Hokkaido Mathematical Journal Vol. 26 (1997) p. 439-449

On the spectrum of Dirac operators with the unbounded potential at infinity

(Dedicated to Professor Kôji Kubota on his sixtieth birthday)

Osanobu YAMADA

(Received April 24, 1996)

Abstract. In this paper we investigate the spectra of Dirac operators

$$H=\sum_{j=1}^{3}\alpha_{j}D_{j}+p(x)\beta+q(x)I_{4}$$

in the Hilbert space $[L^{2}(\mathbf{R}^{3})]^{4}$. We show mainly that if $|p(x)|\longrightarrow\infty$, $q(x)=o(p(x))$ as $|x|\longrightarrow\infty$, then the spectrum of H is purely discrete in the whole line $\mathbf{R}$, and if $p(x)\equiv q(x)\longrightarrow\infty$ as $|x|\longrightarrow\infty$, then the spectrum of H is purely discrete in the half line $\mathbf{R}^{+}$.

Key words: Dirac operators, purely discrete spectrum.

1. Introduction and Results

In this paper we consider the following type of Dirac operators

$$L=\sum_{j=1}^{3}\alpha_{j}D_{j}+p(x)\beta+q(x)I_{4},\quad x\in\mathbf{R}^{3},\quad D_{j}=-i\frac{\partial}{\partial x_{j}},$$

on $\mathcal{H}=[L^{2}(\mathbf{R}^{3})]^{4}$, where $p(x)$ and $q(x)$ are real-valued continuous functions, and

$$\alpha_{j}=\begin{pmatrix}\mathbf{0}&\sigma_{j}\\ \sigma_{j}&\mathbf{0}\end{pmatrix}\quad(1\leq j\leq 3),\quad \beta=\begin{pmatrix}I_{2}&\mathbf{0}\\ \mathbf{0}&-I_{2}\end{pmatrix},\quad I_{4}=\begin{pmatrix}I_{2}&\mathbf{0}\\ \mathbf{0}&I_{2}\end{pmatrix},$$

$$\sigma_{1}=\begin{pmatrix}0&1\\ 1&0\end{pmatrix},\quad \sigma_{2}=\begin{pmatrix}0&-i\\ i&0\end{pmatrix},\quad \sigma_{3}=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix},\quad I_{2}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}.$$

The matrices α_{j} $(1\leq j\leq 3)$ and $\alpha_{4}=\beta$ are Hermitian symmetric matrices satisfying the anti-commutation relations

$$\alpha_{j}\alpha_{k}+\alpha_{k}\alpha_{j}=2\delta_{jk}I_{4}\quad(1\leq j,\ k\leq 4). \tag{1}$$

1991 Mathematics Subject Classification : 35P05.

The symmetric operator L defined on $[C_0^\infty(\mathbf{R}^3)]^4$ is essentially selfadjoint (see, e.g., Jörgens [J]). We denote the unique selfadjoint realization by H.

Our interest here is to investigate the spectrum of the Dirac operator H satisfying

$$|p(x)| \longrightarrow \infty, \quad q(x) = o(p(x)) \ \text{ as } \ |x| \longrightarrow \infty \tag{2}$$

or

$$p(x) \equiv q(x) \longrightarrow \infty \ \text{ as } \ |x| \longrightarrow \infty. \tag{3}$$

Recently, the studies of Dirac operators satisfying (2) or (3) appear in physical articles (see, e.g., Ikhdair–Mustafa–Sever [IMS], Jena–Tripati [JT], Ram–Halasa [RH]). Although the numerical analysis of the eigenvalues is studied there, it seems that the mathematical structure of the spectrum of H is not written explicitely. Their potentials are of types

$p(x) = a|x|^2 + b, \quad q(x) \equiv 0 \quad$ in [RH],

$p(x) \equiv q(x) = a|x|^\nu + b \ \ (\nu = 0.1) \quad$ in [JT] and [IMS],

$p(x) \equiv q(x) = a \log |x| + b \quad$ in [IMS],

where $a > 0$ and b are some real numbers.

If we assume that $p(x) \equiv 1$ and $|q(x)| \longrightarrow \infty$ ($|x| \to \infty$), it is shown by Titchmarsh [T] and Erdélyi [E] that the absolutely continuous spectrum of H covers the whole line, and the singular spectrum of H is empty under the condition that $q(x) = q(|x|)$ is spherically symmetric and

$$\int_R^{+\infty} \frac{|q'(r)|}{q(r)^2} dr < \infty \ \text{ for some } \ R > 0$$

(see also Thaller [Th], Chapter 4 and Schmidt [Sc]). On the other hand, if we assume (2) or (3), we have the different structure of spectrum of H, which we will study in this paper.

Before we state our results, we explain some notations:

$\sigma(H)$ = the spectrum of H, i.e., the complement of the resolvent set of H,

$\sigma_p(H)$ = the set of all the eigenvalues of H,

$\sigma_d(H)$ = the set of all the isolated eigenvalues of H with finite multiplicity,

$\sigma_{ess}(H) = \sigma(H) \setminus \sigma_d(H)$,

$\mathbf{R}^+ = (0, +\infty), \quad \mathbf{R}^- = (-\infty, 0)$.

Our results are as follows;

Theorem 1 *Assume that $p(x) \in C^1$ and $q(x) \in C^1$ satisfy the following conditions*:

(a–1) $|p(x)| \longrightarrow \infty$ as $|x| \longrightarrow \infty$,

(a–2) *There exist positive constants R and $\varepsilon_0 < 1$ such that*

$$|q(x)| \leq \varepsilon_0 |p(x)| \quad (|x| \geq R).$$

(a–3) $|\nabla p(x)| = O(p(x))$, $|\nabla q(x)| = O(p(x))$ *as* $|x| \longrightarrow \infty$.

Then we have $\sigma(H) = \sigma_d(H)$.

Theorem 2 *Assume that $p(x) \equiv q(x) \in C^0$ satisfies*

(b–1) $q(x) \longrightarrow \infty$ *as* $|x| \longrightarrow \infty$.

Then we have $\sigma(H) \cap \mathbf{R}^+ = \sigma_d(H)$.

Concerning the negative spectrum of H under the assumption of Theorem 2, we have a result for a class of potentials $q(x) = O(|x|^2)$ at infinity, as follows.

Proposition 3 *Assume that $p(x) \equiv q(x) \in C^0$ with the radial derivative satisfies*

(c–1) $q(x) \longrightarrow \infty$ *as* $|x| \longrightarrow \infty$,

(c–2) *There are positive constants C, R and $1 \leq \alpha \leq 2$ such that*

$$q(x) \leq C|x|^\alpha, \quad \frac{2(\alpha - 1)}{r} q(x) \leq \frac{\partial q}{\partial r} \quad (|x| \geq R),$$

where $r = |x|$. Then we have $\mathbf{R}^- \subset \sigma_{ess}(H)$ and $\sigma_p(H) \cap (\mathbf{R}^- \cup \{0\}) = \emptyset$.

Theorem 1 will be proved in §2, and Theorem 2 in §3. The proof of Proposition 3 and some remarks will be given in §4. In Theorems 1, 2 and Proposition 3 we may allow some local singularities of $p(x)$ and $q(x)$, which we omit for the sake of simplicity.

Example. If $q(x)$ is a positive homogeneous function of degree $0 < \theta \leq 2$, then $q(x)$ satisfies (c–1) and (c–2) with $\alpha = 1 + (\theta/2)$. $q(x) = \log r$ satisfies (c–1) and (c–2) with $\alpha = 1$.

2. Proof of Theorem 1

We prove under the assumption that the resolvent $(H - i)^{-1}$ is a compact operator in $\mathcal{H} = [L^2(\mathbf{R}^3)]^4$, which yields Theorem 1. Let $\{f_n\}_{n=1,2,\cdots}$ be any bounded sequence in $\mathcal{H}$, say, $\|f_n\| \leq C(n = 1, 2, \cdots)$ for a positive constant C, where $\|\ \ \|$ is the norm in $\mathcal{H}$. Then we set

$$u_n = (H - i)^{-1} f_n \in \mathcal{H}^1_{loc} := [H^1_{loc}]^4 \quad n = 1, 2, \dots,$$

where H^1_{loc} is the local Sobolev space of all functions locally in the Sobolev space H^1. The sequence $\{u_n\}_{n=1,2,\dots}$ clearly satisfies

$$\|u_n\| \leq \|f_n\| \leq C \quad (n = 1, 2, \cdots) \tag{4}$$

Let us put $P(x) = \sqrt{p(x)^2 - q(x)^2}$ $(|x| \geq R)$ and $P(x) = 0$ $(|x| \leq R)$. We show below that the sequence $\{u_n\}_{n=1,2,\cdots}$ is bounded in a Hilbert space

$$\mathcal{H}_P = \left\{ g \in \mathcal{H} \mid \|g\|_P^2 := \|g\|^2 + \|Pg\|^2 + \sum_{j=1}^{3} \|D_j g\|^2 < \infty \right\}$$

with the inner product

$$(f, g)_P = (f, g) + (Pf, Pg) + \sum_{j=1}^{3} (D_j f, D_j g),$$

where $\|\ \ \|$ and $(\ ,\ )$ are the usual norm and the inner product in $\mathcal{H}$, respectively. The assumption (a–2) of Theorem 1 gives

$$P(x)^2 = p(x)^2 - q(x)^2 \leq p(x)^2 \leq \frac{1}{1 - \varepsilon_0^2} P(x)^2 \quad (|x| \geq R). \tag{5}$$

which implies

$$\mathcal{H}_P = \left\{ g \in \mathcal{H} \mid \|g\|^2 + \|pg\|^2 + \sum_{j=1}^{3} \|D_j g\|^2 < \infty \right\}.$$

The sesquilinear forms $(f, g)_P$ and (f, g) are also used for $f \in [\mathcal{D}']^4$ and $g \in \mathcal{D}^4$, where $\mathcal{D} = C_0^\infty(\mathbf{R}^3)$ and $\mathcal{D}'$ is the space of distributions on $\mathbf{R}^3$.

Operating $\vec{\alpha} \cdot \vec{D} = \sum_{j=1}^{3} \alpha_j D_j$ to

$$(\vec{\alpha} \cdot \vec{D}) u_n + p(x) \beta u_n + q(x) u_n - i u_n = f_n$$

and using the anti-commutation relation (1) we have

$$\begin{aligned} -\Delta u_n &+ [p(x)^2 - q(x)^2 + 1] u_n + [2iq(x) + (\vec{\alpha} \cdot \vec{D} p) \beta + (\vec{\alpha} \cdot \vec{D} q)] u_n \\ &= (\vec{\alpha} \cdot \vec{D}) f_n + [p(x) \beta - q(x) + i] f_n. \end{aligned} \tag{6}$$

Take a C^∞ function $\gamma(x)$ such that $\gamma(x) = 1 \quad (|x| \geq R+1)$ and $\gamma(x) = 0 \quad (|x| \leq R)$. For any $\psi \in \mathcal{D}^4$ we have

$$\begin{aligned}(\gamma u_n, \psi)_P &= (-\Delta(\gamma u_n) + [1 + P(x)^2](\gamma u_n), \psi) \\ &= (-(\Delta\gamma)u_n - 2\vec{\nabla}\gamma \cdot \vec{\nabla} u_n - \gamma \Delta u_n \\ &\qquad\qquad\qquad + [1 + P(x)^2](\gamma u_n), \psi),\end{aligned}$$

and, by using (6),

$$\begin{aligned}(\gamma u_n, \psi)_P &= -(u_n, (\Delta\gamma)\psi) + 2(u_n, \vec{\nabla} \cdot [(\vec{\nabla}\gamma)\psi]) \\ &\quad + (f_n, (\vec{\alpha} \cdot \vec{D}\gamma)\psi) + (f_n, \gamma(\vec{\alpha} \cdot \vec{D})\psi + \gamma[p\beta - q - i]\psi) \\ &\quad + (u_n, \gamma[2iq + \beta(\vec{\alpha} \cdot \vec{D}p) + (\vec{\alpha} \cdot \vec{D}q)]\psi).\end{aligned}$$

Therefore we can find a positive constant C_1 from (4), (5) and the assumptions (a–2), (a–3) such that

$$|(\gamma u_n, \psi)_P| \leq C_1(\|f_n\| + \|u_n\|) \cdot \|\psi\|_P \leq 2CC_1\|\psi\|_P \quad (\forall \psi \in \mathcal{D}^4).$$

Since $\mathcal{D}^4$ is dense in $\mathcal{H}_P$, we have $\gamma u_n \in \mathcal{H}_P$ and

$$\|\gamma u_n\|_P \leq 2CC_1, \quad n = 1, 2, \cdots.$$

The above inequality and the assumption (a–1) give the relative compactness of the sequence $\{u_n\}_{n=1,2,\cdots}$ in $\mathcal{H}$ (see, e.g., Reed–Simon [RS], Theorem XIII.65). □

3. Proof of Theorem 2

Let λ be an arbitrary positive number. We show below that λ does not belong to the essential spectrum $\sigma_{ess}(H)$ of H, that is, there is no orhtonormal system $\{u_n\}_{n=1,2,\cdots}$ in $\mathcal{H}$ such that

$$\{u_n\}_{n=1,2,\cdots} \subset D(H), \quad \|Hu_n - \lambda u_n\| \longrightarrow 0 \;\text{ as }\; n \longrightarrow \infty, \tag{7}$$

where $D(H)$ denotes the domain of H. Assume that such an orthonormal system $\{u_n\}_{n=1,2,\cdots}$ would exist. Then write

$$u_n = \begin{pmatrix} v_n \\ w_n \end{pmatrix}, \quad (H - \lambda)u_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix},$$
$$(v_n, w_n, f_n, g_n \in \mathbf{h} := [L^2(\mathbf{R}^3)]^2)$$

Then we have

$$(\vec{\sigma} \cdot \vec{D})w_n + 2q(x)v_n - \lambda v_n = f_n, \tag{8}$$

$$(\vec{\sigma} \cdot \vec{D})v_n - \lambda w_n = g_n, \tag{9}$$

where $(\vec{\sigma} \cdot \vec{D}) = \sum_{j=1}^{3} \sigma_j D_j$. In view of Rellich's theorem we may assume that $\{v_n\}$ and $\{w_n\}$ are strongly convergent in $[L^2(\Omega)]^2$ for any bounded domain Ω, by selecting a subsequence if necessary. Operating $\vec{\sigma} \cdot \vec{D}$ to (9) and using (8), we get

$$-\Delta v_n + 2\lambda q(x)v_n = (\vec{\sigma} \cdot \vec{D})g_n + \lambda f_n + \lambda^2 v_n, \tag{10}$$

Take a positive number R such that $q(x) \geq 1 \ \ (|x| \geq R)$ by means of (b–1), and put

$$Q(x) = \sqrt{2\lambda q(x)} \ \ (|x| \geq R) \ \text{ and } \ Q(x) = 1 \ \ (|x| \leq R).$$

Let us prepare a Hilbert space $\mathbf{h}_Q$:

$$\mathbf{h}_Q = \left\{ g \in \mathbf{h} = [L^2(\mathbf{R}^3)]^2 \mid \|g\|_Q^2 := \|Qg\|_{\mathbf{h}}^2 + \sum_{j=1}^{3} \|D_j g\|_{\mathbf{h}}^2 < \infty \right\}$$

with the inner product

$$(f, g)_Q = \langle Qf, Qg \rangle + \sum_{j=1}^{3} \langle D_j f, D_j g \rangle,$$

where $\| \ \ \|_{\mathbf{h}}$ and $\langle \ , \ \rangle$ are the norm and the inner product in $\mathbf{h}$, respectively. The sesquilinear forms $(f, g)_Q$ and (f, g) are also used as in § 2 for $f \in [\mathcal{D}']^2$ and $g \in \mathcal{D}^2$. Let $\gamma(x)$ be the same function as in §2. Then, for any $\varphi \in \mathcal{D}^2$ we have

$$\begin{aligned} (\gamma v_n, \varphi)_Q &= \langle -\Delta(\gamma v_n) + 2\lambda q(\gamma v_n), \varphi \rangle \\ &= \langle -(\Delta\gamma)v_n - 2\vec{\nabla}\gamma \cdot \vec{\nabla} v_n - \gamma \Delta v_n + 2\lambda q \gamma v_n, \varphi \rangle \end{aligned}$$

and, by using (10),

$$\begin{aligned} (\gamma v_n, \varphi)_Q &= -\langle v_n, (\Delta\gamma)\varphi \rangle + 2\langle v_n, \vec{\nabla} \cdot [(\vec{\nabla}\gamma)\varphi] \rangle \\ &\quad + \langle g_n, (\vec{\sigma} \cdot \vec{D}\gamma)\varphi \rangle + \langle g_n, \gamma(\vec{\sigma} \cdot \vec{D})\varphi \rangle \\ &\quad + \langle \lambda f_n + \lambda^2 v_n, \gamma\varphi \rangle. \end{aligned}$$

Therefore we can find a positive constant C independent of φ such that

$$|(\gamma v_n, \varphi)_Q| \le C(\|f_n\|_{\mathbf{h}} + \|g_n\|_{\mathbf{h}} + \|v_n\|_{\mathbf{h}}) \cdot \|\varphi\|_Q.$$

Noting that $\mathcal{D}^2$ is dense in $\mathbf{h}$, we have $v_n \in \mathbf{h}_Q$ and

$$\|\gamma v_n\|_Q \le C(\|f_n\|_{\mathbf{h}} + \|g_n\|_{\mathbf{h}} + \|v_n\|_{\mathbf{h}}) \quad (n = 1, 2, \cdots). \tag{11}$$

Since $\{v_n\}$, $\{f_n\}$ and $\{g_n\}$ are bounded sequences in $\mathbf{h}$, we select a subsequence $\{v_{n_j}\}_{j=1,2,\cdots}$ of $\{v_n\}$, which is strongly convergent in $\mathbf{h}$, using again Reed–Simon [RS], Thoerem XIII.65. Since $\{u_n\}$ is orhonormal, $\{v_n\}$ converges weakly to 0 in $\mathbf{h}$. Therefore we have

$$v_{n_j} \longrightarrow 0 \ \text{ as } \ j \longrightarrow \infty \tag{12}$$

strongly in $\mathbf{h}$. The above inequality (11) and

$$\|f_n\|_{\mathbf{h}} + \|g_n\|_{\mathbf{h}} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty$$

in view of (7), yield

$$\gamma(\vec{\sigma} \cdot \vec{D})v_{n_j} = (\vec{\sigma} \cdot \vec{D})(\gamma v_{n_j}) - (\vec{\sigma} \cdot \vec{D}\gamma)v_{n_j} \longrightarrow 0$$

strongly in $\mathbf{h}$,. By means of (9) we have

$$\gamma w_{n_j} \longrightarrow 0 \ \text{ as } \ j \longrightarrow \infty$$

strongly in $\mathbf{h}$. Since $\{w_n\}$ is locally strongly convergent in $\mathbf{h}$, the above property implies the strong convergence of $\{w_{n_j}\}$ in $\mathbf{h}$. Moreover, since it converges weakly to 0 in $\mathbf{h}$, we have

$$w_{n_j} \longrightarrow 0 \ \text{ as } \ j \longrightarrow \infty \tag{13}$$

strongly in $\mathbf{h}$. Thus, (12) and (13) give a contradiction to

$$\|u_{n_j}\|^2 = \|v_{n_j}\|^2 + \|w_{n_j}\|^2 = 1, \quad j = 1, 2, \cdots. \qquad \square$$

4. Proof of Proposition 3 and Remarks

We show first the non-existence of eigenvalues of H in $\mathbf{R}^-$. Suppose

$$\lambda \le 0, \quad u = \begin{pmatrix} v \\ w \end{pmatrix} \in D(H) \quad (v, w \in \mathbf{h}) \ \text{ and } \ Hu = \lambda u.$$

Then we have

$$(\vec{\sigma} \cdot \vec{D})w + 2qv = \lambda v,$$

$$(\vec{\sigma} \cdot \vec{D})v = \lambda w. \tag{14}$$

Therefore, v satisfies

$$-\Delta v + 2\lambda q v = \lambda^2 v. \tag{15}$$

It is well known that if $\lambda < 0$, the Schödinger operator $-\Delta + 2\lambda q(x)$ has no eigenfunctions in $L^2(\mathbf{R}^3)$ under the conditions (c–1) and (c–2) (see, e.g., Uchiyama–Yamada [UY]). If $\lambda = 0$, we obtain from (15) that $\Delta v = 0$. Therefore, $v \in \mathbf{h}$ means $v = 0$, which and (14) imply $w = 0$ and $u = 0$.

Finally, we prove $\mathbf{R}^- \subset \sigma(H)$. The proof is given along the same line of Arai–Yamada [AY]. Let us denote

$$\begin{aligned} B_R &= \{x \in \mathbf{R}^3 \mid |x| \le R\}, \\ E_R &= \{x \in \mathbf{R}^3 \mid |x| \ge R\}, \\ \Omega &= B_{R/2}, \end{aligned}$$

where R is the number in the assumption (c–2) , and take a C^∞ function $\rho(x)$ such that

$$\rho(x) = 0 \quad (x \in \Omega) \quad \text{and} \quad \rho(x) = 1 \quad (x \in E_R).$$

Let $\tilde{H}$ be a selfadjoint operator in $\mathcal{H}$ such that

$$\tilde{H} = (\vec{\alpha} \cdot \vec{D}) + \rho(x) q(x)(\beta + I).$$

Since the essential spectrum $\sigma_{ess}(H)$ of H coincides with $\sigma_{ess}(\tilde{H})$ of $\tilde{H}$, it suffices to prove $\mathbf{R}^- \subset \sigma_{ess}(\tilde{H})$. Let $\{\mu_0, \mu_1, \cdots\}$ be the totality of eigenvalues of $-\Delta_{|\Omega}$ with Neumann boundary condition, and $\{\varphi_0, \varphi_1, \cdots\}$ the corresponding complete orthonormal system of the eigenfunctions such that

$$0 = \mu_0 \le \mu_1 \le \cdots \quad \text{and} \quad \varphi_0(x) \equiv [\mathrm{vol}(\Omega)]^{-1/2}.$$

We show below

$$\mathbf{R}^- \setminus \{-\sqrt{\mu_1}, -\sqrt{\mu_2}, \cdots\} \subset \sigma(\tilde{H}),$$

which yields $\mathbf{R}^- \subset \sigma_{ess}(\tilde{H})$. Assume that a negative number λ such that

$$\lambda^2 \in \mathbf{R}^+ \setminus \{\mu_1, \mu_2, \cdots\}.$$

would not belong to $\sigma(\tilde{H})$, that is, λ would belong to the resolvent set of

$\tilde{H}$. Then, for

$$f(x) = {}^t(\varphi_0, \varphi_0) \quad (x \in \Omega) \quad \text{and} \quad f(x) = 0 \quad (x \notin \Omega)$$

we can find a unique solution $u = {}^t(v, w) \in D(\tilde{H}) \subset \mathcal{H}^1_{loc} = [H^1_{loc}]^4$ such that

$$(\tilde{H} - \lambda)u = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}.$$

Then we have

$$(\vec{\sigma} \cdot \vec{D})w + 2\rho(x)q(x)v(x) - \lambda v(x) = f(x),$$

$$(\vec{\sigma} \cdot \vec{D})v - \lambda w(x) = 0.$$

and

$$-\Delta v + 2\lambda\rho(x)q(x)v(x) - \lambda^2 v(x) = \lambda f(x) \quad (x \in \mathbf{R}^n).$$

and, therefore, $v \in [H^2_{loc}]^2$. In view of Sobolev's theorem (Sobolev [So], p.85) $v(r\cdot)$ and $\frac{\partial v}{\partial r}(r\cdot)$ are strongly continuous in $[L^2(S^2)]^2$ with respect to $r > 0$. The conditions (c–1) and (c–2) gives that $-\Delta + 2\lambda q$ has no eigenfunctions in $L^2(E_R)$ without any restriction of boundary conditions (see, e.g., Uchiyama–Yamada [UY]). Therefore, we have $v(x) \equiv 0$ in E_R. By the unique continuation property of elliptic operators (e.g., Eastham–Kalf [EK], §6.5, Corollary 6.5.1) , we have $v(x) \equiv 0$ in $E_{R/2}$ and

$$-\Delta v - \lambda^2 v(x) = \lambda \begin{pmatrix} \varphi_0 \\ \varphi_0 \end{pmatrix} \quad \text{in } \Omega, \quad v = 0 \quad \text{and} \quad \frac{\partial v}{\partial r} = 0 \quad \text{on } \partial\Omega.$$

Since each component of v satisfies the Neumann condition on $\partial\Omega$ as seen above, v can be expanded with $\{\varphi_j\}_{j=1,2,\cdots}$. Noting λ^2 is none of eigenvalues $\{\mu_j\}$, we have

$$v(x) = -\frac{1}{\lambda} \begin{pmatrix} \varphi_0 \\ \varphi_0 \end{pmatrix} \quad \text{in } \Omega,$$

which contradicts to $v = 0$ on $\partial\Omega$. □

Remark 1. In Theorem 1 the discrete spectrum $\sigma_d(H)$ is unbounded above and below. This is proved as follows. For example, assume $\sigma_d(H)$ would be bounded above. Then there exists a positive constant M such that

$$(Hu, u) \leq M\|u\|^2 \quad u \in D(H).$$

Write $u = {}^t(v, w) \in \mathbf{h} \times \mathbf{h}$. Then we obtain

$$\begin{aligned} &2\mathrm{Re}\langle(\vec{\sigma}\cdot\vec{D})v, w\rangle + 2\langle(p+q)v, v\rangle + 2\langle(p-q)w, w\rangle \\ &\qquad \le M(\|v\|_{\mathbf{h}}^2 + \|w\|_{\mathbf{h}}^2). \end{aligned}$$

Since $p(x)$ and $q(x)$ are locally bounded functions, we can find a positive constant C such that

$$2\mathrm{Re}\langle(\vec{\sigma}\cdot\vec{D})v, w\rangle \le C(\|v\|_{\mathbf{h}}^2 + \|w\|_{\mathbf{h}}^2) \quad v, w \in [C_0^\infty(B_1)]^2. \tag{16}$$

Substituting $w = v$ and $w = -v$ in (16), we have

$$|\langle(\vec{\sigma}\cdot\vec{D})v, v\rangle| \le C\|v\|_{\mathbf{h}}^2, \quad v \in [C_0^\infty(B_1)]^2$$

which implies that $\vec{\sigma}\cdot\vec{D}$ in B_1 with Dirichlet boundary condition is a bounded operator in $\mathbf{h}$, that is,

$$\|(\vec{\sigma}\cdot\vec{D})v\|_{\mathbf{h}} = \|\nabla v\|_{\mathbf{h}} \le C\|v\|_{\mathbf{h}}, \quad v \in [C_0^\infty(B_1)]^2.$$

This is a contradiction.

Similarly, we obtain that H in Theorem 2 has the discrete spectrum unbounded in $\mathbf{R}^+$.

Remark 2. The conditions in Proposition 3 can be weakened. For the non-existence theorem of eigenvalues of Schrödinger operators plays an important role in Proposition 3. The non-existence theorem for Schrödinger operators has been studied extensively by many authors (see, e.g., the reference of [UY], where the reader can find some works concerning the non-existence theorem.)

It is conjectured in Proposition 3 that the half line $\mathbf{R}^-$ is included in the absolutely continuous spectrum $\sigma_{ac}(H)$.

Acknowledgment The author would like to thank Prof. M. Arai and the referee for their valuable advices.

References

[AY] Arai M. and Yamada O., *A remark on essential spectrum of magnetic Schrödinger operators with exploding potentials.* Memo. Inst. Sci. Engi. Ritsumeikan Univ. **50** (1991), 21–32.

[EK] Eastham M.S.P. and Kalf H., *Schrödinger–type Operators with Continuous Spectra.* Research Notes in Math. **65**, Pitman Publishing, Boston–London–Melbourne, 1982.

[E] Erdélyi A., *Note on a paper of Titchmarsh.* Quart. J. Math. Oxford (2) **14** (1963), 147–152.

[IMS] Ikhdair S., Mustafa O. and Sever R., *Solution of Dirac equation for vector and scalar potentials and some applications.* Hadronic J. **16** (1993), 57–74.

[J] Jörgens K., *Perturbation of the Dirac operator.* Lecture Notes in Math. **280**, Springer Verlag, Berlin–Heidelberg–New York, 1972, 87–102.

[JT] Jena S.N. and Tripati T., *Dirac bound state spectra of $Q\bar{Q}$, $q\bar{q}$, and $Q\bar{q}$ systems.* Phys. Rev. D **28** (1983), 1780–1782.

[RH] Ram B. and Halasa R., *Dirac equation with a quadratic scalar potential – are quarks confined ?* Lett. AL Nuovo Cimento **26** (1979), 551–554.

[RS] Reed M. and Simon B., *Methods of Modern Mathematical Physics, IV : Analysis of Operators.* Academic Press, New York–San Francisco–London, 1978.

[Sc] Schmidt K.M., *Absolutely continuous spectrum of Dirac systems with potentials infinite at infinity.* to appear in Math. Proc. Camb. Phil. Soc.

[So] Sobolev S.L., *Applications of Functional Analysis in Mathematical Physics.* Translations of Mathematical Monographs **7**, AMS, Providence, 1963.

[Th] Thaller B., *The Dirac Equation.* Texts and Monographs in Physics, Springer Verlag, Berlin–Heidelberg–New York, 1992.

[Ti] Titchmarsh E.C. *On the nature of the spectrum in problems of relativistic quantum mechanics.* Quart. J. Math. Oxford (2) **12** (1961), 227–240.

[UY] Uchiyama J. and Yamada O., *Sharp estimates of lower bounds of polynomial decay order of eigenfunctions.* Publ. RIMS, Kyoto Univ. **26** (1990), 419–449.

Department of Mathematics
Ritsumeikan University
Kusatsu, Shiga 525-77, Japan
E-mail: yamadaos@bkc.ritsumei.ac.jp