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Periodic points of disk homeomorphisms having a pseudo-Anosov component

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Abstract. We consider an orientation-preserving homeomorphism f of the disk. The braid type is one of the topological invariants defined for finite invariant sets of f. We show that if f has a finite invariant set S whose braid type contains a pseudo-Anosov component, then there are at least $2n+3$ periodic points of period $\leq n$ for any n with $n\geq \mathrm{per}(S)$, where $\mathrm{per}(S)$ is the maximum of the periods of the points in S.

Key words: braid, periodic point, pseudo-Anosov homeomorphism, disk homeomorphism.

1. Introduction

Let M be a compact connected orientable surface possibly with boundary, and $f: M\rightarrow M$ an orientation-preserving homeomorphism. Given a finite set S which is contained in the interior of M and is invariant under the map f, its braid type $\mathrm{bt}(S, f)$ is defined as the isotopy class of f relative to the set S [3] (see Section 2 below). From the Nielsen-Thurston classification theory of isotopy classes for surface homeomorphisms [6, 22], braid types are classified into three classes: pseudo-Anosov, periodic, and reducible. Moreover, in the reducible case, any braid type is decomposed into a finite number of irreducible components, each of which is either pseudo-Anosov or periodic.

This classification for braid types has great significance in the theory of 2-dimensional dynamical systems: If f has a finite invariant set S whose braid type $\mathrm{bt}(S, f)$ contains a pseudo-Anosov component, then f must have dynamical complexity, e.g. positive topological entropy and an infinite number of periodic points. Also, the following results have been obtained on the existence and the number of periodic points: The growth rate of the number of periodic points of period n, as n tends to infinity, is positive. (This follows from Jiang [16, Theorem 3.8]. See also [15].) There exist infinitely many period doubling sequences of periodic orbits (Guaschi [12, Theorem 4]). f has a periodic point with any sufficiently large period, provided that

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$\mathrm{bt}(S, f)$ is pseudo-Anosov [9].

Moreover, some further results on periodic points have been obtained in the special case where M is the closed disk D and the braid type $\mathrm{bt}(S, f)$ is pseudo-Anosov. A result of the author [20, Theorem 2] on embeddings of the plane immediately implies that f has a periodic point of every period greater than three, provided S has three points and f is differentiable. (The same result is valid for the plane as well.) Following an example of Gambaudo et al. [10], Kolev [18] has proved that f has periodic points of all periods if S is a periodic orbit of period 3. Guaschi [11] generalized Kolev's result to the case of invariant sets having 3 or 4 points. Results of Franks and Misiurewicz (Propositions 13.2 and 13.4 in [8]) imply that if S is a periodic orbit, then either f has periodic points of all periods greater than $n-3$ or f has periodic points of all periods divisible by n, where n is the period of S.

In this paper, we also restrict ourselves to the case of the disk. However, we allow $\mathrm{bt}(S, f)$ to be reducible. Our main theorem gives an estimate for the number of periodic points with period not exceeding a given number. Let $\mathrm{per}(S)$ denote the maximum of the periods of all periodic points contained in S.

Theorem 1 *Let $f : D \to D$ be an orientation-preserving homeomorphism on the closed disk, and S a finite invariant set of f in the interior of D. Suppose the braid type $\mathrm{bt}(S, f)$ has a pseudo-Anosov component. Then, for every integer $n \geq \mathrm{per}(S)$, f has at least $2n+3$ periodic points whose periods are less than or equal to n.*

The above estimate can be improved if we assume some transversality condition. Let n be a positive integer. We say f is *n-transversal* if f is differentiable and for any positive integer $m \leq n$, the differential of f^m at each fixed point of f^m does not have eigenvalue one.

Theorem 2 *Assume the hypotheses of Theorem 1. Suppose n is an integer such that $n \geq \mathrm{per}(S)$ and f is n-transversal. Then, f has at least $3n+6$ periodic points whose periods are less than or equal to n.*

In the special case of $\mathrm{per}(S) = n = 1$, Theorem 2 has been obtained and also shown to be the best possible estimate by the author [21] (Remark, pp. 461–462). When S has three or four points, one can derive sharper estimates than Theorems 1 and 2 for large n from Guashi's result

[11] mentioned above. The author does not know whether our theorems give the best possible estimates in the remaining case.

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2. Preliminaries

Here we recall the definition of a braid type and the classification theory for braid types. Also, we prove some results which are necessary for the proof of Theorem 1.

Let M be a compact connected orientable surface possibly with boundary ∂M. Let $f : M \to M$ be an orientation-preserving homeomorphism. A point x in M is a fixed point of f if $f(x) = x$. x is called a periodic point of f if it is a fixed point of f^n for some positive integer n. The least such n is called the *period* of the periodic point and denoted by $\mathrm{per}(x, f)$. A periodic point of period n will also be called an *n-periodic point.* Let $\mathrm{Fix}(f)$ denote the set of fixed points of f. Let $P(f)$ denote the set of all periodic points of f, and $P_n(f)$ the set of periodic points of period n. Let

$$P^n(f) = \bigcup_{m \leq n} P_m(f).$$

Namely, $P^n(f)$ is the set of periodic points whose periods are less than or equal to n. Clearly $P^1(f) = P_1(f) = \mathrm{Fix}(f)$. The aim of this paper is to give lower bounds for the cardinality of this set $P^n(f)$ in the case of $M = D$.

A subset S of M is called an invariant set of f if $f(S) = S$. Suppose f, $g : M \to M$ are homeomorphisms. Suppose S is a finite set in the interior $\mathrm{Int}\, M$ of M and it is an invariant set for both f and g. We say f and g are *isotopic relative to* S if there exist homeomorphisms $h_t : M \to M$ for $0 \leq t \leq 1$ such that $h_0 = f$, $h_1 = g$, $h_t(x)$ is continuous in x and t, and $h_t(S) = S$ for any t. An equivalence class under this relation is called an isotopy class relative to S.

Definition 1 Suppose $f : M \to M$ is an orientation-preserving homeomorphism, and S is a finite invariant set of f in $\mathrm{Int}\, M$. Define the *braid type* $\mathrm{bt}(S, f)$ of the set S with respect to f to be the isotopy class of $f : M \to M$ relative to S.

Braid types can be classified according to the Nielsen-Thurston classification theory on surface homeomorphisms up to isotopy. We give a brief account of this theory. (We shall mainly follow [4, Section 7].) Let $\phi : M \to M$ be a homeomorphism. ϕ is said to be *periodic* if $\phi^m = \mathrm{id}$ for some positive integer m, where id denotes the identity map. Let S be a finite invariant set of ϕ in $\mathrm{Int}\, M$. ϕ is said to be *pseudo-Anosov relative to* S, if the following conditions hold:

(a) There exists a pair of transverse foliations $\mathcal{F}^u$, $\mathcal{F}^s$ on M, carrying transverse measures which are uniformly expanded and contracted by ϕ respectively.

(b) Each foliation has a finite number of singularities which coincide in the interior $\mathrm{Int}\, M$ and alternate on the boundary ∂M. Any singularity is p-pronged for some positive integer p, $p \neq 2$.

(c) Any singularity on ∂M is 3-pronged. (We consider segments of the boundary to be prongs.)

(d) 1-prongs are permitted only at points of S.

In the case where S is empty, ϕ is simply said to be pseudo-Anosov. ϕ is called a *generalized pseudo-Anosov* homeomorphism if it is pseudo-Anosov relative to some non-empty finite invariant set (see [6]). We say ϕ is *reducible* relative to S if there exists a disjoint union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ of simple closed curves, called *reducing curves*, in $\mathrm{Int}\, M - S$ such that

(a) Σ is an invariant set of ϕ, and

(b) each connected component of $M - (\Sigma \cup S)$ has negative Euler characteristic.

Theorem 3 (Nielsen-Thurston Classification Theorem) *Let $f : M \to M$ be a homeomorphism, and S a finite invariant set of f in* $\mathrm{Int}\, M$. *Then f is isotopic relative to S to a homeomorphism ϕ which is periodic, pseudo-Anosov relative to S, or reducible relative to S. In the reducible case, ϕ can be chosen to have an invariant open tubular neighborhood $A(\Sigma)$ of Σ such that on each connected component N of the complement of $A(\Sigma)$, $\phi^\mu : N \to N$ is either periodic or pseudo-Anosov relative to $S \cap N$, where μ is the least positive integer such that $\phi^\mu(N) = N$.*

The map ϕ in the above theorem is called a *canonical homeomorphism* (or a canonical form) relative to S. In the case where S is empty, ϕ is simply called a canonical homeomorphism. Each component N is called a *component* of ϕ. Also, each connected component of $A(\Sigma)$ is called a *reducing annulus*. This classification theorem leads to the following definition:

Definition 2 A braid type $\mathrm{bt}(S, f)$ is called *periodic, pseudo-Anosov*, or *reducible* if a canonical homeomorphism $\phi : M \to M$ in the isotopy class of f relative to S is periodic, pseudo-Anosov relative to S, or reducible relative to S, respectively. We say $\mathrm{bt}(S, f)$ contains a pseudo-Anosov component if so does ϕ.

Now let $\phi : M \to M$ be an orientation-preserving pseudo-Anosov homeomorphism relative to S. Let $\mathcal{F}$ be one of the associated foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. For $x \in M$, let $p(x)$ denote the number of prongs of $\mathcal{F}$ at x. (If x is a regular point of $\mathcal{F}$, we regard the two half leaves emanating from x as the prongs at x, and thus set $p(x) = 2$.) Let $\mathrm{Sing}(\mathcal{F})$ denote the set of singularities of $\mathcal{F}$. Then we have the following Euler-Poincaré formula [6, p. 75]:

$$\sum_{x \in \mathrm{Sing}(\mathcal{F})} (2 - p(x)) = 2\chi(M), \tag{1}$$

where $\chi(M)$ denotes the Euler characteristic of M.

Let $x \in \mathrm{Int}\, M$ be an n-periodic point of ϕ. Then ϕ^n induces a permutation on the prongs at x of the foliation $\mathcal{F}$. Let l, a be positive integers. We say x has *type* (l, a) if

(a) l is the least period of a prong at x under the permutation ϕ^n, and

(b) $p(x) = al$.

This definition makes sense, since the prongs at x have the same period under ϕ^n and l, $p(x)$ do not depend on the choice of $\mathcal{F}$. Note also that l always divides $p(x)$ and hence $p(x)$ is written in the form (b) above. For example, if x is a one-pronged singularity, then it has type (1,1). If x is a regular point of $\mathcal{F}$, then it has type (1, 2) or (2, 1).

We recall some facts about the index for a periodic point. Let $f : M \to M$ be a continuous map. Given a fixed point x of f, let $\mathrm{ind}(x, f)$ denote its fixed point index. If x is an n-periodic point, we define its *index* to be $\mathrm{ind}(x, f^n)$. It is easy to see that if x is a periodic point of ϕ in $\mathrm{Int}\, M$ which has period n and type (l, a), then for any $q \geq 1$, we have

$$\mathrm{ind}(x, \phi^{nq}) = \begin{cases} 1 - al & \text{if } q \text{ is a multiple of } l, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

Namely, for an interior fixed point x of an iterate ϕ^m, $\mathrm{ind}(x, \phi^m) = 1$ if

ϕ^m rotates the prongs at x, and $\operatorname{ind}(x,\phi^m) = 1 - p(x)$ if ϕ^m preserves each (equivalently some) prong. In particular, the index of an interior periodic point is zero if and only if it is a one-pronged singularity. If x is a periodic point in ∂M, then it must be a singularity of $\mathcal{F}^s$ or $\mathcal{F}^u$, and

$$\operatorname{ind}(x,\phi^m) = \begin{cases} -1 & \text{if } x \in \operatorname{Sing}(\mathcal{F}^s) \\ 0 & \text{if } x \in \operatorname{Sing}(\mathcal{F}^u), \end{cases} \tag{3}$$

for any m such that ϕ^m fixes x.

Two fixed points x_0 and x_1 of a continuous map $f : M \to M$ are said to be *f-Nielsen equivalent* (or *Nielsen equivalent*, shortly) if there exists a path $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$, and $f \circ \gamma$ is homotopic to γ fixing end points. (See [2, 14].) An equivalence class under this relation is called an *f-Nielsen class* or a *Nielsen class*. The Nielsen class represented by x is denoted by $NC(x, f)$.

Two invariant sets K_0 and K_1 of f are called *f-related* if there exists a path $\gamma : [0,1] \to M$ such that $\gamma(0) \in K_0$, $\gamma(1) \in K_1$, and $f \circ \gamma$ and γ are homotopic via a homotopy $\{\gamma_t\}_{0 \le t \le 1}$ of paths in M such that $\gamma_t(0) \in K_0$ and $\gamma_t(1) \in K_1$ for any t. (See [17, p. 70].) In general, this is not an equivalence relation among invariant subsets. When K_0 and K_1 are single points, it reduces to the Nielsen equivalence relation between fixed points of f.

Suppose $g : M \to M$ is a continuous map homotopic to f. Let $H = \{h_t\} : M \to M$ $(0 \le t \le 1)$ be a homotopy from f to g. Then $x \in \operatorname{Fix}(f)$ and $y \in \operatorname{Fix}(g)$ are said to be *H-related* if there exists a path α in M such that $\alpha(0) = x$, $\alpha(1) = y$, and two paths α and $h_t(\alpha(t))$ are homotopic fixing end points.

Lemma 1 *Suppose K is a subset of M such that $h_t(K) = K$ for any t. Suppose $x \in \operatorname{Fix}(f)$ and $y \in \operatorname{Fix}(g)$ are H-related and y is g-related to K. Then x is f-related to K.*

Proof. Given paths ω, ω' in M which have the same initial point and whose terminal points are both contained in K, we write $\omega \sim \omega'$ if there is a homotopy $\{\omega_t\}$ of paths in M with $\omega_0 = \omega$, $\omega_1 = \omega'$, $\omega_t(0) = \omega(0)$, $\omega_t(1) \in K$ for any t. Since y and K are g-related, there is a path γ in M such that $\gamma(0) = y$, $\gamma(1) \in K$, and $\gamma \sim g \circ \gamma$. Since x and y are H-related, there is a path α in M from x to y such that α and $\bar{\alpha}$, where $\bar{\alpha}$ is given by

$\bar{\alpha}(t) = h_t(\alpha(t))$, are homotopic fixing end points. Thus we have that

$$\alpha\gamma \sim \bar{\alpha}(g \circ \gamma). \tag{4}$$

Define a homotopy $\{\beta_t\}$ $(0 \le t \le 1)$ of paths in M by

$$\beta_t(s) = \begin{cases} h_{2s(1-t)}(\alpha(2s)) & s \le \dfrac{1}{2} \\ h_{1-t}(\gamma(2s-1)) & s \ge \dfrac{1}{2}. \end{cases}$$

Then $\beta_t(0) = x$ and $\beta_t(1) \in K$ for each t, and $\beta_0 = \bar{\alpha}(g \circ \gamma)$, $\beta_1 = (f \circ \alpha)(f \circ \gamma) = f \circ (\alpha\gamma)$. Therefore, $\bar{\alpha}(g \circ \gamma) \sim f \circ (\alpha\gamma)$ and hence by (4), $\alpha\gamma \sim f \circ (\alpha\gamma)$. This means that x is f-related to K. □

The following lemma is closely related to Lemma 3.4 of Jiang and Guo [17] and Proposition 1.5 of Boyland [5].

Lemma 2 *Let M be a compact surface obtained from the disk D by removing finitely many disjoint open disks in D. Let $\psi : M \to M$ be a canonical homeomorphism. Suppose ψ has a pseudo-Anosov component N such that $\psi(N) = N$. Let K_0 and K_1 be disjoint connected invariant sets of ψ. Suppose either*

(a) *K_0 and K_1 are contained in different connected components of $M - \operatorname{Int} N$, or*

(b) *K_0 or K_1 is a single point contained in* $\operatorname{Int} N$. *Then K_0 and K_1 are not ψ-related.*

Proof. The method of the proof here is different from those in [5, 17] which use the covering space technique. Instead, we use the hypothesis that M is a subspace of the disk. Let $X_1, \ldots, X_k$ be all of the connected components of $M - \operatorname{Int} N$ and let $C_i = N \cap X_i$. Then, since N is a component of a canonical homeomorphism of ψ, we see that either $X_i = C_i$ or X_i is a compact surface which has C_i as one of its boundary circles. We can assume, by rearranging indices if necessary, that $X_i = C_i$ if $i < q$, and X_i is a compact surface if $i \ge q$, where $1 \le q \le k+1$. Also, we can assume that ψ preserves each X_i, by taking its iterate if necessary, because ψ-related subsets are ψ^m-related for any $m \ge 1$. If $i \ge q$, then C_i is isotopic to some reducing curve and hence X_i has at least two other boundary circles $C_{i,1}, \ldots, C_{i,b_i}$, where $b_i \ge 2$. Let $\hat{M}$ be a compact surface obtained from M by collapsing each $C_{i,j}$ to a point, where $i \ge q$, $2 \le j \le b_i$. Denote by $\pi : M \to \hat{M}$ the projection map, and let $\hat{X}_i = \pi(X_i)$. Since π is injective on N, we can regard N as a subspace of $\hat{M}$. Then $\hat{M} - \operatorname{Int} N = \hat{X}_1 \cup \cdots \cup \hat{X}_k$. For $i < q$, we have $\hat{X}_i = C_i$. If $i \geq q$, then $\hat{X}_i$ is a closed annulus whose boundary circles are C_i and $C_{i,1}$. Thus N is a strong deformation retract of $\hat{M}$, i.e., there exists a homotopy $r_t : \hat{M} \to \hat{M}$ such that

$$r_0 = \mathrm{id}, \quad r_1(\hat{M}) = N, \quad r_t|_N = \mathrm{id}$$

for any t. Moreover, we can choose r_t so that the image of $\hat{M} - \operatorname{Int} N$ under r_t is disjoint from $\operatorname{Int} N$ for any t. Let $r = r_1 : \hat{M} \to N$. Let $\hat{\psi} : \hat{M} \to \hat{M}$ be a homeomorphism induced by ψ. Then $\hat{\psi} \circ \pi = \pi \circ \psi$, $r(\hat{X}_i) = C_i$ for any i, and $\hat{\psi} = \psi$ on N. Also, $\hat{\psi}(\hat{X}_i) = \hat{X}_i$ since ψ is assumed to preserve X_i.

Suppose K_0 and K_1 were ψ-related. Then there is a path γ in M such that $\gamma(0) \in K_0$, $\gamma(1) \in K_1$, and γ is homotopic to $\psi \circ \gamma$ via a homotopy $\{\gamma_t\}$ of paths in M such that $\gamma_t(0) \in K_0$, $\gamma_t(1) \in K_1$ for any t. Define a path ω in N by $\omega = r \circ \pi \circ \gamma$. We shall make a homotopy $\{\omega_t\}$ of paths in N between ω to $\psi \circ \omega$. Define two homotopies $\{\omega'_t\}$, $\{\omega''_t\}$ of paths in N by

$$\omega'_t = r \circ \pi \circ \gamma_t, \quad \omega''_t = r \circ \hat{\psi} \circ r_t \circ \pi \circ \gamma.$$

Then $\omega'_0 = \omega$, $\omega'_1 = \omega''_0$, $\omega''_1 = r \circ \hat{\psi} \circ \omega = \psi \circ \omega$, since ω is a path in N and $r \circ \hat{\psi} = \psi$ on N. Therefore if we denote by $\{\omega_t\}$ the product of two homotopies $\{\omega'_t\}$ and $\{\omega''_t\}$, (i.e., $\omega_t = \omega'_{2t}$ if $t \leq 1/2$ and $\omega_t = \omega''_{2t-1}$ if $t \geq 1/2$) then it gives the desired homotopy. Note that if $K_\epsilon \subset X_i$, where $\epsilon = 0, 1$, then $\omega'_t(\epsilon) \in r(\hat{X}_i) = C_i$, $\omega''_t(\epsilon) \in r(\hat{\psi}(\hat{X}_i)) = C_i$, and hence $\omega_t(\epsilon) \in C_i$.

Suppose first that the condition (a) holds. Then $K_0 \subset X_{i_0}$, $K_1 \subset X_{i_1}$, where $i_0 \neq i_1$, and hence $\omega_t(\epsilon) \in C_{i_\epsilon}$ for $\epsilon = 0, 1$ and any t. This means that different boundary components C_{i_0}, C_{i_1} of N are related under the pseudo-Anosov homeomorphism $\psi|_N$. This is a contradiction. (See e.g. Jiang and Guo [17, Lemma 2.2] and Boyland [5, Lemma 1.1 (b) (iii)].)

Suppose next the condition (b) holds. We can assume that K_0 consists of a single point, say x_0, in $\operatorname{Int} N$. Then clearly $\omega_t(0) = x_0$. Also, since K_1 is a connected invariant set disjoint from $\{x_0\}$, K_1 is contained in some X_i or is a single point in $\operatorname{Int} N$. These imply that the fixed point x_0 in the interior of N is $\psi|_N$-related to another interior fixed point or some boundary circle C_i of N. This also contradicts the results of [5, 17] cited above. Hence we complete the proof. $\square$

Suppose $f, g : M \to M$ are homotopic continuous maps, and $H = \{h_t\} : M \to M$ is a homotopy from f to g. For a subset E of M, define a subset $\Delta_H(E)$ of $\mathrm{Fix}(g)$ by

$$\Delta_H(E) = \{y \in \mathrm{Fix}(g) \mid \text{some } x \in E \cap \mathrm{Fix}(f) \text{ is } H\text{-related to } y\}.$$

Note that if $x \in \mathrm{Fix}(f)$ is H-related to $y \in \mathrm{Fix}(g)$ and y is g-Nielsen equivalent to another $y' \in \mathrm{Fix}(g)$, then x is also H-related to y'. As a consequence, $\Delta_H(E)$ is empty or a union of g-Nielsen classes. For a point $x \in M$, let $\Delta_H(x) = \Delta_H(\{x\})$. This is a single Nielsen class or an empty set. We say Nielsen classes F of f and G of g are H-related if $\Delta_H(F) = G$, i.e., every $x \in F$ and every $y \in G$ (equivalently some $x \in F$ and some $y \in G$) are H-related. It is known that H-related Nielsen classes have the same fixed point index (see e.g. [2]). In particular, if F is essential (i.e., if its index is nonzero), then $\Delta_H(F)$ is also essential and hence not empty.

Proposition 1 *Let $f : D \to D$ be an orientation-preserving homeomorphism with a finite invariant set S in $\mathrm{Int}\, D$. Let $\phi : D \to D$ be a canonical homeomorphism relative to S which is isotopic to f. Assume ϕ has a pseudo-Anosov component. Let L be the union of all pseudo-Anosov components of ϕ. Then*

(a) *There exists an injective map $\tau : P(\phi) \cap (\mathrm{Int}\, L - S) \to P(f) - S$ which is period-preserving, i.e.* $\mathrm{per}(x, \phi) = \mathrm{per}(\tau(x), f)$ *for every x.*

(b) *$\sharp P^n(f) \geq \sharp(P^n(\phi) \cap (S \cup \mathrm{Int}\, L))$ for any $n \geq 1$, where $\sharp$ denotes the cardinality.*

Proof. (a) We need the notion of the blow-up of a homeomorphism. Let D_S be the surface obtained from $D - S$ by attaching a boundary circle at each point of S. Let Γ be the union of the attached boundary circles, and $\pi_S : D_S \to D$ the projection. Then we have $D_S = (D - S) \cup \Gamma$ and $\Gamma = \pi_S^{-1}(S) = \partial D_S - \partial D$. If $g : D \to D$ is a homeomorphism with $g(S) = S$ and its restriction to $D - S$ is extendable to a homeomorphism $G : D_S \to D_S$, then we call G the *blow-up* of g at S. It is known that any homeomorphism which is smooth at S and whose differential at each point in S is non-singular has the blow-up at S [1, p. 24]. Also, it is not difficult to see that any canonical homeomorphism relative to S has the blow-up at S. The blow-up of a homeomorphism which is pseudo-Anosov relative to S becomes a pseudo-Anosov homeomorphism on D_S.

Now we begin the proof. Let $\phi_S : D_S \to D_S$ be the blow-up of ϕ at S.

For each $x \in S$, we choose an open disk W^x centered at x so that these disks are mutually disjoint. Also, for $x \in S$ and $m \geq 1$, choose open disks V_m^x centered at x so that $V_m^x \subset W^x$ and $f^m(V_m^x) \subset W^{f^m(x)}$. Let $V_m = \bigcup_{x \in S} V_m^x$, and $(V_m)_S = (V_m - S) \cup \Gamma$. Then $(V_m)_S$ is an open neighborhood of Γ in D_S. We isotope f, relative to S, to a homeomorphism $f_m : D \to D$ which is smooth at S and has the non-singular differential at every point in S. Furthermore, f_m can be chosen so that its m-th iterate $(f_m)^m$ coincides with f^m away from V_m. It follows from the assumption on V_m^x that the blow-up $(f_m)_S$ of f_m at S has the property that every fixed point of its m-th iterate $(f_m)_S^m$ in $(V_m)_S$ is $(f_m)_S^m$-related to Γ. Since ϕ^m and $(f_m)^m$ are isotopic relative to S, there is an isotopy $H_m = \{H_{m,t}\} : D_S \to D_S$ from ϕ_S^m to $(f_m)_S^m$.

We shall define the map τ. Suppose x is a periodic point of ϕ in $\operatorname{Int} L - S$. Then x is also a periodic point of ϕ_S. Let $m = \operatorname{per}(x, \phi)$. Write f_S and V_S for $(f_m)_S$ and $(V_m)_S$ respectively for the sake of simplicity. Since $x \notin S$, it follows from (2) that x has nonzero index, and hence $\{x\}$ is an essential ϕ_S^m-Nielsen class. This implies that the f_S^m-Nielsen class $\Delta_{H_m}(x)$ is essential and hence not empty. Choose an element from this class, and denote it by $\tau(x)$. Namely, $\tau(x)$ is a fixed point of f_S^m satisfying

$$NC(\tau(x), f_S^m) = \Delta_{H_m}(x). \tag{5}$$

We show that the orbit of $\tau(x)$ under f_S is contained in $D_S - V_S$. Suppose to the contrary that $f_S^j(\tau(x)) \in V_S$ for some j. This implies that $f_S^j(\tau(x))$ is f_S^m-related to Γ by the property of f_S, and hence $\tau(x)$ is f_S^m-related to Γ. Since x and $\tau(x)$ are H_m-related, it follows from Lemma 1 that x is ϕ_S^m-related to Γ. This contradicts Lemma 2. Hence the orbit of $\tau(x)$ is contained in $D_S - V_S = D - V_m$. Since f^m and $(f_m)^m$ coincide on $D - V_m$, $\tau(x)$ is a fixed point of f^m in $D - S$. Thus we have defined the map τ.

We show τ is period-preserving. Let $x \in P(\phi) \cap (\operatorname{Int} L - S)$, and let m, m' be the periods of x, $\tau(x)$ respectively. Since $\phi_S^{m'}(x)$ is H_m-related to $f_S^{m'}(\tau(x))$, we have by (5)

$$\Delta_{H_m}(\phi_S^{m'}(x)) = NC(f_S^{m'}(\tau(x)), f_S^m) = NC(\tau(x), f_S^m) = \Delta_{H_m}(x).$$

Then x and $\phi_S^{m'}(x)$ are ϕ_S^m-Nielsen equivalent, and so they are identical. Therefore m' must be a multiple of m. On the other hand, since $\tau(x) \in \operatorname{Fix}(f_S^m)$, m is a multiple of m'. Therefore $m = m'$.

We show τ is injective. Suppose $x, y \in P(\phi) \cap (\operatorname{Int} L - S)$ have the same image under τ. Then, since τ is period-preserving, they have the same period, say m, and (5) implies x and y are ϕ_S^m-Nielsen equivalent. Then by Lemma 2, x and y must be equal.

(b) By (a), we have

$$\begin{aligned} \sharp P^n(f) &= \sharp(P^n(f) - S) + \sharp(P^n(f) \cap S) \\ &\geq \sharp(P^n(\phi) \cap (\operatorname{Int} L - S)) + \sharp(P^n(\phi) \cap S) \\ &= \sharp(P^n(\phi) \cap (S \cup \operatorname{Int} L)). \end{aligned}$$

□

Let g be a homeomorphism of the closed annulus $A = S^1 \times [0,1]$ isotopic to the identity. Let $\tilde{g}$ be a lift of g to the universal cover $\tilde{A} = \mathbf{R} \times [0,1]$. The projection onto the first factor is denoted by $\pi_1 : \tilde{A} \to \mathbf{R}$.

Given $x \in A$, choose a lift $\tilde{x} \in \tilde{A}$ and define its rotation number under $\tilde{g}$ as

$$\rho(x, \tilde{g}) = \lim_{n \to \infty} \frac{\pi_1(\tilde{g}^n(\tilde{x})) - \pi_1(\tilde{x})}{n}$$

if this limit exists. It is clear that the rotation number does not depend on the choice of the lift $\tilde{x}$. The set of all rotation numbers under $\tilde{g}$ is denoted by $\rho(\tilde{g})$. On each boundary circle Γ of the annulus A, every point has the same rotation number, since the restriction of g to Γ is an orientation-preserving circle homeomorphism. We denote this rotation number by $\rho(\Gamma, \tilde{g})$. The following theorem is an immediate consequence of Franks [7, Cor. 2.4].

Theorem 4 ([7]) *Let $g : A \to A$ be a generalized pseudo-Anosov homeomorphism of the annulus isotopic to the identity. Let $\tilde{g} : \tilde{A} \to \tilde{A}$ be a lift of g. Suppose α, β are distinct numbers contained in $\rho(\tilde{g})$. Then for every rational number p/q between α and β, where p and q are relatively prime, g has a periodic point of period q and rotation number p/q.*

Let $\phi : D \to D$ be an orientation-preserving generalized pseudo-Anosov homeomorphism. Let x_0 be a fixed point of ϕ in $\operatorname{Int} D$. Let $g : A \to A$ be a homeomorphism obtained by blowing up ϕ at the fixed point x_0. Then g is a generalized pseudo-Anosov homeomorphism on A isotopic to the identity. Let $\tilde{g} : \tilde{A} \to \tilde{A}$ be a lift of g. We have the following lemma:

Lemma 3 *Suppose l and n are positive integers, and x is a periodic point of ϕ in $D - \{x_0\}$. Assume the following conditions:*

(a) $\operatorname{per}(x, \phi) < l \operatorname{per}(x, \phi^l)$.
(b) $\rho(\tilde{g}) \ni k/l$ *for some integer* k *prime to* l.
(c) $n \geq 2l + \operatorname{per}(x, \phi)$.

Then ϕ *has at least* $2n+3$ *periodic points of period* $\leq n$ *in the interior of* D.

Proof. We first show that $\rho(x, \tilde{g}) \neq k/l$. Assume $\rho(x, \tilde{g})$ were equal to k/l. Then $\operatorname{per}(x, \phi)$ must be a multiple of l, and hence $\operatorname{per}(x, \phi^l) = \operatorname{per}(x, \phi)/l$. This contradicts the hypothesis (a). Thus $\rho(x, \tilde{g}) \neq k/l$.

We assume $\rho(x, \tilde{g}) > k/l$, the other case being similar. Write $\rho(x, \tilde{g})$ as $\rho(x, \tilde{g}) = k'/l'$, where k'/l' is expressed in lowest terms. Let I be the interval $[k/l, k'/l']$. Let $\mathcal{F}_n$ be the Farey series of order n, i.e. the ascending series of rational numbers between 0 and 1 whose denominators do not exceed n. Write $\mathcal{F}_n \cap I$ as

$$\mathcal{F}_n \cap I = \{k_1/l_1, \ldots, k_\beta/l_\beta\},$$

where $k_1/l_1 = k/l$, $k_\beta/l_\beta = k'/l'$, $k_i/l_i < k_{i+1}/l_{i+1}$ and k_i/l_i is expressed in lowest terms for every i.

The assumption (c) implies $\beta \geq 4$. In fact, since $2l + l' \leq 2l + \operatorname{per}(x, \phi) \leq n$, the set $\mathcal{F}_n \cap I$ contains, besides k/l, k'/l', two distinct rationals $(k+k')/(l+l')$, $(2k+k')/(2l+l')$.

Since $\rho(\tilde{g})$ contains k/l by (b) and also contains $\rho(x, \tilde{g}) = k'/l'$, we see by Theorem 4 that, for any i with $1 \leq i \leq \beta$, g has an l_i-periodic point $x_i \in A$ with rotation number k_i/l_i. A result of Guaschi [11, Theorem 3] says that, for a generalized pseudo-Anosov homeomorphism on the closed annulus, there are periodic points in the interior whose rotation numbers are those of the boundaries. Hence, we can take all of the periodic points x_i to be contained in the interior of A. Hence, they become periodic points of ϕ in $\operatorname{Int} D - \{x_0\}$. Since the orbit of x_i consists of l_i points, this implies ϕ has at least $l_1 + \cdots + l_\beta$ periodic points of period $\leq n$ except for x_0. Since $\beta \geq 4$ and the sum of denominators of any consecutive two numbers in $\mathcal{F}_n$ is greater than n (see e.g. [13, Theorem 30]), $l_1 + \cdots + l_\beta \geq 2(n+1)$. Hence we have the proof. $\square$

Let $\psi : S^2 \to S^2$ be an orientation-preserving generalized pseudo-Anosov homeomorphism of a sphere S^2. Given positive integers n, l, a, let $P_n(l, a)$ denote the set of n-periodic points of ψ of type (l, a). Let P_n^+ be the set of n-periodic points of ψ with index one. We have the following formula on these numbers:

Proposition 2

$$\sharp P_n^+ = \sum_{a\geq 1}\left(\sum_{\substack{l>1\\ l|n}} al\,\sharp P_{\frac{n}{l}}(l,a) + (a-1)\sharp P_n(1,a)\right) + \epsilon_n,$$

where $l|n$ means that l is a divisor of n, and $\epsilon_n = 0$ or 2 according to $n > 1$ or $n = 1$.

Proof. Let q be a positive integer. Since $\mathrm{Fix}(\psi^q)$ is the disjoint union of $P_n(l,a)$ for all positive integers l, a and all n with $n|q$, we have by (2)

$$\begin{aligned} 2 &= \chi(S^2) = \sum_{x\in \mathrm{Fix}(\psi^q)} \mathrm{ind}(x,\psi^q) \\ &= \sum_{n|q} \sharp P_n(\psi) - \sum_{a\geq 1}\sum_{\substack{l,n\\ ln|q}} al\,\sharp P_n(l,a). \end{aligned}$$

Note that for each a,

$$\sum_{\substack{l,n\\ ln|q}} al\,\sharp P_n(l,a) = \sum_{\substack{l,m\\ l|m|q}} al\,\sharp P_{\frac{m}{l}}(l,a) = \sum_{\substack{l,n\\ l|n|q}} al\,\sharp P_{\frac{n}{l}}(l,a).$$

Therefore if we define an integer Ω_n for any $n \geq 1$ by

$$\Omega_n = \sharp P_n(\psi) - \sum_{a\geq 1}\sum_{l|n} al\,\sharp P_{\frac{n}{l}}(l,a),$$

then we have $\sum_{n|q}\Omega_n = 2$ for any $q \geq 1$. In the case of $q = 1$, this means $\Omega_1 = 2$. Hence we have $\sum_{n>1,n|q}\Omega_n = 0$ for any $q > 1$. Therefore we can easily prove that $\Omega_n = 0$ for $n > 1$ by induction on n. Thus we have proved that

$$\Omega_n = \epsilon_n \quad \text{for any} \quad n \geq 1. \tag{6}$$

Since $P_n(l,a) \subset P_n^+$ if and only if $l \geq 2$, $P_n(\psi)$ is the disjoint union of P_n^+ and $\bigcup_a P_n(1,a)$ and hence $\sharp P_n^+ = \sharp P_n(\psi) - \sum_a \sharp P_n(1,a)$. Therefore by (6), the proof is completed. □

3. Proof of Theorem 1

Let $\phi : D \to D$ be a canonical homeomorphism which is isotopic to f relative to S. Since $\mathrm{bt}(S,f)$ contains a pseudo-Anosov component, so does ϕ. Let L be the union of all pseudo-Anosov components of ϕ. By Proposition 1, it suffices for the proof to show that

$$\sharp(P^n(\phi) \cap (S \cup \operatorname{Int} L)) \geq 2n + 3 \tag{7}$$

for every $n \geq \operatorname{per}(S)$. We shall fix an integer n with $n \geq \operatorname{per}(S)$. The proof is divided into two parts (I), (II).

(I) Here we consider the case where $\operatorname{bt}(S, f)$ is pseudo-Anosov, that is, the canonical homeomorphism ϕ is pseudo-Anosov relative to S. Note that, in this case, $S \cup \operatorname{Int} L = \operatorname{Int} D$ and the inequality (7) is equivalent to

$$\sharp(P^n(\phi) \cap \operatorname{Int} D) \geq 2n + 3. \tag{8}$$

Since the Euler characteristic of the disk is one, ϕ has a fixed point x_1 in $\operatorname{Int} D$ of index 1. Choose a singularity x_2 of $\mathcal{F}^s$ in ∂D. By (3), x_2 is a periodic point with index -1. Let (l_1, a_1) denote the type of x_1, and l_2 the period of x_2. Note that $l_1 > 1$ since x_1 has index 1, but l_2 may be 1.

We claim that ϕ has an l_1-periodic point y_1 and an l_2-periodic point y_2 in $\operatorname{Int} D - \{x_1\}$ which have index 1 and belong to different orbits. Let $\psi : S^2 \to S^2$ be the generalized pseudo-Anosov homeomorphism of the sphere induced by ϕ by collapsing the boundary of D to a point denoted by ∞. Then ∞ is a fixed point of ψ of type (l_2, a_2) for some $a_2 \geq 1$. By Proposition 2, we have

$$\sharp P_n^+ \geq \begin{cases} \displaystyle\sum_a an\sharp P_1(n,a) & \text{if } n > 1 \\ 2 & \text{if } n = 1. \end{cases} \tag{9}$$

Since $P_1(l_1, a_1) \ni x_1$, this implies that $\sharp P_{l_1}^+ \geq a_1 l_1 \sharp P_1(l_1, a_1) > 0$. Therefore, there exists an l_1-periodic point y_1 of ψ with index 1. Since $l_1 > 1$, we see $y_1 \neq \infty$ and y_1 is a periodic point of ϕ in $\operatorname{Int} D - \{x_1\}$. Let Y_1 be the orbit of y_1 under the map ϕ, and let $Q = P_{l_2}^+ - (Y_1 \cup \{x_1, \infty\})$. To show the existence of the desired l_2-periodic point y_2, it suffices to see that $\sharp Q > 0$. Consider first the case of $l_2 \neq l_1$. If $l_2 \geq 2$, then $Q = P_{l_2}^+$. Therefore, since $P_1(l_2, a_2) \ni \infty$, we have $\sharp Q > 0$ by (9). If $l_2 = 1$, then the fixed point ∞ has index $1 - a_2 \leq 0$. Therefore Q is equal to $P_1^+ - \{x_1\}$ and hence $\sharp Q > 0$ by (9). Consider next the case of $l_2 = l_1$. Then $\bigcup_a P_1(l_2, a) \supset \{x_1, \infty\}$, and $l_2 > 1$. Hence $\sharp Q = \sharp(P_{l_2}^+ - Y_1) \geq a_1 l_1 + a_2 l_2 - l_1 > 0$. Thus we have verified the existence of y_2. Let Y_2 be the orbit of y_2. For $i = 1, 2$, let (m_i, b_i) be the type of y_i. Since y_i has index 1, $m_i \geq 2$.

Arguing similarly as above, we can show that for $i = 1, 2$, ϕ has an $l_i m_i$-periodic point z_i in $\operatorname{Int} D - (Y_1 \cup Y_2)$ of index 1 such that z_1 and z_2 belong to different orbits. Let Z_i be the orbit of z_i. It is clear that $\{x_1\}$, Y_1, Y_2, Z_1, Z_2 are mutually disjoint.

There are two cases to consider:

(1) $n \geq l_1(m_1 + 2)$ or $n \geq l_2(m_2 + 2)$.

(2) $n < l_i(m_i + 2)$ for $i = 1$ and 2.

Case (1): In this case $n \geq l_{i_0}(m_{i_0} + 2)$, where $i_0 = 1$ or 2. To simplify the notation, let $l = l_{i_0}$, $a = a_{i_0}$, $m = m_{i_0}$, $b = b_{i_0}$. Let A be the annulus obtained from D by blowing up the point y_{i_0} into a circle Γ. Let $g : A \to A$ be the blow-up of ϕ^l at y_{i_0}. Since x_1 is a fixed point of g, $\rho(x_1, \tilde{g})$ is an integer. By choosing an appropriate lift $\tilde{g}$ of g, we can assume $\rho(x_1, \tilde{g}) = 0$. Since y_{i_0} has type (m, b), $\rho(\Gamma, \tilde{g}) = d/m$ for some integer d prime to m. These imply that $\rho(\tilde{g})$ contains 0 and d/m, and hence it contains $[0, 1/m]$ or $[-1/m, 0]$. We assume $\rho(\tilde{g}) \supset [0, 1/m]$, the other case being similar. Then, by Theorem 4, for any $s > m$, g has an s-periodic point u_s with $\rho(u_s, \tilde{g}) = 1/s$. Since $s > m$, $u_s \notin \Gamma$ and hence u_s is an s-periodic point of ϕ^l and so $\operatorname{per}(u_s, \phi)$ is a divisor of ls.

There are two subcases:

(i) $\operatorname{per}(u_s, \phi) < ls$ for some s with $m < s \leq n/l$.

(ii) $\operatorname{per}(u_s, \phi) = ls$ for every s with $m < s \leq n/l$.

Case (i): Choose an s such that $m < s \leq n/l$ and $\operatorname{per}(u_s, \phi) < ls$. Let A' be the annulus obtained by blowing up the point x_1 to a circle Γ', and $g' : A' \to A'$ the blow-up of ϕ at x_1. Let $\tilde{g}' : \tilde{A}' \to \tilde{A}'$ be a lift of g' to the universal cover $\tilde{A}'$ of A'. We show that the hypotheses (a), (b), (c) of Lemma 3 are satisfied if we replace x_0, x, g there by x_1, u_s, g' respectively.

(a) This is the assumption of Case (i).

(b) If $i_0 = 2$, then $\rho(\tilde{g}') \ni \rho(\partial D, \tilde{g}') = k/l$, where k is an integer prime to l. If $i_0 = 1$, then since x_1 is of type (l, a), $\rho(\tilde{g}') \ni \rho(\Gamma', \tilde{g}') = k/l$ for some k prime to l.

(c) Since $\operatorname{per}(u_s, \phi)$ is a proper divisor of ls, it does not exceed $ls/2$. Therefore, since $m \geq 2$ and the inequality $n \geq l(m + 2)$ is assumed,

$$2l + \operatorname{per}(u_s, \phi) \leq \frac{l(m+2)}{2} + \frac{ls}{2} \leq \frac{n}{2} + \frac{n}{2} = n.$$

Thus, all the hypotheses of Lemma 3 have been verified, and hence (8) holds.

Case (ii): Let $\alpha = [n/l]$, where [ ] denotes the Gauss symbol, i.e. α is the greatest integer which does not exceed n/l. The hypothesis $n \geq l(m+2)$ implies that $\alpha \geq m+2 \geq 4$. Since $\sharp(P^n(\phi) \cap \operatorname{Int} D)$ is equal to the sum of periods of all interior periodic orbits with period $\leq n$, we have

$$\begin{aligned}
\sharp(P^n(\phi) \cap \operatorname{Int} D) &\geq \operatorname{per}(x_1, \phi) + \operatorname{per}(y_{i_0}, \phi) + \operatorname{per}(z_{i_0}, \phi) \\
&\quad + \sum_{s=m+1}^{\alpha} \operatorname{per}(u_s, \phi) \\
&\geq 1 + l + \sum_{s=m}^{\alpha} ls \geq 1 + l + \sum_{s=\alpha-2}^{\alpha} ls \\
&\geq 3\alpha l - 2l + 1.
\end{aligned}$$

Since $\alpha l \geq n-l+1$ and $\alpha \geq 4$, we have $3\alpha l - 2l + 1 \geq 2(n-l+1) + (\alpha-2)l + 1 \geq 2n+3$. Therefore (8) holds.

Case (2): Let $\mathcal{S}$ (resp. $\mathcal{S}_1$) be the set of singularities (resp. one-pronged singularities) of $\mathcal{F}^s$. Note that $\mathcal{S}_1$ is a subset of S by the definition of a pseudo-Anosov homeomorphism relative to S. Also, $\mathcal{S}_1$ has no intersectin with the disjoint union of $\{x_1\}$, Y_1, Y_2, Z_1, Z_2, since every point in $\mathcal{S}_1$ has index zero. Clearly $P^n(\phi) \cap \operatorname{Int} D$ contains x_1. Also it contains $\mathcal{S}_1$, since n is taken to satisfy $n \geq \operatorname{per}(S)$. Note that $P^n(\phi) \cap \operatorname{Int} D \supset Y_i$ (resp. Z_i) if $n \geq l_i$ (resp. $n \geq l_i m_i$). Therefore, if we let $\delta(k) = 1$ or 0 according to whether $k \leq n$ or $k > n$, then we have

$$\sharp(P^n(\phi) \cap \operatorname{Int} D) \geq 1 + \sharp\mathcal{S}_1 + \sum_{i=1,2} \left(l_i \delta(l_i) + l_i m_i \delta(l_i m_i)\right). \tag{10}$$

Since the number of prongs at x_1, x_2, y_i, and any point in $\mathcal{S}_1$ are $a_1 l_1$, 3, $b_i m_i$, 1 respectively, the formula (1) implies the following:

$$-2 = \sum_{x \in \mathcal{S}} (p(x) - 2) \geq (a_1 l_1 - 2) + a_2 l_2 + \sum_{i=1,2} l_i (b_i m_i - 2) - \sharp\mathcal{S}_1.$$

Hence

$$\sharp\mathcal{S}_1 \geq \sum_{i=1,2} l_i (m_i - 1). \tag{11}$$

Let

$$\mu_i = l_i(m_i - 1) + l_i\delta(l_i) + l_i m_i \delta(l_i m_i).$$

Then we have by (10), (11),

$$\sharp(P^n(\phi) \cap \operatorname{Int} D) \geq \mu_1 + \mu_2 + 1. \tag{12}$$

We have by the definition of μ_i:

$$\mu_i = \begin{cases} l_i(m_i - 1) & \text{if } \ n < l_i \\ l_i m_i & \text{if } \ l_i \leq n < l_i m_i \\ 2 l_i m_i & \text{if } \ l_i m_i \leq n < l_i(m_i + 2). \end{cases}$$

Since $l_i(m_i - 1) \geq l_i$ and $2l_i m_i \geq l_i(m_i + 2) > n$, this implies that $\mu_i \geq n+1$. Hence (12) implies (8).

(II) We consider here the case where $\operatorname{bt}(S, f)$ is reducible and contains a pseudo-Anosov component. Choose a pseudo-Anosov component of ϕ and denote it by N. Let μ be the least positive integer such that $\phi^\mu(N) = N$. Let d be the number of boundary circles of N. Since N is homeomorphic to a disk possibly with finitely many open disks removed, N has a boundary circle C_0 such that all the other boundary circles (called the inner boudary circles) lie inside of C_0. Denote the inner boundary circles of N by $C_1, \ldots, C_{d-1}$. Let D' be the disk obtained from N by collapsing each C_i ($i = 1, \ldots, d-1$) to a point, which we denote by c_i. Let $S' = (S \cap N) \cup \{c_i\}_{i=1}^{d-1}$. Let $\phi' : D' \to D'$ be a pseudo-Anosov homeomorphism relative to S' induced by $\phi^\mu : N \to N$.

Let $n' = [n/\mu]$. We show $n' \geq \operatorname{per}(S')$, where $\operatorname{per}(S') = \max_{x \in S'} \operatorname{per}(x, \phi')$. If $x \in S \cap N$, clearly $\operatorname{per}(x, \phi') = \operatorname{per}(x, \phi)/\mu \leq \operatorname{per}(S)/\mu$. Let X_i be the disk bounded by the simple closed curve C_i. Then $S \cap X_i$ has at least two points, since C_i is isotopic to a reducing curve in $D - S$. Choose a point w_i in $S \cap X_i$. Then C_i must go back to itself under $\operatorname{per}(w_i, \phi)$ times iterate of ϕ. Hence $\operatorname{per}(c_i, \phi') \leq \operatorname{per}(w_i, \phi)/\mu \leq \operatorname{per}(S)/\mu$. Therefore we have $\operatorname{per}(S') \leq [\operatorname{per}(S)/\mu] \leq n'$.

Thus, replacing ϕ, D, n in the estimate (8) by ϕ', D', n' respectively, we have $\sharp(P^{n'}(\phi') \cap \operatorname{Int} D') \geq 2n' + 3$. Since $\operatorname{Int} N = \operatorname{Int} D' - \{c_i\}_{i=1}^{d-1}$, this implies $\sharp(P^{n'}(\phi^\mu) \cap \operatorname{Int} N) \geq 2n' + 4 - d$. Therefore, letting $\bar{N}$ be the disjoint union of $N, \phi(N), \ldots, \phi^{\mu-1}(N)$, we have the following: (Note that

$\sharp(P^n(\phi)\cap\phi^i(\mathrm{Int}\,N))$ is independent of i.)

$$\begin{aligned}\sharp(P^n(\phi)\cap\mathrm{Int}\,\bar{N}) &= \mu\sharp(P^n(\phi)\cap\mathrm{Int}\,N)\geq\mu\sharp(P^{n'}(\phi^\mu)\cap\mathrm{Int}\,N)\\ &\geq \mu(2n'+4-d).\end{aligned}$$

Therefore, since $L\supset\bar{N}$, $S\subset P^n(\phi)$, and $\sharp(S-\mathrm{Int}\,\bar{N})\geq\mu\sum_{i=1}^{d-1}\sharp(S\cap X_i)\geq 2\mu(d-1)$, we have

$$\begin{aligned}\sharp(P^n(\phi)\cap(S\cup\mathrm{Int}\,L)) &\geq \sharp(P^n(\phi)\cap\mathrm{Int}\,\bar{N})+\sharp(S-\mathrm{Int}\,\bar{N})\\ &\geq \mu(2n'+4-d)+2\mu(d-1)\\ &\geq \mu(2n'+3).\end{aligned}$$

Since $n'\geq(n-\mu+1)/\mu$, we have $\mu(2n'+3)\geq 2n+\mu+2\geq 2n+3$. Thus (7) is proved and the proof is completed.

4. Transversal homeomorphisms

We prove here some results on n-transversal homeomorphisms on the disk which will be needed to prove Theorem 2. Let S be a finite set in $\mathrm{Int}\,D$. Let $\phi: D\to D$ be a canonical homeomorphism relative to S with a pseudo-Anosov component. Let L be the union of all pseudo-Anosov components of ϕ.

Given a periodic point $x\in L$ of ϕ with period m, define integers $\nu^n(x,\phi)$ for $n\geq 1$ by

$$\nu^n(x,\phi)=\begin{cases}\max\limits_{1\leq k\leq n/m}|\mathrm{ind}(x,\phi^{mk})| & \text{if } n\geq m \text{ and } x\notin\mathcal{S}_1,\\ 2 & \text{if } n\geq m \text{ and } x\in\mathcal{S}_1,\\ 0 & \text{if } n<m.\end{cases}$$

Note that if x is in $\mathrm{Int}\,L-\mathcal{S}_1$ and has type (l,a), then by (2), $\nu^n(x,\phi)=al-1$ if $n\geq ml$ and $\nu^n(x,\phi)=1$ if $m\leq n<ml$.

We can assume without loss of generality that ϕ has no periodic points on the open tubular neighborhood $A(\Sigma)$, and moreover that ϕ sends every point in $A(\Sigma)-\Sigma$ to a point having a greater distance from Σ. Then any singular point of $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) on ∂L has index -1 (resp. 0) as a periodic point of ϕ. This implies that if x is an m-periodic point in ∂L, then for any $n\geq m$, we have $\nu^n(x,\phi)=1$ or 0 according to whether $x\in\mathrm{Sing}(\mathcal{F}^s)$ or $x\in\mathrm{Sing}(\mathcal{F}^u)$.

We extend the definition of $\nu^n(x,\phi)$ for all points in L by setting

$\nu^n(x,\phi)=0$ for every non-periodic point x. For a subset E of L, let

$$\nu^n(E,\phi)=\sum_{x\in E}\nu^n(x,\phi).$$

This sum can be defined, since $P^n(\phi)$ has finitely many points and so $\nu^n(x,\phi)=0$ except for finitely many x. To simplify the notation, we shall write $\nu^n(x)$, $\nu^n(E)$ for $\nu^n(x,\phi)$, $\nu^n(E,\phi)$ respectively. We have:

Proposition 3 *Let n be a positive integer. Suppose $f:D\to D$ is an orientation-preserving n-transversal homeomorphism with $f(S)=S$ and* $\mathrm{per}(S)\le n$. *Suppose f is isotopic to ϕ relative to S. Let N be a pseudo-Anosov component of ϕ. Let μ be the least positive integer such that $\phi^\mu(N)=N$. Then we have*

$$\sharp P^n(f)\ge \mu(\nu^n(N)+2d_0-d_1),$$

where d_0 (resp. d_1) is the number of inner boundary circles C of N which satisfy the condition that the common period of the periodic points on C is greater than n (resp. less than or equal to n).

Proof. Let $\bar N=\bigcup_{j=0}^{\mu-1}\phi^j(N)$. Let d be the number of boundary circles of N. Let C_0 be the outer boundary circle of N, and $C_1,\dots,C_{d-1}$ the inner boundary circles. Let $\bar C_0=\bigcup_{j=0}^{\mu-1}\phi^j(C_0)$. Let X_0 be the outer connected component of $D-\mathrm{Int}\,\bar N$, i.e., the connected component of $D-\mathrm{Int}\,\bar N$ which contains ∂D. Then $C_0=X_0\cap N$, and either $X_0=C_0=\partial D$ or X_0 is a compact surface with boundary $\bar C_0\cup\partial D$. For $1\le i\le d-1$, let X_i be the connected component of $D-\mathrm{Int}\,\bar N$ whose boundary circle is C_i. Namely, X_i is a disk bounded by the simple closed curve C_i. Clearly, for every $i>0$, X_i lies inside of C_0.

Let D_S be the compactification of $D-S$, $\pi_S:D_S\to D$ the projection, and Γ the union of the attached boundary circles. For $x\in S$, let $\Gamma_x=\pi_S^{-1}(x)$. Let $N_S=\pi_S^{-1}(N)$. N_S is a compact surface whose boundary circles are $C_0,C_1,\dots,C_{d-1}$, Γ_x $(x\in S\cap N)$. Let $Y_i=\pi_S^{-1}(X_i)$ for $0\le i\le d-1$. For $i>0$, Y_i is a compact surface whose boundary is the union of C_i and Γ_x, $x\in S\cap X_i$.

Let $\phi_S:D_S\to D_S$ be the blow-up of ϕ at S. Let $\mathcal{E}$ be the collection of the following subsets of D_S :

$$\{x\}\ (x\in\mathrm{Int}\,N-S),\quad \Gamma_x\ (x\in S\cap N),\quad Y_i\ (1\le i\le d-1).$$

Let

$$\bar{\mathcal{E}} = \{Y_0\} \cup \{\phi_S^j(E) \mid E \in \mathcal{E},\ 0 \le j \le \mu - 1\}.$$

Then D_S is the disjoint union of the sets in $\bar{\mathcal{E}}$.

An important property of $\bar{\mathcal{E}}$ is that for any $m > 0$, every ϕ_S^m-Nielsen class is contained in exactly one of the elements of $\bar{\mathcal{E}}$. This follows easily from Lemma 2. Furthermore, if two invariant sets of ϕ_S^m are ϕ_S^m-related, then they are contained in the same element of $\bar{\mathcal{E}}$.

For $x \in S$ and $m \ge 1$, let V_m^x be the open disk centered at x introduced in the proof of Proposition 1. Since f is n-transversal, any fixed point of f^m is isolated for all $m \le n$. Hence we can assume that x is the only point in the set $P^n(f) \cap V_m^x$. Let

$$\begin{aligned} V^x &= \bigcap_{m=1}^{n} V_m^x, \quad V = \bigcup_{x \in S} V^x, \quad V_S^x = (V^x - \{x\}) \cup \Gamma_x, \\ V_S &= (V - S) \cup \Gamma. \end{aligned}$$

These sets are open neighborhoods of x, S, Γ_x and Γ respectively. Perturbing f in a sufficiently small neighborhood of S contained in V, we obtain a homeomorphism $f' : D \to D$ which is isotopic to f relative to S and has the blow-up, which we denote by f_S, at S. We can also require that for all $m \le n$, f^m and $(f')^m$ coincide away from V. Then for all $m \le n$, we have $(f')^m(V^x) = f^m(V^x) \subset W^{f^m(x)}$ and hence every fixed point of f_S^m in V_S^x is f_S^m-related to Γ_x. Note also that $P^n(f_S) - V_S = P^n(f) - V$. Let $H = \{H_t\} : D_S \to D_S$ be the isotopy from ϕ_S to f_S.

For $E \in \bar{\mathcal{E}}$ and $m \ge 1$, define a subset $\Delta_m(E)$ of $\mathrm{Fix}(f_S^m)$ by

$$\Delta_m(E) = \Delta_{H^m}(E)\ (= \Delta_{H^m}(E \cap \mathrm{Fix}(\phi_S^m))),$$

where $H^m = \{H_t^m\}$. If E, E' are distinct elements of $\bar{\mathcal{E}}$, then $\Delta_m(E)$ and $\Delta_{m'}(E')$ are disjoint for any m, m'. In fact, if they have an intersection for some m, m', then some $x \in E \cap \mathrm{Fix}(\phi_S^m)$ and some $x' \in E' \cap \mathrm{Fix}(\phi_S^{m'})$ are $H^{mm'}$-related to the same point, and hence x and x' are $\phi_S^{mm'}$-Nielsen equivalent. This contradicts the property of $\bar{\mathcal{E}}$ stated above.

Let E be an element of $\bar{\mathcal{E}}$. Since $E \cap \mathrm{Fix}(\phi_S^m)$ is empty or a union of ϕ_S^m-Nielsen classes and Δ_{H^m} preserves the fixed point index for each ϕ_S^m-Nielsen class, we have

$$\mathrm{ind}(\Delta_m(E), f_S^m) = \mathrm{ind}(E \cap \mathrm{Fix}(\phi_S^m), \phi_S^m). \tag{13}$$

For $m \geq 1$, let $\Theta_m(E)$ be the set of fixed points x of f_S^m satisfying the condition that x is f_S^m-related to some f_S^m-invariant, connected component of $\Gamma \cap E$. Note that $\Theta_m(E) - \Delta_m(E)$ is a union of f_S^m-Nielsen classes, and moreover all of these Nielsen classes are inessential. In fact, suppose F is an essential f_S^m-Nielsen class contained in $\Theta_m(E)$. If we denote by F' the unique ϕ_S^m-Nielsen class satisfying $\Delta_m(F') = F$, then F' is ϕ_S^m-related to $\Gamma \cap E$ by Lemma 1 and so $F' \subset E$. This implies $F \subset \Delta_m(E)$, and hence $\Theta_m(E) - \Delta_m(E)$ contains no essential Nielsen classes. Thus the set $\Delta_m(E) \cup \Theta_m(E)$ has the same fixed point index under f_S^m as the set $\Delta_m(E)$. Note also that $\Delta_m(\Gamma \cap E) \subset \Theta_m(E)$. If E is a single point in $\operatorname{Int} N - S$, then $\Theta_m(E)$ is empty for every m, since $\Gamma \cap E$ is empty in this case. Let $V_S^E = \bigcup_x V_S^x$, where the union is taken over all $x \in S$ with $\Gamma_x \subset E$. It is clear that if E is a single point in $\operatorname{Int} N - S$, then V_S^E is empty. Also, if E is Γ_x, where $x \in S$, then $V_S^E = V_S^x$. We show that

$$(\Delta_m(E) \cup \Theta_m(E)) \cap V_S \subset V_S^E. \tag{14}$$

Suppose first that y is a point in $\Delta_m(E) \cap V_S$. Then there is a point $x \in E \cap \operatorname{Fix}(\phi_S^m)$ which is H^m-related to y. Also, $y \in V_S^z$ for some $z \in S$. It follows from the property of V^z that y and Γ_z are f_S^m-related. Therefore by Lemma 1, x and Γ_z are ϕ_S^m-related, and hence by the property of $\bar{\mathcal{E}}$, they are contained in the same element of $\bar{\mathcal{E}}$. Since $x \in E$, this element must be E and hence $\Gamma_z \subset E$. Thus, $y \in V_S^E$. Suppose next that $y \in \Theta_m(E) \cap V_S$. Then $y \in V_S^z$ for some z, which implies y is f_S^m-related to Γ_z. On the other hand, $y \in \Theta_m(E)$ implies y is f_S^m-related to a component of $\Gamma \cap E$. These imply Γ_z is f_S^m-related, and so ϕ_S^m-related, to $\Gamma \cap E$. Hence $\Gamma_z \subset E$ and $y \in V_S^E$. Thus (14) is proved.

For $E \in \bar{\mathcal{E}}$, define a subset $F^n(E)$ of $P^n(f_S) - V_S$ by

$$F^n(E) = \bigcup_{m=1}^{n} (\Delta_m(E) \cup \Theta_m(E)) - V_S.$$

Then it follows from the property of $\bar{\mathcal{E}}$ that $F^n(E)$, $E \in \bar{\mathcal{E}}$, are disjoint subsets of $P^n(f) - S$. We can assume that $V^{f^j(x)} \subset f^j(V^x)$ for every $x \in S$ inside C_0 and for all positive integers $j < \mu$. Then, for any $E \in \mathcal{E}$ and every positive $j < \mu$, we have $F^n(\phi_S^j(E)) \supset f_S^j(F^n(E))$ and hence $\sharp F^n(\phi_S^j(E)) \geq \sharp F^n(E)$. Thus we obtain the following inequality:

$$\sharp P^n(f) \geq \mu \sum_{E \in \mathcal{E}} \sharp F^n(E) + \sharp F^n(Y_0) + \sharp S. \tag{15}$$

We now estimate each term in the right-hand side of this inequality. Suppose m is a positive integer with $m \le n$. Since f is n-transversal, any fixed point x of f^m has index 1 or -1 if $x \in \operatorname{Int} D$ (see 3.2 (2) in [14, p. 12]), and index 1, -1, or 0 if $x \in \partial D$. Therefore, for any set K of fixed points of f_S^m in $D_S - V_S$, we have

$$\sharp K \ge |\operatorname{ind}(K, f_S^m)|. \tag{16}$$

Also, since for any $x \in S$, $(f')^m$ and f^m have the same fixed point index on V^x and $\operatorname{Fix}(f^m) \cap V^x = \{x\}$, we have for any $x \in \operatorname{Fix}(f^m) \cap S$, that

$$\operatorname{ind}(\operatorname{Fix}(f_S^m) \cap V_S^x, f_S^m) = \operatorname{ind}(x, f^m) - 1 = 0 \quad \text{or} \quad -2. \tag{17}$$

By (16), we see that if $\Delta_m(E)$ is disjoint from V_S, then

$$\sharp\Delta_m(E) \ge |\operatorname{ind}(\Delta_m(E), f_S^m)| = |\operatorname{ind}(E \cap \operatorname{Fix}(\phi_S^m), \phi_S^m)|. \tag{18}$$

Suppose $x \in \operatorname{Int} N - S$. We show

$$\sharp F^n(\{x\}) \ge \nu^n(x). \tag{19}$$

If $x \notin P^n(\phi)$, then clearly $F^n(\{x\})$ is empty and hence $\sharp F^n(\{x\}) = 0 = \nu^n(x)$. Suppose $x \in P^n(\phi)$. Let $m = \operatorname{per}(x, \phi)$. By (14), $\Delta_j(\{x\}) \cap V_S = \emptyset$ for any $j > 0$, and hence $F^n(\{x\}) \supset \bigcup_{k \le n/m} \Delta_{mk}(\{x\})$. Also, by (18) we have

$$\sharp\Delta_{mk}(\{x\}) \ge |\operatorname{ind}(x, \phi_S^{mk})| = |\operatorname{ind}(x, \phi^{mk})|$$

for any k. Therefore

$$\sharp F^n(\{x\}) \ge \max_{k \le \frac{n}{m}} |\operatorname{ind}(x, \phi^{mk})| = \nu^n(x).$$

Thus (19) is proved.

Suppose $x \in S \cap N$. Let (l, a) be its type with respect to ϕ and $m = l \operatorname{per}(x, \phi)$. Then m is the period of the periodic points of ϕ_S on Γ_x. Let $Q = \Delta_m(\Gamma_x) \cup \Theta_m(\Gamma_x)$, $Q' = Q \cap V_S^x$. Since $Q \cap V_S \subset Q'$ by (14), we have $Q - V_S \supset Q - Q'$ and hence by (16),

$$\sharp F^n(\Gamma_x) \ge \delta(m) \sharp (Q - V_S) \ge \delta(m) |\operatorname{ind}(Q, f_S^m) - \operatorname{ind}(Q', f_S^m)|,$$

where $\delta(m)$ is the number defined in I, Case (2) of Section 3. Since ϕ_S^m has al fixed points of index -1 on Γ_x, we see by (13) that

$$\operatorname{ind}(Q, f_S^m) = \operatorname{ind}(\Delta_m(\Gamma_x), f_S^m) = \operatorname{ind}(\Gamma_x \cap \operatorname{Fix}(\phi_S^m), \phi_S^m) = -al.$$

Since $Q' = \mathrm{Fix}(f_S^m) \cap V_S^x$, we have by (17) that if $m \leq n$ then $\mathrm{ind}(Q', f_S^m) = 0, -2$. As a consequence, if $x \notin \mathcal{S}_1$, then $\sharp F^n(\Gamma_x) \geq \delta(m)(al-2) \geq \nu^n(x)-1$. Suppose $x \in \mathcal{S}_1$. Then since $m = \mathrm{per}(x, \phi) \leq \mathrm{per}(S) \leq n$, we have $\delta(m) = 1$. Also $\mathrm{ind}(Q, f_S^m) = -al = -1$. Therefore we have $\sharp F^n(\Gamma_x) \geq 1 = \nu^n(x) - 1$. Thus, in either case, we have proved that

$$\sharp F^n(\Gamma_x) \geq \nu^n(x) - 1. \tag{20}$$

Let i be an integer with $0 \leq i \leq d-1$. Let n_i denote the period of a periodic point of ϕ on C_i. Note that n_i does not depend on the choice of a periodic point. Note also that d_0 (resp. d_1) is equal to the number of integers i such that $1 \leq i \leq d-1$ and $n < n_i$ (resp. $n \geq n_i$). Let $u_i = \sharp(\mathrm{Sing}(\mathcal{F}^s) \cap C_i)$, i.e., the cardinality of the set of periodic points of ϕ_S on C_i with index -1. Then $u_i > 0$, since $\mathcal{F}^s$ has at least one singularity on each boundary circle. Let λ_i be the number of points in $S \cap X_i$ which are fixed by ϕ^{n_i} (or equivalently by f^{n_i}). Then λ_i is equal to the number of connected components of $\Gamma \cap Y_i$ which are invariant under $\phi_S^{n_i}$. Therefore, since $\Theta_{n_i}(Y_i) \cap V_S$ is identical to $\mathrm{Fix}(f_S^{n_i}) \cap V_S^{Y_i}$, which is the union of $\mathrm{Fix}(f_S^{n_i}) \cap V_S^x$ for all $x \in S \cap X_i \cap \mathrm{Fix}(f^{n_i})$, we have by (17) that if $n \geq n_i$ then

$$0 \geq \mathrm{ind}(\Theta_{n_i}(Y_i) \cap V_S, f_S^{n_i}) \geq -2\lambda_i. \tag{21}$$

Consider the case of $i > 0$. Let A_i be the reducing annulus which is contained in Y_i and is adjacent to N. Let Σ_i be the reducing curve lying in A_i. Let A_i' be the connected components of $A_i - \Sigma_i$ which is adjacent to N, and let $Y_i^0 = Y_i - A_i'$. Then, Y_i^0 is a compact surface which has Σ_i as a boundary circle, and $\phi_S^{n_i}$ maps Y_i^0 onto itself. Let η be the restriction of $\phi_S^{n_i}$ to Y_i^0. Since ϕ is assumed to have no periodic points on A_i, $\mathrm{ind}((Y_i - C_i) \cap \mathrm{Fix}(\phi_S^{n_i}), \phi_S^{n_i})$ is equal to $\mathrm{ind}(Y_i^0 \cap \mathrm{Fix}(\phi_S^{n_i}), \eta) = \mathrm{ind}(\mathrm{Fix}(\eta), \eta)$, and hence equal to the Lefschetz number of η. Since this Lefschetz number is easily seen to be $1 - \lambda_i$, we have by (13)

$$\begin{aligned}
\mathrm{ind}(\Delta_{n_i}(Y_i), f_S^{n_i}) &= \mathrm{ind}(Y_i \cap \mathrm{Fix}(\phi_S^{n_i}), \phi_S^{n_i}) \\
&= \mathrm{ind}(C_i \cap \mathrm{Fix}(\phi_S^{n_i}), \phi_S^{n_i}) \\
&\quad + \mathrm{ind}((Y_i - C_i) \cap \mathrm{Fix}(\phi_S^{n_i}), \phi_S^{n_i}) \\
&= -u_i + (1 - \lambda_i).
\end{aligned}$$

Therefore, since $(\Delta_{n_i}(Y_i) \cup \Theta_{n_i}(Y_i)) \cap V_S = \Theta_{n_i}(Y_i) \cap V_S$, and since $\Delta_{n_i}(Y_i) \cup$

$\Theta_{n_i}(Y_i)$ has the same fixed point index as $\Delta_{n_i}(Y_i)$, we see by (16), (21) that if $n \geq n_i$ then

$$\begin{aligned}\sharp(\Delta_{n_i}(Y_i) \cup \Theta_{n_i}(Y_i) - V_S) \;\geq\; & \operatorname{ind}(\Theta_{n_i}(Y_i) \cap V_S, f_S^{n_i}) \\ & - \operatorname{ind}(\Delta_{n_i}(Y_i), f_S^{n_i}) \\ \geq\; & u_i - \lambda_i - 1.\end{aligned}$$

Hence

$$\sharp F^n(Y_i) \geq \delta(n_i)(u_i - \lambda_i - 1) = \nu^n(C_i) - \delta(n_i)(\lambda_i + 1). \tag{22}$$

Now consider the case of $i = 0$. Then, in the same way as in the case of $i > 0$, we can prove that

$$\begin{aligned}\operatorname{ind}(\Delta_{n_0}(Y_0), f_S^{n_0}) \;=\; & \operatorname{ind}(\bar{C}_0 \cap \operatorname{Fix}(\phi_S^{n_0}), \phi_S^{n_0}) \\ & + \operatorname{ind}((Y_0 - \bar{C}_0) \cap \operatorname{Fix}(\phi_S^{n_0}), \phi_S^{n_0}) \\ =\; & -\mu u_0 + (1 - \mu - \lambda_0).\end{aligned}$$

Hence, if $n \geq n_0$, then by (16), (21), $\sharp(\Delta_{n_0}(Y_0) \cup \Theta_{n_0}(Y_0) - V_S) \geq -2\lambda_0 - (-\mu u_0 + 1 - \mu - \lambda_0) \geq \mu u_0 - \lambda_0$. Therefore

$$\begin{aligned}\sharp F^n(Y_0) \geq \delta(n_0)(\mu u_0 - \lambda_0) \;\geq\; & \mu \nu^n(C_0) - \delta(n_0)\lambda_0 \\ \geq\; & \mu \nu^n(C_0) - \lambda_0.\end{aligned} \tag{23}$$

Since S has at least $\max\{\lambda_i, 2\}$ points in X_i,

$$\sharp S \geq \mu \sharp(S \cap N) + \mu \sum_{i>0} \max\{\lambda_i, 2\} + \lambda_0.$$

This implies that

$$-\lambda_0 + \sharp S \geq \mu\left(\sharp(S \cap N) + \sum_{i>0} \delta(n_i)(\lambda_i + 1) + 2d_0 - d_1\right),$$

since $\max\{\lambda_i, 2\} - \delta(n_i)(\lambda_i + 1) \geq 2$ or -1 according to $n < n_i$ or $n \geq n_i$. Thus by (15), (19), (20), (22), (23),

$$\begin{aligned}\sharp P^n(f) \;\geq\; & \mu\left(\sum_{x \in \operatorname{Int} N - S} \nu^n(x) + \sum_{x \in S \cap N} (\nu^n(x) - 1)\right. \\ & \left. + \sum_{i>0} (\nu^n(C_i) - \delta(n_i)(\lambda_i + 1))\right) + \mu \nu^n(C_0) - \lambda_0 + \sharp S\end{aligned}$$

$$\geq \mu\Big(\nu^n(N) - \sharp(S \cap N) - \sum_{i>0} \delta(n_i)(\lambda_i + 1)\Big) - \lambda_0 + \sharp S$$
$$\geq \mu(\nu^n(N) + 2d_0 - d_1).$$

□

In the rest of this section, we return to the situation of Lemma 3 in Section 2, i.e. $\phi : D \to D$ is an orientation-preserving generalized pseudo-Anosov homeomorphism, $x_0 \in \operatorname{Int} D$ a fixed point of ϕ, $g : A \to A$ the blow-up of ϕ at x_0, and $\tilde{g} : \tilde{A} \to \tilde{A}$ a lift of g. Given a positive integer n and an integer k prime to n, let

$$P_{n,k}(g) = \{x \in P_n(g) \mid \rho(x, \tilde{g}) = k/n\}.$$

Each $P_{n,k}(g)$ is identical to some g^n-Nielsen class. Since g is homotopic to the identity map, every g^n-Nielsen class has index zero. Therefore, we have:

$$\operatorname{ind}(P_{n,k}(g), g^n) = 0. \tag{24}$$

Lemma 4 *Assume all the hypotheses of Lemma* 3. *Then* $\nu^n(D) \geq 3n+8$.

Proof. Let $k_1/l_1, \ldots, k_\beta/l_\beta$ be the rational numbers and $\{x_i\}_{i=1}^{\beta}$ the interior periodic points of g in the proof of Lemma 3. Let X_i denote the orbit of x_i. Then X_i is a periodic orbit of ϕ in $\operatorname{Int} D - \{x_0\}$ with period $l_i \leq n$. If $X_i \subset \mathcal{S}_1$, then $\nu^n(X_i) = 2l_i$ by the definition of ν^n. Suppose X_i is not contained in $\mathcal{S}_1$. Since $x_i \in P_{l_i,k_i}(g)$ and by (2) $\operatorname{ind}(x_i, g^{l_i}) \neq 0$, it follows from (24) that there is another l_i-periodic point x_i' in A with nonzero index and rotation number k_i/l_i. Let X_i' be the orbit of x_i'. Since x_i and x_i' have nonzero index, $\nu^n(X_i)$, $\nu^n(X_i') \geq l_i$. Let Γ be the boundary circle of A corresponding to x_0. Divide $\{1, \ldots, \beta\}$ into three subsets J_1, J_2, J_3 given by

$$J_1 = \{i \mid X_i \subset \mathcal{S}_1\}, \quad J_2 = \{i \notin J_1 \mid X_i' \cap \Gamma = \emptyset\},$$
$$J_3 = \{i \notin J_1 \mid X_i' \subset \Gamma\}.$$

Since $X_i' \subset \Gamma$ implies $k_i/l_i = \rho(\Gamma, \tilde{g})$, it is impossible that J_3 contains more than one element. Therefore, J_3 is empty or consists of a single integer, say i_0. Consider first the case of $J_3 = \{i_0\}$. Then x_0 has type (l_{i_0}, a) for some $a \geq 1$. Since $n \geq l_{i_0}$, this implies that $\nu^n(x_0) \geq l_{i_0} - 1$. Therefore, since

$l_1 + \cdots + l_\beta \geq 2n + 2$,

$$\nu^n(D) \geq \sum_{i \in J_1} \nu^n(X_i) + \sum_{i \in J_2} \nu^n(X_i \cup X_i') + \nu^n(X_{i_0}) + \nu^n(x_0)$$
$$\geq 2 \sum_{i \neq i_0} l_i + l_{i_0} + (l_{i_0} - 1) \geq 4n + 3.$$

Since the assumption (a) implies $l \geq 2$, we have $n \geq 5$ by (c) and hence $4n + 3 \geq 3n + 8$. Thus we have the proof in this case. The proof for the case of J_3 empty is easier. □

5. Proof of Theorem 2

The proof is divided into two parts (I), (II).

(I) Consider the case where $\mathrm{bt}(S, f)$ is pseudo-Anosov. Then the canonical homeomorphism ϕ isotopic to f is pseudo-Anosov relative to S, and we have $\sharp P^n(f) \geq \nu^n(D)$ by Proposition 3. Therefore, it is enough to show that

$$\nu^n(D) \geq 3n + 6. \tag{25}$$

Let x_i, y_i, z_i $(i = 1, 2)$ be the periodic points of ϕ introduced in Section 3. x_1 is a fixed point of index 1, and its type is denoted by (l_1, a_1). x_2 is one of the periodic points on ∂D with index -1; its period is l_2 and there are $a_2 l_2$ periodic points on ∂D with index -1. y_i is an l_i-periodic point of index 1, and its type is (m_i, b_i). z_i is an $l_i m_i$-periodic point of index 1. Let X_i, Y_i, Z_i be the orbits of x_i, y_i, z_i respectively.

We have two cases to consider:

(1) $n \geq l_{i_0}(m_{i_0} + 2)$, where i_0 is 1 or 2.

(2) $n < l_i(m_i + 2)$ for $i = 1$ and 2.

Case (1): Let $l = l_{i_0}$, $m = m_{i_0}$. Let $g : A \to A$ be the blow-up of ϕ^l at y_{i_0}. Let Γ be a circle corresponding to y_{i_0}, and $\alpha = [n/l]$. For s with $m < s \leq \alpha$, let $U_s = P_{s,1}(g) \cup P_{s,-1}(g)$. We have shown in Section 3 that U_s is not empty for any s, if we choose the lift $\tilde{g}$ appropriately.

Consider first the case where, for some s, U_s contains an element u with $\mathrm{per}(u, \phi) < ls$. Then, as has been shown in Case (1), (i) of Section 3, all the hypotheses of Lemma 3 are satisfied if x_0, x are replaced by x_1, u respectively. Thus by Lemma 4, (25) is proved.

Consider next the case where $\mathrm{per}(u, \phi) = ls$ for any s and any $u \in U_s$.

Let $W = X_{i_0} \cup Y_{i_0} \cup \mathrm{Fix}(\phi)$. Since $\rho(\Gamma, \tilde{g}) = d/m$ for some d prime to m, we see $U_s \cap \Gamma$ is empty. Therefore U_s can be thought of as a subset of D. Moreover, $\{U_s\}_{s>m}$, Z_{i_0}, and W are mutually disjoint subsets of D. Thus

$$\nu^n(D) \geq \sum_{s=m+1}^{\alpha} \nu^n(U_s) + \nu^n(Z_{i_0}) + \nu^n(W). \tag{26}$$

Note that $\nu^n(Z_{i_0}) \geq lm$. We show $\nu^n(U_s) \geq 2ls$. In fact, if $U_s \subset \mathcal{S}_1$, then by the definition of ν^n, $\nu^n(U_s) \geq 2ls$. Also if U_s is disjoint from $\mathcal{S}_1$, then every point in U_s has nonzero index, and hence by (24), U_s contains at least two ls-periodic orbits with nonzero index. Thus $\nu^n(U_s) \geq 2ls$. Therefore, since $n \geq l(m+2)$ and hence $\alpha - 1 \geq m+1$, we have

$$\sum_{s=m+1}^{\alpha} \nu^n(U_s) \geq \sum_{s=\alpha-1}^{\alpha} 2ls \geq 2l(2\alpha - 1).$$

We show $\nu^n(W) \geq 2l + \epsilon$, where $\epsilon = 1, -1$ according to $l = 1$ or $l \geq 2$. If $l \geq 2$, then

$$\nu^n(W) \geq \nu^n(X_{i_0}) + \nu^n(Y_{i_0}) \geq (al - 1) + l(bm - 1) \geq 2l - 1.$$

In the case of $l = 1$, we have $\nu^n(W) \geq 3$. In fact, $l = 1$ is possible only if $i_0 = 2$. Then x_2 is a fixed point of ϕ on ∂D with index -1. Therefore, since $\chi(D) = 1$, there must be a fixed point of nonzero index different from x_1 and x_2. Thus, $\nu^n(W) \geq \nu^n(\mathrm{Fix}(\phi)) \geq 3$.

The above estimates together with (26) imply $\nu^n(D) \geq 4l\alpha + lm + \epsilon$. Since $l\alpha \geq n - l + 1$ and $n \geq l(m+2)$, this implies $\nu^n(D) - (3n+6) \geq 2l(m-1) - 2 + \epsilon > 0$. Hence (25) holds.

Case (2): Let

$$\begin{aligned} \nu_1 &= 2l_1(m_1 - 1) + (2l_1 - 2)\delta(l_1) + 2l_1(m_1 - 1)\delta(l_1 m_1) + 1, \\ \nu_2 &= 2l_2(m_2 - 1) + 2l_2\delta(l_2) + 2l_2(m_2 - 1)\delta(l_2 m_2). \end{aligned}$$

For $i = 1, 2$, let $Q_i = X_i \cup Y_i \cup Z_i$. Then we have:

Lemma 5 $\nu^n(\mathcal{S}_1 \cup Q_1 \cup Q_2) \geq \nu_1 + \nu_2$.

Proof. Since $\mathrm{ind}(x_1, \phi^{l_1}) = 1 - a_1 l_1$, $\mathrm{ind}(x_2, \phi^{l_2}) = -1$, $\mathrm{ind}(y_i, \phi^{l_i m_i}) =$

$1 - b_i m_i$, we have

$$\begin{aligned} \nu^n(X_1) &= 1 + (a_1 l_1 - 2)\delta(l_1), & \nu^n(X_2) &= a_2 l_2 \delta(l_2), \\ \nu^n(Y_i) &= l_i \delta(l_i) + l_i(b_i m_i - 2)\delta(l_i m_i), & \nu^n(Z_i) &\geq l_i m_i \delta(l_i m_i). \end{aligned} \tag{27}$$

Also by (11), we have $\nu^n(\mathcal{S}_1) = 2\sharp\mathcal{S}_1 \geq 2\sum_{i=1}^{2} l_i(m_i - 1)$. Therefore, since $\mathcal{S}_1$, Q_1, Q_2 are mutually disjoint, we have the lemma. □

Lemma 6

$$\nu_1 \geq \begin{cases} 2n+3 & \textit{if} \quad n \leq l_1 \\ 2n+1 & \textit{if} \quad n < 3l_1 \quad \textit{or} \quad m_1 \geq 3 \\ n+4 & \textit{for any} \quad n \geq 1, \end{cases}$$

$$\nu_2 \geq \begin{cases} 2n+2 & \textit{if} \quad n < 3l_2 \quad \textit{or} \quad m_2 \geq 3 \\ n+5 & \textit{if} \quad n \geq l_2 \geq 2 \\ n+3 & \textit{if} \quad l_2 = 1. \end{cases}$$

Proof. These follow from the following, which can be easily verified:

$$\nu_1 = \begin{cases} 2l_1(m_1 - 1) + 1 \geq 2l_1 + 1 & \text{if} \quad n < l_1 \\ 2l_1 m_1 - 1 & \text{if} \quad l_1 \leq n < l_1 m_1 \\ 4l_1 m_1 - 2l_1 - 1 & \text{if} \quad l_1 m_1 \leq n < l_1(m_1 + 2), \end{cases}$$

$$\nu_2 = \begin{cases} 2l_2(m_2 - 1) \geq 2l_2 & \text{if} \quad n < l_2 \\ 2l_2 m_2 & \text{if} \quad l_2 \leq n < l_2 m_2 \\ l_2(4m_2 - 2) & \text{if} \quad l_2 m_2 \leq n < l_2(m_2 + 2). \end{cases}$$

□

Let Λ be the set of pairs (l, m) of integers which satisfy $n < 3l$ or $m \geq 3$. There are four subcases.

(i) $(l_2, m_2) \in \Lambda$.
(ii) $(l_1, m_1) \in \Lambda$, $(l_2, m_2) \notin \Lambda$, and $l_2 \geq 2$.
(iii) $(l_1, m_1) \in \Lambda$, $(l_2, m_2) \notin \Lambda$, and $l_2 = 1$.
(iv) $(l_1, m_1), (l_2, m_2) \notin \Lambda$.

Case (i): By Lemma 6, $\nu_1 \geq n + 4$ and $\nu_2 \geq 2n + 2$. Then Lemma 5 implies (25).

Case (ii): Since $(l_2, m_2) \notin \Lambda$, we have $n \geq 3l_2$. Therefore, by Lemma 6, $\nu_1 \geq 2n+1$ and $\nu_2 \geq n+5$, and so (25) holds.

Case (iii): This case is possible only if $n = 3$. In fact, since $n \geq 3l_2$, $m_2 = 2$, and the inequality $n < l_2(m_2+2)$ is assumed, the hypothesis $l_2 = 1$ implies $n = 3$. Note that by (11) we see that $\mathcal{S}_1$ has at least three points. There are three subcases:

(iii-a) $l_1 \geq 3$.
(iii-b) $l_1 = 2$ and $\sharp\mathcal{S}_1 \geq 4$.
(iii-c) $\sharp\mathcal{S}_1 = 3$.

Case (iii-a): Since $n = 3 \leq l_1$, by Lemma 6, $\nu_1 \geq 2n+3$. Therefore, $\nu_2 \geq n+3$ implies (25).

Case (iii-b): By (27), we have $\nu^3(X_1 \cup X_2) = a_1 l_1 - 1 + a_2 l_2 \geq 2$, $\nu^3(Y_1 \cup Y_2) \geq l_1 + l_2 = 3$, and $\nu^3(Z_2) \geq l_2 m_2 = 2$. Therefore, since $\nu^3(\mathcal{S}_1) \geq 8$, we see $\nu^3(\mathcal{S}_1 \cup Q_1 \cup Q_2) \geq 15 = 3n+6$.

Case (iii-c): Let ψ be the homeomorphism on the sphere S^2 induced by ϕ by collapsing ∂D to a point ∞. Then ψ is pseudo-Anosov relative to the invariant set $\mathcal{S}_1 \cup \{\infty\}$. Since this invariant set has four points, ψ^q has $\operatorname{tr} A^q$ fixed points on S^2 for every $q \geq 1$, where A is a 2×2 matrix whose entries are all positive integers and $\operatorname{tr} A \geq 3$. (See [19, Theorem 2.4] or [20, p. 70].) This implies $\sharp\mathrm{Fix}(\psi^3) \geq \operatorname{tr} A^3 \geq 18$, and hence the set $\mathrm{Fix}(\phi^3) \cap \mathrm{Int}\, D$ has at least 17 points. Since $\nu^3(x) > 0$ for any x in this set, we have $\nu^3(D) \geq 17$ and (25) holds.

Case (iv): The hypotheses imply that $m_i = 2$ and

$$3l_i \leq n < l_i(m_i+2) = 4l_i \tag{28}$$

for $i = 1$ and 2. Note that, since $l_1 \geq 2$, (28) implies $l_2 \geq 2$ and Q_2 contains no fixed points. Therefore, if we let $R = \mathrm{Fix}(\phi) - (\mathcal{S}_1 \cup X_1)$, then R is disjoint from $\mathcal{S}_1$, Q_1, Q_2. Since $l_1 m_1 = 2l_1 < n \leq 4l_1 - 1$ by (28), we have $\nu^n(Q_1) \geq 4l_1 - 1 \geq n$ by (27). Similarly $\nu^n(Q_2) \geq 4l_2 \geq n+1$. Therefore

$$\nu^n(\mathcal{S}_1 \cup Q_1 \cup Q_2 \cup R) \geq 2\sharp\mathcal{S}_1 + 2n + 1 + \nu^n(R). \tag{29}$$

There are three subcases.
(iv-a) $l_2 \geq 2$ and $\sharp\mathcal{S}_1 \geq 2l_2 + 2$.
(iv-b) $l_2 = 2$ and $\sharp\mathcal{S}_1 = 5$.

(iv-c) $l_2 \geq 3$ and $\sharp\mathcal{S}_1 \leq 2l_2+1$, or $l_2 = 2$ and $\sharp\mathcal{S}_1 \leq 4$.

Case (iv-a): In this case, $2\sharp\mathcal{S}_1 \geq 4l_2+4 \geq n+5$. Therefore (25) follows from (29).

Case (iv-b): Let $\mathcal{S}' = \mathcal{S} - (\mathcal{S}_1 \cup \partial D)$. Then by (1) the sum $\sum_{x\in\mathcal{S}'}(2-p(x))$ is equal to $2 - \sharp\mathcal{S}_1 + a_2 l_2 = 2a_2 - 3$, and hence equal to -1 or a positive number. Since $2-p(x)$ is negative for any $x \in \mathcal{S}'$, this sum must be -1. This means that there is one and only one singularity in $\mathcal{S}'$ and moreover it must be 3-pronged. We denote this singurality by x_0. Then, since x_0 is 3-pronged, $\mathrm{ind}(x_0,\phi) = 1$ or -2. Since $l_2 = 2$, $l_1 = 2$ by (28) and hence x_1 is a regular point of $\mathcal{F}^s$. This implies $x_0 \neq x_1$, and so $x_0 \in R$. Since x_1 has index one and any point of $\mathcal{S}_1$ has index zero, R has fixed point index zero. Therefore, R has a fixed point of nonzero index different from x_0. This implies $\nu^n(R) \geq 2$. Therefore, since $n = 6$ or 7 by (28), we have $2\sharp\mathcal{S}_1 + \nu^n(R) \geq 12 \geq n+5$. Thus (25) follows from (29).

Case (iv-c): Let $g : A \to A$ be the blow-up of ϕ^{l_2} at y_2. Since y_2 has type $(m_2, b_2) = (2, b_2)$, we see $\rho(\Gamma, \tilde{g}) = d/2$, where Γ is a boundary circle corresponding to y_2 and d is odd. Also, $\rho(\partial D, \tilde{g})$ is an integer. Therefore, we can assume that $\rho(\tilde{g})$ contains $[0,1/2]$ or $[-1/2,0]$. We assume $\rho(\tilde{g}) \supset [0,1/2]$, the other case being similar. Then by Theorem 4, there is a 3-periodic point of g with rotation number $1/3$. Choose such a point w. It is easy to see that w is an interior periodic point of ϕ and $w \in \mathrm{Fix}(\phi^{3l_2}) - \mathrm{Fix}(\phi^{l_2})$. Hence $\mathrm{per}(w,\phi)$ is written as $\mathrm{per}(w,\phi) = 3l_2/r$, where r is a divisor of l_2 and $r \neq 3$.

If $r \geq 4$, then $\mathrm{per}(w,\phi) < l_2 \leq l_2\ \mathrm{per}(w,\phi^{l_2})$. Also by (28), $2l_2 + \mathrm{per}(w,\phi) < 3l_2 \leq n$. Let $g' : A' \to A'$ be the blow-up of ϕ at x_1. Then $\rho(\tilde{g}')$ contains $\rho(x_2, \tilde{g}')$ which is equal to k/l_2 for some k prime to l_2. Therefore by Lemma 4, (25) holds.

Assume $r \leq 2$. Consider first the case of $w \in \mathcal{S}_1$. Then its orbit $\mathrm{Orb}(w)$ under ϕ is contained in $\mathcal{S}_1$. By a hypothesis, $\sharp\mathcal{S}_1 \leq 2l_2 + \epsilon$, where $\epsilon = 1$ or 0 according to $l_2 \geq 3$ or $l_2 = 2$. In the present case, we have $r = 2$, because if $r = 1$, the number $\mathrm{per}(w,\phi) = 3l_2$ could exceed the cardinality of $\mathcal{S}_1$. Thus $\mathrm{per}(w,\phi) = 3l_2/2$. By (11) and (28), $\sharp\mathcal{S}_1 \geq l_1 + l_2 > 7l_2/4$. Therefore, we have

$$\frac{l_2}{4} < \sharp(\mathcal{S}_1 - \mathrm{Orb}(w)) = \sharp\mathcal{S}_1 - \mathrm{per}(w,\phi) \leq (2l_2+\epsilon) - \frac{3}{2}l_2$$

$$= \frac{l_2}{2} + \epsilon < l_2.$$

This implies that there is a periodic point $y \in \mathcal{S}_1$ of ϕ with period less than l_2. Then $\mathrm{per}(y,\phi) < l_2\ \mathrm{per}(y,\phi^{l_2})$, and $2l_2 + \mathrm{per}(y,\phi) < 3l_2 \le n$. Also we showed above that $\rho(\tilde{g}') \ni k/l_2$. Hence by Lemma 4, we have (25).

Consider the case of $w \notin \mathcal{S}_1$. Then $\mathrm{ind}(w, g^3) \neq 0$, since any periodic point of index zero is contained in $\mathcal{S}_1$ or ∂D. Therefore by (24), there is another 3-periodic point $w' \notin \mathcal{S}_1$ of g with $\rho(w',\tilde{g}) = 1/3$. We show w, $w' \notin Q_1 \cup Q_2$. In fact, if $w \in Q_2$, then $\mathrm{per}(w,\phi) = l_2$ or $2l_2$, since x_2, y_2, z_2 have periods l_2, l_2, $2l_2$ respectively. Then $3l_2/r = l_2$ or $2l_2$, which contradicts that $r = 1, 2$. Suppose $w \in Q_1$. Then $3l_2/r = \mathrm{per}(w,\phi) = l_1$ or $2l_1$. Therefore since $r = 1,2$ and $3l_2 < 4l_1$ by (28), we have $3l_2 = l_1$ or $2l_1$. This contradicts that $3l_1 < 4l_2$. In the same way, we can verify $w' \notin Q_1 \cup Q_2$.

Let W, W' be the orbits of w, w' respectively. We can assume that $\mathrm{per}(w',\phi) \ge \mathrm{per}(w,\phi)$ without loss of generality. Then, $\nu^n(W), \nu^n(W') \ge 3l_2/2$. Therefore, we have by Lemma 5,

$$\nu^n(\mathcal{S}_1 \cup Q_1 \cup Q_2) + \nu^n(W) + \nu^n(W') \ge \nu_1 + \nu_2 + 3l_2. \tag{30}$$

Since $n \ge 3l_2$ and $m_2 = 2$, we have $\nu_2 = 6l_2$ by the definition of ν_2. Also, by Lemma 6, $\nu_1 \ge n + 4$. Therefore, since W, W', $\mathcal{S}_1 \cup Q_1 \cup Q_2$ are mutually disjoint, we have $\nu^n(D) \ge n + 4 + 9l_2$ by (30). Since $4l_2 \ge n + 1$, this implies (25).

(II) Here we consider the general case; namely, the case where $\mathrm{bt}(S, f)$ is reducible and contains a pseudo-Anosov component. We shall use the same notations as in (II) of Section 3. Since $\phi' : D' \to D'$ is pseudo-Anosov relative to S' and $n' \ge \mathrm{per}(S')$, applying the result in (I) of this section to ϕ', we have $\nu^{n'}(D', \phi') \ge 3n' + 6$. Let n_i (resp. u_i) be the period (resp. the number) of periodic points of ϕ on C_i. Note that $\nu^{n'}(c_i, \phi') = 1$ or $u_i - 1$ according to whether $n' < n_i/\mu$ (equivalently $n < n_i$) or $n' \ge n_i/\mu$. Hence we have $\nu^n(C_i, \phi) = \nu^{n'}(c_i, \phi') - 1 + 2\delta(n_i)$. Therefore, we have

$$\begin{aligned}
\nu^n(N) &= \nu^n\left(N - \bigcup_{i=1}^{d-1} C_i\right) + \sum_{i=1}^{d-1} \nu^n(C_i) \\
&= \nu^{n'}(D' - \{c_i\}_{i=1}^{d-1}, \phi') + \sum_{i=1}^{d-1} (\nu^{n'}(c_i, \phi') - 1 + 2\delta(n_i))
\end{aligned}$$

$$= \nu^{n'}(D', \phi') - d_0 + d_1.$$

Therefore by Proposition 3,

$$\begin{aligned}\sharp P^n(f) &\geq \mu(\nu^n(N) + 2d_0 - d_1) \geq \mu(\nu^{n'}(D', \phi') + d_0) \\ &\geq \mu(3n' + 6).\end{aligned}$$

Since $\mu n' \geq n - \mu + 1$, this shows $\sharp P^n(f) \geq 3n + 6$, and the proof is completed.

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