1(w)-i_1(v)}\cdots p_l^{k_l+i_l(w)-i_l(v)}.$$

5. 6. EXAMPLES FOR $n=p=2$: Let L be a compact abelian group, a an involutive automorphism on L, and $r\in L$ such that $a(r)=r$. Then there exists (up to isomorphism) a unique compact group G containing L as normal subgroup of index 2 such that there exists $b\in G-L$ satisfying $b^2=r$ and $b^{-1}xb=a(x)$ for all $x\in L$ (cf. Satz I.14.2 In Huppert [12]). In particular, if $r=e$ is the neutral element, then G is the semidirect product of L and $\boldsymbol{Z}(2):=\{id,a\}$.

We here remark that, by Eq. (5. 5), the dual hypergroup $\hat{G}$ does not depend on r which means that there exist non-isomorphic compact groups having isomorphic dual hypergroups. This reflects the fact that the groups under consideration are disconnected; in fact, it hat been shown in McMullen [16] that connected compact groups having isomorphic dual hypergroups are isomorphic.

We next record some concrete examples:

5.7. If $L=\boldsymbol{Z}(m)$ is the cyclic group of order m, if the automorphism a is given by $a(x)=x^{-1}$, and if $r=0$ is the identity element, then G is the dihedral group D_m. The above results yield that

$$\hat{D}_m \simeq \begin{cases} S(\boldsymbol{Z}(m)^{\boldsymbol{Z}(2)}, \{0\} \to \boldsymbol{Z}(2)) & \text{if } m \text{ is odd} \\ S(\boldsymbol{Z}(m)^{\boldsymbol{Z}(2)}, \{0, m/2\} \to \{0, m/2\} \times \boldsymbol{Z}(2)) & \text{if } m \text{ is even} \end{cases} \tag{5.8}$$

An explicit computation of all irreducible representations of D_m can be found in Section 27.62(d) of Hewitt and Ross [11].

5.8. If $L=\boldsymbol{Z}(2l)$ is the cyclic group of order $2l \geq 4$, if the automorphism a is given by $a(x)=x^{-1}$, and if $r=l$, then G is the generalized quaternion group Q_l. By (5.5), $\hat{Q}_l$ is isomorphic to $\hat{D}_{2l}$ although the groups Q_l and D_{2l} fail to be isomorphic.

5.9. Take $l \in \boldsymbol{N}$, $l \geq 3$, and fix $m := l^2 - 1$. Then $a(k) := kl \bmod m$ defines an involutive automorphism on the cyclic group $L=\boldsymbol{Z}(m)$. Identifying the dual $\hat{L}$ with $\boldsymbol{Z}(m)$, we get $\{\alpha \in \boldsymbol{Z}(m) = \hat{L} : \alpha \circ a = \alpha\} = \{j(l+1) : j=0, 1, \ldots, l-2\} =: F \simeq \boldsymbol{Z}(l-1)$. Hence, regarding F as subgroup of the orbit hypergroup $\hat{L}^{\{id,a\}} = \{\{\alpha, \alpha \circ a\} : \alpha \in \hat{L}\}$, the dual hypergroup $\hat{G}$ of the semidirect product $G : \boldsymbol{Z}(m) \ltimes \{id, a\}$ satisfies

$$\hat{G} \simeq S(\boldsymbol{Z}(m)^{\{id,a\}}, F \to F \times \boldsymbol{Z}(2)). \tag{5.9}$$

5.10. Let $L := \{z \in \boldsymbol{Z} : |z| = 1\}$ be the group of complex numbers of modulus 1. Let a be the involutive automorphism on L given by $a(z)=\bar{z}$. Taking $r=+1$ and $r=-1$ we obtain two (non-isomorphic) groups G_1 and G_2. In particular, $G_1 = L \ltimes \boldsymbol{Z}(2)$ is the group generated by all rotations and reflections of $\boldsymbol{R}^2$ which preserve the origin. Using (5.5), we see that $\hat{G}_1 \simeq \hat{G}_2 \simeq S(\boldsymbol{Z}^{\boldsymbol{Z}(2)}, (2\boldsymbol{Z})^{\boldsymbol{Z}(2)} \to (2\boldsymbol{Z})^{\boldsymbol{Z}(2)} \times \boldsymbol{Z}(2))$. In other words, the duals $\hat{G}_1$ and $\hat{G}_2$ are both isomorphic to the hypergroup that appears when the subhypergroup $2\boldsymbol{N}_0$ of the polynomial hypergroup $(\boldsymbol{N}_0, *)$ associated with the Tchebichef polynomials of the first kind will be substituted by $(2\boldsymbol{N}_0) \times \boldsymbol{Z}(2)$ (for details on polynomial hypergroups see Lasser [15]).

6. Hypergroups having subgroups of index 2

In this section we shall use substitution to describe all hypergroups having subgroups of index 2. It is clear that it is possible to generalize this classification to fixed finite indices $n \geq 3$; these cases, however, split into a great number of subcases and are difficult to handle explicitely. Restricted to groups, this classification is well known by a theorem of Schreier (cf. Huppert [12], Section I. 14).

Before we shall deal with subgroups of index 2 in Theorem 6. 4, we first consider cosets of a closed subgroup G of a hypergroup K. In this case, the sets $\{x\}*\{y\}$ and $\{y\}*\{x\}$ consist exactly of one element for all $x \in G$ and $y \in K$. We denote this element by xy and yx respectively. Similarly, we write xW and Wx instead of $\{x\}*W$ and $W*\{x\}$ respectively for $x \in G$ and $W \subset K$.

6. 1. PROPOSITION. *Let G be a closed normal subgroup of a hypergroup K. For each coset $xG \in K/G$, the set $H_{xG} := \{y \in G : yx = x\}$ does not depend on the representative x of the coset. H_{xG} is a compact normal subgroup of G, and G/H_{xG} is homeomorphic to xG. The mapping $\tau : K/G \to \mathscr{C}(G)$, $xG \mapsto H_{xG}$, is continuous.*

PROOF. Clearly H_{xG} is a subgroup of G independent of the representative x of the coset xG. As $(u^{-1}yu)x = u^{-1}(y(ux)) = u^{-1}(ux) = x$ for all $u \in G$ and $y \in H_{xG}$, we see that H_{xG} is normal in G. As $G \to xG$, $y \mapsto xy$, is continuous, H_{xG} is closed. As the natural projection p from G onto G/H_{xG} is open and continuous, the mapping $G/H_{xG} \to xG$, $yH_{xG} \mapsto xy$, is continuous and bijective. To prove that the inverse mapping is continuous, we consider the following mappings:

$$\begin{array}{ccccccc} xG & \to & \mathscr{C}(K) & \to & \mathscr{C}(G) & \to & \mathscr{C}(G/H_{xG}) \\ xy & \mapsto & \{\bar{x}\}*\{xy\} & \mapsto & (\{\bar{x}\}*\{xy\}) \cap G & \mapsto & p((\{\bar{x}\}*\{xy\}) \cap G) \end{array}$$

The first mapping is continuous by our assumption; the continuity of the second is clear while the third one is continuous by Theorem 5. 10. 1 of [17]. Lemma 4. 1B of [14] shows that $(\{\bar{x}\}*\{xy\}) \cap G = yH_{xG}$. Hence, $xG \to G/H_{xG}$, $xy \mapsto yH_{xG}$, is continuous. The continuity of $K/G \to \mathscr{C}(G)$, $xG \to H_{xG}$, is a consequence of $H_{xG} = (\{\bar{x}\}*\{x\}) \cap G$ and of the continuity of the convolution with respect to the Michael topology.

By Proposition 2. 2 Zeuner [32], any hypergroup on the torus T or on the real line $\boldsymbol{R}$ is isomorphic to the usual group on $(T, \cdot)$ or $(\boldsymbol{R}, +)$ respectively. As each closed subgroup H of T is either finite (and thus $T/H \simeq T$) or equal to T, and as $\{0\}$ is the only compact subgroup of $\boldsymbol{R}$, Proposition 6. 1 has the following consequences:

6. 2. COROLLARY. *Let G be a normal subhypergroup of a hypergroup K such that G is homeomorphic to the torus T. Then G is a group*

isomorphic to the usual group on T. Each coset of G in K is either homeomorphic to T or it consists of exactly one point. Moreover, for each $n\in\boldsymbol{N}$, the set $\{xG\in K/G:|H_{xG}|=n\}$ is open. In particular, the set of all $x\in K$ satisfying $xG=\{x\}$ is a closed subset of K.

6. 3. COROLLARY. *Let G be a normal subhypergroup of a hypergroup K such that G is homeomorphic to $\boldsymbol{R}$. Then G is a group isomorphic to the usual group on $\boldsymbol{R}$, $H_{xG}=\{e\}$ for each $x\in K$, and each coset of G in K is homeomorphic to $\boldsymbol{R}$.*

6. 4. THEOREM. *Let G be a locally compact group.*

Take a compact normal subgroup H of G, and let $\pi: G\rightarrow G/H$ be the natural projection. Fix an (continuous) automorphism h on G/H and $r\in G$ such that

$$h(rH)=rH \quad \text{and} \quad h(h(xH))=r^{-1}xrH \quad \text{for all } x\in G.$$

Moreover, take a measure $\rho\in M_b^+(G/H)$ having compact support such that $\|\rho\|<1$,

$$h(\rho)=\rho=\rho^{-}*\delta_{r^{-1}H} \quad \text{and} \quad \delta_{h(x)H}*\rho=\rho*\delta_{xH} \quad \text{for all } x\in G. \tag{6. 1}$$

Let $K:=G\cup G/H$ be the disjoint union of G and G/H, both sets being embedded as open subsets. We define a convolution $\bullet$ of Dirac measures on K as follows by

$$\begin{aligned}&\delta_x\bullet\delta_y=\delta_{xy}, \quad \delta_x\bullet\delta_{yH}=\delta_{xyH}, \quad \delta_{xH}\bullet\delta_y=\delta_{xH\,h(yH)},\\ &\delta_{xH}\bullet\delta_{yH}=(1-\|\rho\|)\cdot\tilde{\pi}^{-1}(\delta_{xH\,h(yH)r^{-1}H})+\delta_{xH\,h(yH)}*\rho\end{aligned} \tag{6. 2}$$

where $\tilde{\pi}^{-1}$ is the inverse of $\tilde{\pi}: M_b(G|H)\rightarrow M_b(G/H)$ (cf. Section 1. 7). The definition of $\bullet$ is independent of the representatives of H-cosets. $(K,\bullet):=K(G,H,h,r,\rho)$ is a hypergroup containing G as normal subgroup of index 2. The hypergroup involution $^{-}$ on $K-G=G/H$ is given by $(xH)^{-}:=h(rx^{-1}H)$.

Conversely, if $\tilde{K}$ is a hypergroup containing G as subgroup of index 2, then there exist H, h, r and ρ as described above such that $K(G,H,h,r,\rho)$ is isomorphic with $\tilde{K}$.

PROOF. Assume first that H, h, r and ρ are given as above. To show that $(K,\bullet)$ it a hypergroup, it suffices to consider the case $H=\{e\}$. In fact, the general case then follows from Theorem 2. 1 and

$$K(G,H,h,r,\rho)\simeq S(K(G/H,\{e\},h,r,\rho),\ G/H\xrightarrow{\pi}G)$$

by Eq. (6. 2). Suppose now that $H=\{e\}$. To avoid confusion of notation, we take $x_0 \in K-G=G/H$, and use the homeomorphism $p: G \to K-G, x \mapsto xx_0$. Then the convolution (6. 2) may be written as follows:

$$\delta_x \bullet \delta_y = \delta_{xy}, \quad \delta_x \bullet \delta_{p(y)} = \delta_{p(xy)}, \quad \delta_{p(x)} \bullet \delta_y = \delta_{p(xh(y))},$$
$$\delta_{p(x)} \bullet \delta_{p(y)} = (1-\|\rho\|)\, \delta_{xh(y)r^{-1}} + p(\delta_{xh(y)} * \rho) \quad (x, y \in G). \tag{6.3}$$

$K \times K \to M^1(K)$, $(x, y) \mapsto \delta_x \bullet \delta_y$, is weakly continuous and $\bullet$ can be extended to a bilinear mapping on $M_b(K)$ which is weakly continuous when restricted to $M_b^+(K)$.

To check that $\bullet$ is associative, we fix $x, y, z \in G$. By (6. 3), it is clear that

$$(\delta_x \bullet \delta_y) \bullet \delta_{p(z)} = \delta_x \bullet (\delta_y \bullet \delta_{p(z)}),\ (\delta_x \bullet \delta_{p(y)}) \bullet \delta_z = \delta_x \bullet (\delta_{p(y)} \bullet \delta_z),$$
$$(\delta_{p(x)} \bullet \delta_y) \bullet \delta_z = \delta_{p(x)} \bullet (\delta_y \bullet \delta_z), \text{ and } (\delta_x \bullet \delta_{p(y)}) \bullet \delta_{p(z)} = \delta_x \bullet (\delta_{p(y)} \bullet \delta_{p(z)}).$$

Moreover, as h is a group homomorphism, we have

$$\delta_{p(x)} \bullet (\delta_y \bullet \delta_{p(z)}) = (1-\|\rho\|)\, \delta_{xh(yz)r^{-1}} + p(\delta_{xh(yz)} * \rho) = (\delta_{p(x)} \bullet \delta_y) \bullet \delta_{p(z)}.$$

Furthermore, using $\rho * \delta_u = \delta_{h(u)} * \rho$ $(u \in G)$, we obtain that

$$(\delta_{p(x)} \bullet \delta_{p(y)}) \bullet \delta_z = (1-\|\rho\|)\, \delta_{xh(y)r^{-1}z} + p(\delta_{xh(y)} * \rho * \delta_{h(z)}) = \delta_{p(x)} \bullet (\delta_{p(y)} \bullet \delta_z).$$

Finally, as $h(\rho) = \rho$ and $\rho * \delta_u = \delta_{h(u)} * \rho$ for $u \in G$, we have

$$\begin{aligned}(\delta_{p(x)} \bullet \delta_{p(y)}) \bullet \delta_{p(z)} &= (1-\|\rho\|) p(\delta_{xh(y)r^{-1}z}) + (1-\|\rho\|)\, \delta_{xh(y)} * \rho * \delta_{h(z)r^{-1}} + \\ &\quad + p(\delta_{xh(y)} * \rho * \delta_{h(z)} * \rho) \\ &= \delta_{p(x)} \bullet (\delta_{p(y)} \bullet \delta_{p(z)}).\end{aligned}$$

Thus the proof of $\bullet$ being associative is complete. Obviously, the identity of G is the identity of K and the involution is given by $\bar{x} := x^{-1}$ and $\overline{p(x)} := p(r\, h(x^{-1}))$ for $x \in G$. It is easy to check that $(xy)^- = \bar{y}\bar{x}$, $(p(x)y)^- = \bar{y}\,\overline{p(x)}$ and $(x\, p(y))^- = \overline{p(y)}\,\bar{x}$ for all $x, y \in G$. Finally, $\delta_r * h(\rho^-) * \delta_{r^{-1}} = \delta_r * \rho = \rho * \delta_r$ yields

$$\begin{aligned}\delta_{\overline{p(y)}} \bullet \delta_{\overline{p(x)}} &= (1-\|\rho\|\, \delta_{rh(y^{-1}h(rh(x^{-1}))r^{-1}} + p(\delta_{rh(y^{-1})h(rh(x^{-1}))} * \rho) \\ &= (1-\|\rho\|)\, \delta_{rh(y^{-1})x^{-1}} + p(\delta_r * h(\rho^-) * \delta_{h(h(y^{-1})x^{-1})}) = (\delta_{p(x)} \bullet \delta_{p(y)})^-\end{aligned}$$

for all $x, y \in G$. This completes the proof of K being a hypergroup.

Now let $\tilde{K}$ be a hypergroup containing G as normal subgroup of index 2. Using Proposition 6. 1, we define $H := H_{xG}$ for $x \in \tilde{K} - G$. Since $xH = (\bar{H}\bar{x})^- = (H\bar{x})^- = \{x\} = Hx$ for $x \in \tilde{K} - G$, and since H is normal in G, H is normal in $\tilde{K}$. We may assume that $H = \{e\}$, since otherwise we could investigate G/H and $\tilde{K}/H$ instead of G and $\tilde{K}$, and since $\tilde{K} \simeq S(\tilde{K}/H, G/H \to G)$ holds (see Proposition 2. 4).
Suppose now that $H = \{e\}$. Fix $x_0 \in \tilde{K} - G$. Then $y \mapsto \tilde{y} := yx_0$ and

$y \mapsto x_0 y$ are homeomorphism from G onto $\tilde{K}-G$ by Proposition 6.1. Hence, there is a unique homeomorphisms $h: G \to G$ with $h(y)x_0 = x_0 y$ for all $y \in G$. As $h(yz)x_0 = x_0 yz = h(y)x_0 z = h(y)h(z)x_0$ for all $y, z \in G$, h is automorphism of G. Moreover, there is a unique $r \in G$ with $rx_0 = \bar{x}_0$. As $h(r)x_0 = rx_0$, we get $h(r) = r$. Moreover, for $y \in G$ we have $h(h(y))x_0 = x_0 h(y) = (h(y^{-1}\bar{x}_0))^- = (x_0 y^{-1} r)^- = r^{-1} y \bar{x}_0 = r^{-1} y r x_0$ and thus $h(h(y)) = r^{-1} y r$. We find a unique $\rho \in M_b^+(G)$ with $\rho * \delta_{x_0} = (\delta_{x_0} * \delta_{x_0})|_{\tilde{K}-G}$. As $\tilde{K}/G$ is a hypergroup, $\|\rho\| < 1$ holds. Moreover $rx_0 = \bar{x}_0$ yields $\operatorname{supp}(\{x_0\} * \{x_0\}) \cap G = \{r^{-1}\}$. In summary, the convolution on $\tilde{K}$ is given by

$$\begin{aligned} &\delta_z \bullet \delta_y = \delta_{zy}, \quad \delta_z \bullet \delta_{yx_0} = \delta_{zyx_0}, \quad \delta_{zx_0} \bullet \delta_y = \delta_{zh(y)x_0}, \\ &\delta_{zx_0} \bullet \delta_{yx_0} = (1 - \|\rho\|)\, \delta_{zh(y)r^{-1}} + \delta_{zh(y)} * \rho * \delta_{x_0} \quad (y, z \in G). \end{aligned} \tag{6.4}$$

Comparing (6.4) and (6.2), we see that $\tilde{K}$ and $K(G, \{e\}, h, r, \rho)$ are isomorphic.

It remains to check that ρ satisfies (6.1). For $y \in G$ we have

$$\rho * \delta_{h(y)} * \rho_{x_0} = \rho * \delta_{x_0} * \delta_y = (\delta_{x_0} * \delta_{x_0})|_{\tilde{K}-G} * \delta_y = \delta_{h(h(y))} * (\delta_{x_0} * \delta_{x_0})|_{\tilde{K}-G} = \delta_{h(h(y))} * \rho * \delta_{x_0}$$

and thus $\rho * \delta_y = \delta_{h(y)} * \rho$. In a similar way,

$$\rho * \delta_{r^{-1}} = (\rho * \delta_{x_0} * \delta_{x_0})|_G = (\delta_{x_0} * (\delta_{x_0} * \delta_{x_0})|_{\tilde{K}-G})|_G = h(\rho) * \delta_{r^{-1}}$$

and thus $\rho = h(\rho)$. Eq. (6.4) shows that the hypergroup involution on $\tilde{K}$ is given by $\bar{y} = y^{-1}$ and $(yx_0)^- = rh(y^{-1})x_0$. In particular, $\delta_{r^2} * \rho * \delta_{x_0} = \delta_r * h(\rho^-) * \delta_{x_0}$ yields $\delta_r * \rho = h(\rho^-)$. As $\delta_r * \rho = \rho * \delta_r$ and $h(\rho) = \rho$, it follows that $\rho^- = \rho * \delta_r$. Thus, the proof is complete.

6.5. Remarks. We keep the notation of Theorem 6.4.

(1) If ω_G is a left Haar measure of G, then $\pi(\omega_G) \in M^+(G/H)$ is a left Haar measure on G/H. Moreover, $\omega_G + \frac{1}{1-\|\rho\|}\pi(\omega_G) \in M^+(K)$ is a left Haar measure on K. This follows from Eq. (6.2) and the fact that $h(\pi(\omega_G)) = \pi(\omega_G)$ as a consequence of $h \circ h$ being an inner automorphism. A corresponding result holds for right Haar measures. Moreover, K is unimodular if and only if G is.

(2) K is commutative if and only if G is abelian and h is the identity. To determine $\hat{K}$ in the commutative case in terms of $\hat{G}$, we first notice that then $\rho \in M_b^+(G/H)$ and $r \in G$ satisfy $\rho^- * \delta_{rH} = \rho$ and $\|\rho\| < 1$. For $\alpha \in (G/H)^\wedge$ we define

$$u_\pm(\alpha) := u_\pm(\alpha; r, \rho) := \frac{1}{2}\Big(\hat{\rho}(\alpha) \pm \sqrt{\alpha(r^{-1}H)} \cdot \sqrt{|\hat{\rho}(\alpha)|^2 + 4(1 - \|\rho\|)}\Big) \tag{6.5}$$

for $\alpha \in \hat{G}$ where the, possibly complex, roots are taken arbitrarily but fixed and where $\hat{\rho} \in C_b(\hat{G})$ is the usual Fourier transform of ρ.
Let the mappings $E: \hat{G} \to C_b(K)$ and E_+, $E_-: (G/H)^\wedge \to C_b(K)$ be given by

$$E(\beta)(x) := \begin{cases} \beta(x) & \text{if } x \in G \\ 0 & \text{if } x \in K-G \end{cases} \quad \text{and}$$
$$E_\pm(\alpha)(x) := \begin{cases} \alpha(xH) & \text{if } x \in G \\ \alpha(x) u_\pm(\alpha) & \text{if } x \in K-G \end{cases}$$

Then the definition of the convolution on K and Theorem 3.5 show that

$$\hat{K} = E(\hat{G} - A(\hat{G}, H)) \cup E_+((G/H)^\wedge) \cup E_-((G/H)^\wedge)$$

where $E(\hat{G}-A(\hat{G}, H))$ is an open subset of $\hat{K}$ homeomorphic to $\hat{G}-A(\hat{G}, H)$.

We next discuss when the hypergroups K of Theorem 6.4 are isomorphic for different H, h, r and ρ. We may restrict our attention to the case $H=\{e\}$.

6.6. LEMMA. *Let G be a locally compact group and $H := \{e\}$. Let h, r, ρ and $\tilde{h}, \tilde{r}, \tilde{\rho}$ be sets of parameters each of them satisfying the assumptions of Theorem 6.1. Then there exists a hypergroup isomorphism*

$$\pi: K := K(G, \{e\}, h, r, \rho) \to \tilde{K} := K(G, \{e\}, \tilde{h}, \tilde{r}, \tilde{\rho}) \quad \textit{satisfying } \pi(G) = G$$

if and only if there exists an automorphism θ on G and $c \in G$ such that

$$\tilde{r} = \theta(r) c \tilde{h}(c), \quad \tilde{\rho} = \delta_{\tilde{h}(c^{-1})c^{-1}} * \theta(\rho) * \delta_c, \quad \textit{and}$$
$$\tilde{h}(\theta(x)) = c^{-1} \theta(h(x)) c \quad \textit{for all } x \in G.$$

PROOF. We identify $K-G$ and $\tilde{K}_G$ and retain the homeomorphism p between G and $K-G=\tilde{K}-G$ as in the proof of Theorem 6.4.

Assume first that π is an isomorphism as demanded in the lemma. Then $\theta := \pi|_G$ is an automorphism of G. Take $c \in G$ such that $cp(e) = p(c) = \pi(p(e))$. Then, for $x \in G$, $\pi(p(x)) = \theta(x) cp(e)$, $c\tilde{h}(\theta(x)) p(e) = \theta(h(x)) c p(e)$, and thus $\tilde{h}(\theta(x)) = c^{-1} \theta(h(x)) c$. Moreover,

$$\delta_{\theta(r^{-1})} + \theta(\rho) * \delta_c * \delta_{p(e)} = \delta_{c\tilde{h}(c)\tilde{r}^{-1}} + \delta_{c\tilde{h}(c)} * \tilde{\rho} * \delta_{p(e)}$$

proves that $\tilde{r} = \theta(r) c \tilde{h}(c)$ and $\tilde{\rho} = \delta_{\tilde{h}(c^{-1})c^{-1}} * \theta(\rho) * \delta_c$ as claimed.

Conversely, if $\theta \in \mathrm{Aut}(G)$ and $c \in G$ have the properties mentioned above, then $\pi(x) := \theta(x)$ and $\pi(p(x)) := \theta(x) cp(e)$ $(x \in G)$ defines a homeomorphism between K and $\tilde{K}$. When reading the equations of the first part of the proof backwards, it becomes clear that π is a hypergroup homeomorphism. This completes the proof.

6.7. COROLLARY. *Consider the following hypergroup structures on* $K:=T\times\{0,1\}$:

(1) *For* $c\in[0,1[$, $n\in\boldsymbol{N}$ *and* $r\in\{1,-1\}$, *let the hypergroup convolution on* $K=K_1(c,n,r)$ *be generated by*

$$\delta_{(x,0)}*\delta_{(y,0)}=\delta_{(xy,0)},\quad \delta_{(x,0)}*\delta_{(y,1)}=\delta_{(xy,1)},\quad \delta_{(x,1)}*\delta_{(y,0)}=\delta_{(xy^{-1},1)},$$

and $$\delta_{(x,1)}*\delta_{(y,1)}=\frac{1-c}{n}\cdot\sum_{k=0}^{n-1}\delta_{(xy^{-1}r)^{1/n}\exp(2\pi ik/n)}+c\cdot\int_T\delta_{(t,1)}\,d\omega(t)$$

for $x, y\in T$ *where* ω *stands for the normalized Haar measure on* T *and* $z^{1/n}$ *for an arbitrary but fixed* n-*th complex root of* $z\in T$.

(2) *Take* $n\in\boldsymbol{N}$ *and* $\rho\in M_b^+(T)$ *such that* ρ *has compact support,* $\rho=\rho^-$ *and* $\|\rho\|<1$. *Let the commutative hypergroup* $K_2(n,\rho)$ *be generated by*

$$\delta_{(x,0)}*\delta_{(y,0)}=\delta_{(xy,0)},\quad \delta_{(x,0)}*\delta_{(y,1)}=\delta_{(x,1)}*\delta_{(y,0)}=\delta_{(xy,1)},\quad \textit{and}$$

$$\delta_{(x,1)}*\delta_{(y,1)}=\frac{1-\|\rho\|}{n}\cdot\sum_{k=0}^{n-1}\delta_{(xy)^{1/n}\exp(2\pi ik/n)}+\int_T\delta_{(xyt,1)}\,d\rho(t)\quad \textit{for all } x, y\in T.$$

Then $K_1(c,n,r)$ *is isomorphic to* $K_1(\tilde{c},\tilde{n},\tilde{r})$ *if and only if* $c=\tilde{c}$, $n=\tilde{n}$ *and* $r=\tilde{r}$. *Moreover,* $K_2(n,\rho)$ *is isomorphic to* $K_2(\tilde{n},\tilde{\rho})$ *if and only if* $n=\tilde{n}$ *and* $\tilde{\rho}\in\{\rho,\rho*\delta_{-1}\}$.

Furthermore, if K *is any hypergroup structure on* $T\times\{0,1\}$, *then* K *is isomorphic to a hypergroup introduced either in part* (1) *or in part* (2).

PPOOF. Theorem 6.4 ensures that the convolutions above generate hypergroups. The statement regarding isomorphism is obvious in the first case as the hypergroup involution is the identity on $\{(x,1):x\in T\}$ if and only if $r=1$. The corresponding result in the second case follows from Lemma 6.6 and the facts that each automorphism on T is equal either to the identity or to the complex conjugation.

Now let K be a hypergroup on $T\times\{0,1\}$. We assume that $C_e:=\{(x,0):x\in T\}$ is the connected component of the identity element. By Proposition 2.2 of Zeuner [32] and Theorem 1.3 of Vrem [31], C_e is a normal subgroup of K isomorphic to the usual group on T. By Theorem 6.4, K is isomorphic to $K(T,H,h,r,\rho)$ for suitable indices H, h, r and ρ. As $T/H\simeq T$, H is the finite cyclic subgroup of T of a certain order $n\in\boldsymbol{N}$. Moreover, the involutive automorphism h on $T/H\simeq T$ is either the complex conjugation or the identity. In the first case, we conclude from

Eq. (6.1) that either $\rho=0$ or that ρ is a Haar measure on $T/H\simeq T$ with $c:=\|\rho\|<1$. As $r=\pm 1$ by Theorem 6.4, K is isomorphic to a hypergroup considered in case (1). Assume now that h is the identity mapping. Then $r\in T$ admits a square root in T, and the hypergroups $K(T,\{e\}, id, r, \rho)$ and $K(T,\{e\}, id, 0, \delta_{r^{1/2}}*\rho)$ are isomorphic by Lemma 6.6. Hence, $K(T, H, id, r, \rho)$ and $K(T, H, id, 0, \delta_{r^{1/2}}*\rho)$ are isomorphic which ensures that K is isomorphic to a hypergroup as considered in case (2). Thus the proof is complete.

The following result can be derived in the same way from the fact that each compact subgroup of $\boldsymbol{R}$ is trivial.

6.8. COROLLARY. *Consider the following hypergroups on the set* $K:=\boldsymbol{R}\times\{0,1\}$:

(1) $K=\boldsymbol{R}\ltimes\boldsymbol{Z}(2)$ *is the semidirect product of the groups* $\boldsymbol{R}$ *and* $\boldsymbol{Z}(2)$ *(where the non-trivial element of* $\boldsymbol{Z}(2)$ *acts on* $\boldsymbol{R}$ *by taking the inverse element).*

(2) *Take* $\rho\in M_b^+(\boldsymbol{R})$ *such that* ρ *has compact support and* $\rho=\rho^-$ *and* $\|\rho\|<1$ *hold. Let the commutative hypergroup convolution on* $K=K(\rho)$ *be generated by*

$$\delta_{(x,0)}*\delta_{(y,0)}=\delta_{(x+y,0)},\quad \delta_{(x,0)}*\delta_{(y,1)}=\delta_{(x,1)}*\delta_{(y,0)}=\delta_{(x+y,1)},\quad \textit{and}$$
$$\delta_{(x,1)}*\delta_{(y,1)}=(1-\|\rho\|)\cdot\delta_{(x+y,0)}+\int_{\boldsymbol{R}}\delta_{(x+y+t,1)}\,d\rho(t)\quad \textit{for all } x,y\in\boldsymbol{R}.$$

Then $K(\rho)$ *is isomorphic to* $K(\tilde{\rho})$ *if and only if there exists* $a>0$ *such that* $\tilde{\rho}(A)=\rho(a\cdot A)$ *for all Borel set* $A\subset\boldsymbol{R}$ *(where* $a\cdot A=\{ax: x\in A\}$*).*

Furthermore, if K *is any hypergroup structure on* $\boldsymbol{R}\times\{0,1\}$*, then* K *is isomorphic either to the group* $\boldsymbol{R}\ltimes\boldsymbol{Z}(2)$ *or to a hypergroup* $K(\rho)$ *as introduced in* (2).

6.9. Consider the following hypergroups on $K:=\boldsymbol{Z}\times\{0,1\}$:

(1) $K=\boldsymbol{Z}\ltimes\boldsymbol{Z}_2$ is the semidirect product of the groups $\boldsymbol{Z}$ and $\boldsymbol{Z}_2$ (where the nontrivial element of $\boldsymbol{Z}_2$ acts on $\boldsymbol{Z}$ by taking the inverse element).

(2) Take $r\in\{0,1\}$ and $\rho\in M_b^+(\boldsymbol{Z})$ such that ρ is finitely supported, $\rho=\rho^-*\delta_{-r}$, and $\|\rho\|<1$. Let the hypergroup convolution on $K=K(r,\rho)$ be generated by

$$\delta_{(m,0)}*\delta_{(n,0)}=\delta_{(m+n,0)},\quad \delta_{(m,0)}*\delta_{(n,1)}=\delta_{(m,1)}*\delta_{(n,0)}=\delta_{(m+n,1)},\quad \text{and}$$
$$\delta_{(m,1)}*\delta_{(n,1)}=(1-\|\rho\|)\cdot\delta_{(m+n-r,0)}+\int_{\boldsymbol{Z}}\delta_{(m+n+k,1)}\,d\rho(k)\quad \text{for all } m,n\in\boldsymbol{Z}.$$

It can be shown as in Corollary 6.7 that (1) and (2) define hypergroups, and that the hypergroups $K(r, \rho)$ and $K(\tilde{r}, \tilde{\rho})$ are isomorphic if and only if $r=\tilde{r}$ and $\rho=\tilde{\rho}$.

Moreover, if K is any hypergroup containing the group $\boldsymbol{Z}$ as normal subgroup of index 2, then K is isomorphic either to the group $\boldsymbol{Z}\ltimes\boldsymbol{Z}_2$ or to a commutative hypergroup as introduced in (2). This follows again from Theorem 6.4.

To determine the dual of $K(r, \rho)$, we have to consider two different cases:

If $r=0$, then Remark 6.5(2) shows that all characters of $K(0, \rho)$ are given by

$$\alpha_{\pm,z}(n, i)=\begin{cases} z^n & \text{if } i=0 \\ z^n(\hat{\rho}(z)\pm\sqrt{|\hat{\rho}(z)|^2+4(1-\|\rho\|)})/2 & \text{if } i=1 \end{cases} \tag{6.6}$$

where $n\in\boldsymbol{Z}$, $z\in T$. Obviously, $K(0, \rho)^{\wedge}$ is homeomorphic to $T\times\{0, 1\}$.

If $r=1$, then all characters of $K(1, \rho)$ are given by

$$\alpha_z(n, i)=\begin{cases} z^{2n} & \text{if } i=0 \\ z^{2n}\left(\hat{\rho}(z^2)+z\sqrt{|\hat{\rho}(z^2)|^2+4(1-\|\rho\|)}\right)/2 & \text{if } i=1 \end{cases} \tag{6.7}$$

where $n\in\boldsymbol{Z}$ and $z\in T$. This is a consequence of Remark 6.5 and a suitable parametrization of $K(1, \rho)^{\wedge}$. In particular $K(1, \rho)^{\wedge}$ is homeomorphic to the torus T.

As each hypergroup structure on T is isomorphic to the usual group structure on T by Proposition 2, 2 of Zeuner [32], $K(1, \rho)^{\wedge}$ carries a dual hypergroup structure if and only if $K(1, \rho)$ is a group isomorphic to $\boldsymbol{Z}$, i.e., $\rho=0$.

References

[1] BLOOM, W., HEYER, H.: The Fourier transformation of probability measures on hypergroups. Rend. Math. 2, 315-334 (1982).

[2] BLOOM, W., HEYER, H.: Convolution semigroups and resolvent families of measures on hypergroups. Math. Z. 188, 449-474 (1985).

[3] CONNETT, W.C., SCHWARTZ, A.L.: Product formulas, hypergroups and the Jacobi polynomials. Bull. Amer. Math. Soc. 22, 91-96 (1990).

[4] DUNKL, C.F.: Structure hypergroups for measure algebras. Pacific J. Math. 47, 413-425 (1973).

[5] DUNKL, C.F.: The measure algebra of a locally compact hypergroup. Trans. Amer. Math. Soc. 179, 331-348 (1973).

[6] DUNKL, C.F., RAMIREZ, D.E.: A family of countable compact P*-hypergroups. Trans. Amer. Math. Soc. 202, 339-356 (1975).

[7] FOURNIER, J.J.F., ROSS, K.A.: Random Fourier series on compact Abelian hypergroups. J. Austral. Math. Soc. (Series A) 37, 45-81 (1984).
[8] HARTMANN, K.: $[FIA]_{\bar{B}}$-Gruppen und Hypergruppen. Mh. Math. 89, 9-17 (1980).
[9] HARTMANN, K., HENRICHS, R.W., LASSER, R.: Duals of orbit spaces in groups with relatively compact inner automorphism groups are hypergroups. Mh. Math. 88, 229-238 (1979).
[10] HERMANN, P.: Induced representations of hypergroups. Math. Z. 210, 687-699 (1992).
[11] HEWITT, E., ROSS, K.A.: Abstract Harmonic Analysis I, II. Berlin - Heideberg - New York : Springer 1979, 1790.
[12] HUPPERT, B.: Endliche Gruppen I. Berlin - Heidelberg - New York : Springer 1967.
[13] ILTIS, R.: Some algebraic structure in the dual of a compact group. Canad. J. Math. 20, 1499-1510 (1968).
[14] JEWETT, R.I.: Spaces with an abstract convolution of measures. Adv. Math. 18, 1-101 (1975).
[15] LASSER, R.: Orthogonal polynomials and hypergroups. Rend. Math. Appl. 2, 185-209 (1983).
[16] MCMULLEN, J.R.: On the dual object of a compact connected group. Math. Z. 185, 539-552 (1984).
[17] MICHAEL, E.: Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71, 152-182 (1951).
]18] ROSS, K.A.: Hypergroups and centers of measure algebras. Symposia Math. 22, 189-203 (1977).
[19] ROSS, K.A.: Centers of hypergroups. Trans. Amer. Math. Soc. 243, 251-269 (1978).
[20] SCHWARTZ, A.L.; Classification of one-dimensional hypergroups. Proc. Amer. Math. Soc. 103, 1073-1081 (1988).
[21] SERRE, J.P.: Linear Representations of Finite Groups. Berlin - Heidelberg - New York : Springer 1977.
[22] SKANTHARAJAH, M.: Amenable hypergroups. Illinois J. Math. 36, 15-46 (1992).
[23] SPECTOR, R.: Mesures invariants sur les hypergroups. Trans. Amer. Math. Soc. 239, 147-166 (1978).
[24] VOIT, M.: Positive characters on commutative hypergroups and some applications. Math. Z. 198, 405-421 (1988).
[25] VOIT, M.: Factorization of probability measures on symmetric hypergroups. J. Austral. Math. Soc. (Series A) 50, 417-467 (1991).
[26] VOIT, M.: Duals of subhypergroups and quotients of commutative hypergroups. Math. Z. 210, 289-304 (1992).
[27] VOIT, M.: A generalization of orbital morphisms of hypergroups. In : Probability Measures on Groups X. Proc. Conf., Oberwolfach, 1990, 425-433. Plenum Press 1992.
[28] VOIT, M.: Projective and inductive limits of hypergroups. Proc. London Math. Soc., 67, 617-648 (1993).
[29] VREM, R.: Harmonic analysis on compact hypergroups. Pacific J. Math. 85, 239-251 (1979).
[30] VREM, R.: Hypergroup joins and their dual objects. Pacific J. Math. 111, 483-495 (1984).
[31] VREM, R.: Connectivity and supernormality results for hypergroups. Math. Z. 195, 419-428 (1987).
[32] ZEUNER, Hm.: One-dimensonal hypergroups. Adv. Math. 76, 1-18 (1989).

[33] ZEUNER, Hm.: Duality of commutative hypergroups. In: Probability Measures on Groups X. Proc. Conf., Oberwolfach, 1990, 467-488. Plenum Press 1992.

[34] ZEUMER, Hm.: Polynomial hypergroups in several variables. Arch. Math. 58, 425-434 (1992).

Mathematisches Institut
Technische Universität München
Arcisstr. 21, 80333 München
Germany