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A class of univalent functions

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Abstract. In this paper we consider starlikeness of the class of functions $f(z)=z+a_2z^2+\cdots$ which are analytic in the unit disc and satisfy the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<\lambda, \quad 0<\mu<1,\ 0<\lambda<1.$$

Key words: univalent, starlike, starlike of order β.

1. Introduction and preliminaries

Let H denote the class of functions analytic in the unit disc $U=\{z:|z|<1\}$ and let $A\subset H$ be the class of normalized analytic functions f in U such that $f(0)=f'(0)-1=0$. Further, let

$$S^{*}(\beta)=\left\{f\in A:\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\beta,\ 0\leq\beta<1,\ z\in U\right\}$$

denote the class of *starlike functions of order* β. We put $S^{*}\equiv S^{*}(0)$ (the class of *starlike functions*). It is well-known that these classes belong to the class of univalent functions in U (see, for example [2]). Also, it is known that the class

$$B_1(\mu)=\left\{f\in A:\mathrm{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\}>0,\ \mu>0,\ z\in U\right\} \tag{1}$$

is the class of univalent functions in U ([1]).

Recently, Ponnusamy ([5]) has shown that the stronger condition than in (1) given by

$$\left|f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right|<\lambda, \quad \mu>0,\ z\in U, \tag{2}$$

and appropriate $0<\lambda<1$, implies starlikeness in U.

In this paper we consider starlikeness of the class of functions $f\in A$

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defined by the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1 \right| < \lambda, \quad 0 < \mu < 1,\ 0 < \lambda < 1,\ z \in U, \tag{3}$$

i.e. for $-1 < \mu < 0$ in (2).

For our results we need the following lemmas.

Lemma A ([3]). *Let ω be a nonconstant and analytic function in U with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r$ at z_0, we have $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.*

Lemma B ([6]). *Let $0 < \lambda_1 < \lambda < 1$ and let $\mathcal{Q}$ be analytic in U satisfying*

$$\mathcal{Q}(z) \prec 1 + \lambda_1 z, \quad \mathcal{Q}(0) = 1.$$

(a) *If $p \in H$, $p(0) = 1$ and satisfies*

$$\mathcal{Q}(z)[\beta + (1-\beta)p(z)] \prec 1 + \lambda z,$$

where

$$\beta = \begin{cases} \dfrac{1-\lambda}{1+\lambda_1}, & 0 < \lambda + \lambda_1 \leq 1 \\[2ex] \dfrac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)}, & \lambda^2 + \lambda_1^2 \leq 1 \leq \lambda + \lambda_1 \end{cases}, \tag{4}$$

then $\mathrm{Re}\{p(z)\} > 0$, $z \in U$.

(b) *If $\omega \in H$, $\omega(0) = 0$ and*

$$\mathcal{Q}(z)[1 + \omega(z)] \prec 1 + \lambda z,$$

then

$$|\omega(z)| \leq \frac{\lambda + \lambda_1}{1 - \lambda_1} = r \leq 1, \quad \lambda + 2\lambda_1 \leq 1. \tag{5}$$

The value of β given by (4) and the bound (5) are the best possible.

2. Results and consequences

In the beginning we prove the following

Lemma 1 *Let $p \in H$, $p(0) = 1$ and satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda z, \quad 0 < \mu < 1, \quad 0 < \lambda \le 1. \tag{6}$$

Then

$$p(z) \prec 1 + \lambda_1 z, \tag{7}$$

where

$$\lambda_1 = \lambda \frac{\mu}{1-\mu}. \tag{8}$$

Proof. Let's put

$$p(z) = 1 + \lambda_1 \omega(z), \tag{9}$$

where λ_1 is given by (8). We want to show that $|\omega(z)| < 1$, $z \in U$. If not, by Lemma A there exists a z_0, $|z_0| < 1$, such that $|\omega(z_0)| = 1$, $z_0 \omega'(z_0) = k\omega(z_0)$, $k \ge 1$. If we put $\omega(z_0) = e^{i\theta}$, then we get

$$\begin{aligned} \left| p(z_0) - \frac{1}{\mu} z_0 p'(z_0) - 1 \right| &= \left| \lambda_1 \omega(z_0) - \frac{1}{\mu} \lambda_1 z_0 \omega'(z_0) \right| \\ &= \left| \lambda_1 e^{i\theta} - \frac{\lambda_1}{\mu} k e^{i\theta} \right| = \lambda_1 \left| 1 - \frac{k}{\mu} \right| \\ &\ge \lambda_1 \left(\frac{1}{\mu} - 1 \right) = \lambda \end{aligned}$$

which is a contradiction to (6). Now, it means that $|\omega(z)| < 1$, $z \in U$, and by (9) we have (7). □

Theorem 1 *If $f \in A$ satisfies the condition (3) with $0 < \mu < 1$ and $0 < \lambda \le \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}$, then $f \in S^*$.*

Proof. If we put $Q(z) = \left(\frac{z}{f(z)}\right)^{\mu}$, then by some transformations and (3) we get

$$Q(z) - \frac{1}{\mu} z Q'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z.$$

From there by Lemma 1 we obtain

$$Q(z) \prec 1 + \lambda_1 z, \quad \lambda_1 = \lambda \frac{\mu}{1-\mu}. \tag{10}$$

From the conditions (3) and (10) we have

$$\left|\arg f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right| < \operatorname{arctg}\frac{\lambda}{\sqrt{1-\lambda^2}}$$

and

$$\left|\arg\left(\frac{f(z)}{z}\right)^{\mu}\right| = \left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \operatorname{arctg}\frac{\lambda_1}{\sqrt{1-\lambda_1^2}},$$

which give

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \left|\arg f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right| + \left|\arg\left(\frac{f(z)}{z}\right)^{\mu}\right| \\ &\leq \operatorname{arctg}\frac{\lambda}{\sqrt{1-\lambda^2}} + \operatorname{arctg}\frac{\lambda_1}{\sqrt{1-\lambda_1^2}} \\ &= \operatorname{arctg}\frac{\frac{\lambda}{\sqrt{1-\lambda^2}} + \frac{\lambda_1}{\sqrt{1-\lambda_1^2}}}{1-\frac{\lambda\lambda_1}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_1^2}}} \leq \frac{\pi}{2}\end{aligned}$$

since $1-\frac{\lambda\lambda_1}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_1^2}} \geq 0\left(\Leftrightarrow \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}\right)$ is true by hypothesis. It means that $f \in S^*$. □

Especially for $\mu = 1/2$ we have

Corollary 1 *Let $f \in A$ satisfy the condition*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\frac{3}{2}} - 1\right| < \frac{\sqrt{2}}{2}, \quad z \in U,$$

then $f \in S^*$.

By using Lemma B for $0 < \mu < 1/2$ we can get a better result as the following theorem shows.

Theorem 2 *Let $f \in A$ satisfy the condition* (3) *for* $0 < \mu < 1/2$. *If λ_1 is given by* (8), *then*

(a) $f \in S^*(\beta)$, *where*

$$\beta = \begin{cases} \dfrac{1-\lambda}{1+\lambda_1}, & 0 < \lambda \le 1-\mu \\[2ex] \dfrac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)}, & 1-\mu \le \lambda \le \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}. \end{cases}$$

(b) $\left|\frac{zf'(z)}{f(z)} - 1\right| < r, \quad z \in U,$
where

$$r = \frac{\lambda}{1-\mu-\lambda\mu} \le 1, \quad 0 < \lambda \le \frac{1-\mu}{1+\mu}.$$

Proof. Let's put $\mathcal{Q}(z) = \left(\frac{z}{f(z)}\right)^{\mu}$, $p(z) = \frac{zf'(z)}{f(z)}$, $\omega(z) = \frac{zf'(z)}{f(z)} - 1$. Then by (10) of Theorem 1 we have $\mathcal{Q}(z) \prec 1 + \lambda_1 z$, where $0 < \lambda_1 = \lambda\frac{\mu}{1-\mu} < \lambda < 1$ since $0 < \mu < 1/2$. Also, since the condition (3) is equivalent to

$$\mathcal{Q}(z)\left[\beta + (1-\beta)\frac{p(z)-\beta}{1-\beta}\right] \prec 1 + \lambda z,$$

where β is given by (4) and as

$$\mathcal{Q}(z)[1+\omega(z)] \prec 1 + \lambda z,$$

then the statements of the theorem directly follows from Lemma B. $\square$

Theorem 3 *Let $f \in A$ satisfy the condition* (3) *and let*

$$F(z) = z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} t^{c-\mu-1}dt\right]^{-\frac{1}{\mu}}, \tag{11}$$

where $c - \mu > 0$. Then

(a) $F \in S^*$ *for* $\frac{(c-\mu)\lambda}{1+c-\mu} \le \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}$, $0 < \mu < 1$.

(b) $F \in S^*(\beta)$, *where*

$$\beta = \begin{cases} \dfrac{1-\lambda_1}{1+\lambda_2}, & 0 < \lambda_1 < 1-\mu \\[2ex] \dfrac{1-(\lambda_1^2+\lambda_2^2)}{2(1-\lambda_2^2)}, & 1-\mu \le \lambda_1 \le \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \end{cases},$$

and $\lambda_1 = \frac{(c-\mu)\lambda}{1+c-\mu}$, $\lambda_2 = \lambda_1\frac{\mu}{1-\mu}$, $0 < \mu < \frac{1}{2}$.

(c) $\left|\frac{zF'(z)}{F(z)}-1\right|<r, \quad z\in U,$

where $r=\frac{\lambda_1}{1-\mu-\lambda_1\mu}\le 1$, $\lambda_1=\frac{c-\mu}{1+c-\mu}$, $0<\lambda_1\le\frac{1-\mu}{1+\mu}$, $0<\mu<\frac{1}{2}$.

Proof. If we put $\mathcal{Q}(z)=F'(z)\left(\frac{z}{F(z)}\right)^{1+\mu}$, then from (11) and (3), after some transformations we obtain

$$\mathcal{Q}(z)+\frac{1}{c-\mu}z\mathcal{Q}'(z)=f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\prec 1+\lambda z,$$

and from there, similar as in Lemma 1, we have that

$$\mathcal{Q}(z)\prec 1+\lambda_2 z, \quad \lambda_2=\frac{(c-\mu)\lambda}{1+c-\mu}$$

(also see the proof of Theorem 1 in [5]). The statements of the theorem now easily follows from Theorem 1 and Theorem 2. □

In connection with the previous results we can pose the following

Questions For the limit cases, i.e. for $\mu=0$ or $\mu=1$ and $\lambda=1$ we have the classes of functions defined by the conditions $\left|\frac{zf'(z)}{f(z)}-1\right|<1$, and $\left|\frac{z^2f'(z)}{f^2(z)}-1\right|<1$, respectively. The first class is the subclass of S^*, the second is the class of univalent functions in U (see [4]).

Does the condition $\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<1$, $0<\mu<1$, $z\in U$ imply univalence in U? Generally speaking, can we find the region E in the complex plain such that $f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\in E$, $z\in U$, $0<\mu<1$, provides univalence in the unit disc U?

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References

[1] Bazilevič I.E., *On a case of integrability in quadratures of the Loewner-Kufarev equation.* Mat. Sb. **37**(79) (1955), 471–476 (Russian).

[2] Duren P.L., *Univalent functions.* Springer-Verlag, 1983.

[3] Jack I.S., *Functions starlike and convex of order* α. J. London Math. Soc. (2)**3** (1971), 469–474.

[4] Ozaki M.O. and Nunokawa M., *The Schwarzian derivative and univalent functions.* Proc. Amer. Math. Soc. **33**(2) (1972), 392–394.

[5] Ponnusamy S., *Polya-Schoenberg conjecture by Caratheodory functions.* J. London Math. Soc. (2)**51** (1995), 93–104.

[6] Ponnusamy S. and Sing V., *Convolution properties of some classes of analytic functions.* SPIC Science Foundation (1991).

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