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Bounded subsets in spaces of distributions of L^{p}-growth

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Abstract

In this paper we characterize bounded subsets of the spaces $\mathcal{D}'_{L^{p}}$, $1\leq p\leq\infty$, of distributions of L^{p} growth. Moreover, we give necessary and sufficient conditions on a sequence in $\mathcal{D}'_{L^{p}}$ to converge to 0.

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The space $\mathcal{D}'_{L^{p}}$ of distributions of L^{p} growth has been studied by several authors in the past few years. Pahk [6] gave necessary and sufficient conditions for a convolution operator in $\mathcal{D}'_{L^{p}}$ to be hypoelliptic. Ortner and Wagner [5] considered the convolution of elements in these and corresponding weghted spaces and some other related questions. In [1] the spaces of convolution operators and multipliers of these spaces and their topologies are studied. The characterization of bounded sets in ultradistribution spaces $\mathcal{D}'^{(M_{p})}_{L^{p}}$ is given in [7].

In this work we characterize bounded subsets of the spaces $\mathcal{D}'_{L^{p}}$, and characterize convergent sequences in these spaces.

We use the standard notations as in [7] and [3]. We consider q in the interval $[1, \infty]$ and $p=q/(q-1)$ is its conjugate number; if $q=1$ then $p=\infty$. Recall [9], the space $\mathcal{D}_{L^{q}}$, $q\in[1,\infty]$, consists of all the functions ϕ in $C^{\infty}(\mathbf{R}^{n})$ such that $D^{\alpha}\phi$ in L^{q} for all α in $\mathbf{N}_{0}^{n}$ ($\mathbf{N}_{0}=\mathbf{N}\cup\{0\}$), provided with the topology defined by the seminorms

$$\|\phi\|_{m,q}=\sup_{\alpha\leq m}\|D^{\alpha}\phi\|_{L^{q}},\ m\in\mathbf{N}_{0}.$$

$\dot{\mathcal{B}}$ is the completion of $\mathcal{D}$ in $\mathcal{D}_{L^{\infty}}$; its dual is $\mathcal{D}'_{L^{1}}$. The dual of $\mathcal{D}_{L^{q}}$, $q\in[1, \infty)$, is denoted by $\mathcal{D}'_{L^{p}}$, where p is the conjugate number for q.

In the main Theorem 2 we shall use the fact that $\mathcal{D}_{K}$ is dense in $\mathcal{D}_{K,r}$, $r\in\mathbf{N}$, where K is a compact set in $\mathbf{R}^{n}$ and $\mathcal{D}_{K,r}$ is the space of functions supported by K which have all the derivatives up to r continuous, supplied with the usual norm.

THEOREM 1. *Let B' be a subset of $\mathcal{D}'_{L^{p}}$, $p\in[1, \infty]$. The following conditions are equivalent:*

(i) *B' is bounded.*

(ii) *For every bounded subset B of $\mathcal{D}_{L^q}$, $p\in(1,\infty]$, $q=p/(p-1)$, (reap. of $\dot{\mathcal{B}}$ if $p=1$)*

$$\sup\{|T*\phi(x)|\,;\ T\in B',\ \phi\in B,\ x\in\mathbf{R}^n\}<\infty.$$

(iii) *For every bounded open set $\Omega\subset\mathbf{R}^n$ and every $\phi\in\mathcal{D}_{L^q}$, $p\in(1,\infty]$, $q=p/(p-1)$, (resp. $\phi\in\dot{\mathcal{B}}$ if $p=1$)*

$$\sup\{|T*\phi)(x)|,\ T\in B',\ x\in\Omega\}<\infty.$$

PROOF. The spaces $\mathcal{D}_{L^q}$, $q\in[1,\infty)$ and $\dot{\mathcal{B}}$ are barrelled which implies that the weak and strong boundedness in the corresponding strong duals are equivalent. Also, this implies that a set B' is bounded in the strong dual topology if and only if for every bounded set B in the basic space

$$\sup\{|<T,\phi>|\,;\ T\in B',\ \phi\in B\}<\infty.$$

Since $B\subset\mathcal{D}_{L^q}$, $q\in[1,\infty)$(resp. $B\subset\mathcal{B}$), is bounded if and only if

$$\{\phi(x-\cdot),\ \phi\in B,\ x\in\mathbf{R}^n\}$$

is bounded in $\mathcal{D}_{L^q}$ (resp. in $\dot{\mathcal{B}}$), the proof of the theorem simply follows.

The following theorem characterizes the convergence in $\mathcal{D}'_{L^p}$.

THEOREM 2. *Let $p\in[1,\infty]$ and T_j, $j\in\mathbf{N}$, be a sequence in $\mathcal{D}'$ such that for every $\psi\in\mathcal{D}$, $T_j*\psi$, $j\in\mathbf{N}$, is a sequence from $\mathcal{D}'_{L^p}$ which converges to 0 in $\mathcal{D}'_{L^p}$ as $j\to\infty$. Then T_j converges to 0 in $\mathcal{D}'_{L^p}$.*

PROOF. By [8], any $\psi\in\mathcal{D}$ is of the form

$$\psi=\sum_{i=1}^{N}\psi_i*\phi_i,\quad \psi_i,\ \phi_i\in\mathcal{D},\quad i=1,\dots,N. \tag{1}$$

This implies that $T_j*\psi=\sum_{i=1}^{N}(T_j*\psi_i)*\phi_i$, $j\in\mathbf{N}$, and by [9], for every $i=1,\dots,N$

$$T_j*\psi_i=\sum_{s=0}^{m_i}T_{j,i,s}^{(s)},\qquad j\in\mathbf{N},$$

where $T_{j,i,s}$, $j\in\mathbf{N}$, is a sequence in L^p which converges to 0 in L^p. This implies that for every $s=0,1,\dots,m_i$, $i=1,\dots,N$

$$T_{j,i,s}*\phi_i\to 0 \text{ in } L^p \text{ as } j\to\infty.$$

Thus, the assumption of the theorem implies that for every $\psi\in\mathcal{D}$, $T_j*\psi$, $j\in\mathbf{N}$, is a sequence from L^p which converges to 0 in L^p. By using (1) again, we have $T_j\to 0$ in $\mathcal{D}'$, $j\to\infty$.

Let K be a compact set in $\mathbf{R}^n$ and

$$B_1=\{\theta\in L^p\,;\ \|\theta\|_{L^p}=1\}.$$

Let $\varphi\in\mathfrak{D}_K$. Because $\{T_j*\varphi,\ j\in\mathbf{N}\}$ is bounded in L^p,

$$\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}|<T_j*\psi,\varphi>|=\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}|<T_j*\varphi,\ \psi>|<\infty.$$

Thus, $\{T_j*\psi\,;j\in\mathbf{N},\ \psi\in\mathfrak{D}\cap B_1\}$ is equicontinuous in $\mathfrak{D}'_K$ and there exists a neighbourhood of zero in $\mathfrak{D}_K$

$$V_r(\epsilon)=\{\theta\in\mathfrak{D}_K\,;\ \|\theta\|_{K,r}\leq\epsilon\}$$

such that

$$\theta\in V_r(\epsilon)\Rightarrow\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}|<T_j*\check{\psi}\ \check{\theta}>|=\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}|<T_j*\theta,\ \psi>|\leq 1.$$

The same holds for the closure of $V_r(\epsilon)$ in $\mathfrak{D}_{K,r}$ since $\mathfrak{D}\cap B_1$ is dense in B_1. This implies that for every $\theta\in\mathfrak{D}_{K,r}$, $T_j*\theta\in L^p$, $j\in\mathbf{N}$, and there exists $C>0$ such that

$$\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}\|<T_j*\check{\psi}\ \check{\theta}>|=\sup_{\substack{j\in\mathbf{N}\\ \psi\in\mathfrak{D}\cap B_1}}|T_j*\theta,\ \psi>|\leq C.$$

Thus, for any $\psi\in\mathfrak{D}$ we have

$$\sup_{j\in\mathbf{N}}|<T_j*\theta,\ \psi>|\leq C\|\psi\|_{L^p},$$

i. e. for every $\theta\in\mathfrak{D}_{K,r}$ the set $\{T_j*\theta\,;\ j\in\mathbf{N}\}$ is bounded in L^p. By using [9] we have that for suitable compact neigbourhood of zero ω, $\bar{\omega}=K$, $r\in\mathbf{N}$ and $m\in\mathbf{N}$, there are $\theta\in\mathfrak{D}_{K,r}$ and $\phi\in\mathfrak{D}_K$ such that

$$T_j=\Delta^m T_j*\theta+T_j*\phi,\ j\in\mathbf{N}.$$

This implies that $\{T_j,j\in\mathbf{N}\}$ is bounded in $\mathfrak{D}'_{L^p}$.

From this theorem and its proof we have:

COROLLARY 1. *Let T_j be a sequence in $\mathfrak{D}'_{L^p}$, $p\in[1,\infty]$. It coverges to 0 in D'_{L^p} if and only if $T_j*\psi$ converges to 0 in L^∞ for every $\psi\in\mathfrak{D}(j\to\infty)$.*

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