Hokkaido Mathematical Journal Vol. 27 (1998) p. 273–301

On the extension properties of Triebel-Lizorkin spaces

(Dedicated to Professor Kyûya Masuda on the occasion of his sixtieth birthday)

Akihiko MIYACHI

(Received December 11, 1996)

Abstract. Extension of functions originally defined on a domain of a Euclidean space to the whole Euclidean space is considered. Two results on the extension of functions in A. Seeger's generalized Triebel-Lizorkin spaces are proved.

Key words: extension theorem, Triebel-Lizorkin space, (ε, δ)-domain.

1. Introduction

In [S], Seeger introduced some function spaces on domains of $\mathbb{R}^n$, which can be considered as a natural generalization of the Triebel-Lizorkin space to the case of spaces on domains, and gave an extension theorem for those spaces ([*ibid.*; Theorem 2]). The purpose of the present paper is to give some results concerning the extension properties of Seeger's function spaces. In this section, after fixing several notations, we shall state the main results of this paper.

Throughout this paper we use the letters n, Ω, k, p, q, r, and α in the following fixed meanings: n is a positive integer and denotes the dimension of the Euclidean space $\mathbb{R}^n$; Ω denotes an open subset of $\mathbb{R}^n$; k denotes a nonnegative integer; p, q, and r denote positive real numbers or ∞; α denotes a nonnegative real number.

We also use the following notations. The set

$$Q = Q(x,t) = \{(y_i) \in \mathbb{R}^n \mid \max |y_i - x_i| \leqq t\},$$

where $x = (x_i) \in \mathbb{R}^n$ and $0 < t < \infty$, is called a cube with center x and sidelength $2t$. The center of a cube Q is denoted by x_Q and the sidelength by $\ell(Q)$. If $Q = Q(x,t)$ and $0 < a < \infty$, then the cube $Q(x, at)$ is simply denoted by aQ. The Lebesgue measure of a cube Q is denoted by $|Q|$; thus $|Q| = \ell(Q)^n$. A dyadic cube is a cube of the form $\{(x_i) \in \mathbb{R}^n \mid 2^m k_i \leqq x_i \leqq 2^m(k_i + 1),\ i = 1, \ldots, n\}$ with m and k_i integers. The set of all dyadic

1991 Mathematics Subject Classification : 46E35.

cubes is denoted by $\mathcal{D}$. We call a Lebesgue measurable function merely a function. For functions f on a Lebesgue measurable set $E \subset \mathbb{R}^n$, we write

$$\|f\|_{p,E} = \left(\int_E |f(x)|^p dx \right)^{1/p} ;$$

if $p = \infty$, we use the usual modification that we replace the $(\int (\cdots)^p dx)^{1/p}$ by the essential supremum norm. For functions f on Ω, we define

$$M_p(f)(x) = \sup_{\substack{Q:\text{cube} \\ Q \ni x}} \{|Q|^{-1/p} \|f\|_{p,Q\cap\Omega}\}, \quad x \in \mathbb{R}^n.$$

If $\mathcal{A}$ is a finite set, $\#\mathcal{A}$ denotes the number of elements of $\mathcal{A}$. We use the letter c to denote various positive constants, which may be different in each occasion. If a constant c depends only on the parameters β, γ, $\delta, \ldots$, and if we want to indicate this dependence explicitly, then we write it as $c(\beta, \gamma, \delta, \ldots)$.

Now let f be a function on Ω. For cubes $Q \subset \Omega$, we set

$$v_r^k(f, Q) = \inf\{|Q|^{-1/r} \|f - P\|_{r,Q} \mid P \in \mathcal{P}_k\},$$

where $\mathcal{P}_k$ denotes the set of polynomial functions on $\mathbb{R}^n$ of degree not exceeding k. For $x \in \Omega$, we write

$$\rho_\Omega(x) = \sup\{t > 0 \mid Q(x,t) \subset \Omega\}.$$

Taking ϵ with $0 < \epsilon < 1$, we define

$$G_{q,r}^{\alpha,k}(f)(x) = \left(\int_0^{\epsilon\rho_\Omega(x)} \left(t^{-\alpha} v_r^k(f, Q(x,t)) \right)^q \frac{dt}{t} \right)^{1/q}, \quad x \in \Omega. \quad (1.1)$$

It is easy to see that $(x,t) \mapsto v_r^k(f, Q(x,t))$ is a lower semicontinuous function on the set $\{(x,t) \mid x \in \Omega,\ 0 < t < \rho_\Omega(x)\}$ and $G_{q,r}^{\alpha,k}(f)$ is a lower semicontinuous function on Ω.

Suppose either $q = \infty$ or $0 < p,\ q,\ \alpha < \infty$. Taking r satisfying

$$\frac{1}{r} + \frac{\alpha}{n} > \max\left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad (1.2)$$

we define

$$|f; E_{p,q}^{\alpha,k}(\Omega)| = \|G_{q,r}^{\alpha,k}(f)\|_{p,\Omega}. \quad (1.3)$$

It can be shown that the choices of ϵ in (1.1) and r in (1.3) do not affect

$|\cdot\,; E_{p,q}^{\alpha,k}(\Omega)|$ up to the equivalence (see Propositions 1 and 4 in Section 2).

Remark. For $q=\infty$, the function $G_{q,r}^{\alpha,k}(f)$ and the quasinorm $|f; E_{p,q}^{\alpha,k}(\Omega)|$ are slight modifications of $f_\alpha^\#$ and $|f|_{C_p^\alpha(\Omega)}$ of DeVore and Sharpley [DS]; see the last part of Section 2. In the case $q<\infty$, they are given by Seeger [S] (the notations are different). Seeger [*ibid.*] proved that the quasinorm

$$\|f\| = |f; E_{p,q}^{\alpha,k}(\Omega)| + \|f\|_{p,\Omega} \tag{1.4}$$

is equivalent to the quasinorm of the Triebel-Lizorkin space $F_{p,q}^\alpha(\Omega)$ if $0<p$, $\alpha<\infty$, $0<q\leqq\infty$, $k+1>\alpha$, and $1+\alpha/n>\max\{1/p,\,1/q\}$ and if Ω is an (ϵ,δ)-domain.

We say that Ω is an *extension domain* for $E_{p,q}^{\alpha,k}$ if every function f on Ω with $|f; E_{p,q}^{\alpha,k}(\Omega)|<\infty$ can be extended to a function F on $\mathbb{R}^n$ such that

$$|F; E_{p,q}^{\alpha,k}(\mathbb{R}^n)| \leqq A|f; E_{p,q}^{\alpha,k}(\Omega)| \tag{1.5}$$

with a constant A, $1\leqq A<\infty$, independent of f.

The following is our first main theorem.

Theorem 1 *Suppose $0<p$, q_0, $\alpha<\infty$ and $k+1>\alpha$ and suppose Ω is an extension domain for $E_{p,q_0}^{\alpha,k}$. Then Ω is also an extension domain for $E_{p,q}^{\alpha,k}$ for q in the range $q_0<q\leqq\infty$.*

We shall prove this theorem in Section 3, where we shall first prove a result concerning the extension which holds for arbitrary Ω (Proposition 7) and then we shall deduce Theorem 1.

We next consider the extension property for the (ϵ,δ)-domains. We say that Ω (an open subset of $\mathbb{R}^n$) is an (ϵ,δ)-*domain* ($0<\epsilon\leqq 1$, $0<\delta\leqq\infty$) if Ω is connected and if for each x, $y\in\Omega$ with $|x-y|<\delta$ there exists a rectifiable curve γ in Ω joining x to y and satisfying

$$(\text{the length of } \gamma) \leqq \epsilon^{-1}|x-y|$$

and

$$\operatorname{dis}(z,\Omega^c) \geqq \epsilon\min\{|z-x|,|z-y|\} \quad \text{for all } z \text{ on } \gamma.$$

This concept is due to P. Jones [J2].

The following theorem is known.

Theorem A *An* (ϵ, ∞)*-domain is an extension domain for* $E_{p,q}^{\alpha,k}$ *if either*

$$q = \infty \quad and \quad k + 1 \geqq \alpha \tag{1.6}$$

or

$$0 < p,\ q,\ \alpha < \infty \quad and \quad k + 1 > \alpha. \tag{1.7}$$

Similar extension theorem for (ϵ, δ)-domains with $\delta < \infty$ involving the mod 0 quasinorm of (1.4) is also known. In the case (1.6), Theorem A, together with its mod 0 version for (ϵ, δ)-domain, was proved by Jones [J1] ($p = \infty$, $\alpha = 0$), Christ [C] ($p > 1$, $\alpha > 0$), and Miyachi [M] ($p \leqq 1$, $\alpha > 0$). The present author does not know a literature which contains an explicit statement of Theorem A for the case $p < \infty = q$ and $\alpha = 0$. This case, however, can be proved by the same method as in [J1], [C], or [M]; see Section 4 of the present paper. Theorem A for the case (1.7) was given by Seeger [S] (without detailed proof).

In this paper, we shall be interested in the problem whether the extension operator for the (ϵ, δ)-domain can be made linear. The extension operators of [J1] and [C] are linear. A slight modification of the extension method of [S] gives a linear extension operator in the case p, $q \geqq 1$. In [M], a linear extension operator is given in the case $q = \infty$, $k + 1 \geqq \alpha > 0$, and $1 + \alpha/n > 1/p$. We shall extend these results; the following is the second main theorem of this paper.

Theorem 2 *Suppose* p, q, α, *and* k *satisfy either* (1.6) *or* (1.7). *Also suppose* $1 + \alpha/n > \max\{1/p, 1/q\}$. *Then:*

(1) *If* Ω *is an* (ϵ, ∞)*-domain,* $0 < \epsilon \leqq 1$, *then there exists a linear operator* T_1 *which associates with each function* f *on* Ω *a function* $T_1 f$ *on* $\mathbb{R}^n$ *such that* $(T_1 f) \,|\, \Omega = f$ *and* $F = T_1 f$ *satisfies* (1.5) *with* $A = c(n, k, \alpha, p, q, \epsilon)$.

(2) *If* Ω *is an* (ϵ, δ)*-domain,* $0 < \epsilon \leqq 1$, $0 < \delta < \infty$, *then there exists a linear operator* T_2 *which associates with each function* f *on* Ω *a function* $T_2 f$ *on* $\mathbb{R}^n$ *such that* $(T_2 f) \,|\, \Omega = f$ *and*

$$|T_2 f; E_{p,q}^{\alpha,k}(\mathbb{R}^n)| + \|T_2 f\|_{p,\mathbb{R}^n} \leqq c_{\epsilon,\delta}(|f; E_{p,q}^{\alpha,k}(\Omega)| + \|f\|_{p,\Omega}),$$

where $c_{\epsilon,\delta} = c(n, k, \alpha, p, q, \epsilon, \min\{\delta, \operatorname{diam} \Omega\})$.

This theorem will be proved in Section 4.

The existence of a linear extension operator has, besides practical use in analysis, the following geometric meaning. For simplicity, let us consider the mod 0 case; we define, temporarily, the space $E^{\alpha,k}_{p,q}(\Omega)$ as the class of those functions f on Ω for which the quasinorm $\|f\|$ of (1.4) is finite. Then $E^{\alpha,k}_{p,q}(\Omega)$ equipped with this quasinorm $\|\cdot\|$ is a quasi-Banach space. For any Ω, the restriction operator $\sigma : F \mapsto F|\Omega$ is a continuous linear mapping of $E^{\alpha,k}_{p,q}(\mathbb{R}^n)$ to $E^{\alpha,k}_{p,q}(\Omega)$. The assertion that each function of $E^{\alpha,k}_{p,q}(\Omega)$ can be extended to a function of $E^{\alpha,k}_{p,q}(\mathbb{R}^n)$ is equivalent to the assertion that σ is onto. The assertion that a bounded linear extension operator $E^{\alpha,k}_{p,q}(\Omega) \to E^{\alpha,k}_{p,q}(\mathbb{R}^n)$ exists is equivalent to the assertion that σ is onto and the kernel of σ has a closed complementary subspace in $E^{\alpha,k}_{p,q}(\mathbb{R}^n)$.

2. Basic results

In this section, we give some basic properties of $G^{\alpha,k}_{q,r}(\,\cdot\,)$ and $|\cdot\,; E^{\alpha,k}_{p,q}(\Omega)|$. Most of the results in this section are slight modifications of the known ones. For proofs, we sometimes give only outline or suggestions.

The following lemma is elementary and well known.

Lemma 1 (1) *For polynomials $P \in \mathcal{P}_k$ and for cubes Q, we have*

$$\|P\|_{\infty,aQ} \leqq c(n,k,a)\|P\|_{\infty,Q}, \quad 1 < a < \infty,$$

$$\|\partial^\nu P\|_{\infty,Q} \leqq c(n,k)\ell(Q)^{-|\nu|}\|P\|_{\infty,Q}.$$

(2) *If Q and R are cubes satisfying $Q \subset R$, then*

$$v^k_r(f,Q) \leqq (|R|/|Q|)^{1/r} v^k_r(f,R).$$

For functions f defined on a cube Q and for $1 \leqq A < \infty$, we define $\Pi^A_k(f,r,Q)$ as the set of those polynomials π in $\mathcal{P}_k$ such that

$$\|f-\pi\|_{r,Q} \leqq A\inf\{\|f-P\|_{r,Q} \mid P \in \mathcal{P}_k\}.$$

Lemma 2 (1) $\Pi^A_k(f,r,Q) \neq \emptyset$.

(2) *If $\pi \in \Pi^A_k(f,r,Q)$, then $\|\pi\|_{\infty,Q} \leqq c(n,k,r)A|Q|^{-1/r}\|f\|_{r,Q}$.*

(3) *If Q_1 and Q_2 are cubes such that $Q_1 \cap Q_2 \neq \emptyset$, $bQ_i \subset \Omega$ $(i = 1, 2)$ with $1 < b < \infty$, and $B^{-1} \leqq \ell(Q_1)/\ell(Q_2) \leqq B$ with $1 \leqq B < \infty$, and if f is a function on Ω and $\pi_i \in \Pi^A_k(f,r,Q_i)$ $(i = 1, 2)$, then*

$$\|\pi_1 - \pi_2\|_{\infty,Q_1} \leqq c(n,k,r,b,B)A(v^k_r(f,bQ_1) + v^k_r(f,bQ_2)).$$

Proof. The assertion (1) follows from the fact that $\mathcal{P}_k$ is finite dimensional. For (2), see *e.g.* [DS; p. 23]. Proof of (3) is easy if one of Q_i is included in the other (see *e.g.* [DS; p. 24]); the general case can be reduced to this easy case by taking a cube R which satisfy $R \subset bQ_1 \cap bQ_2$ and $\ell(R) \approx \ell(Q_1) \approx \ell(Q_2)$. Details are left to the reader. □

We now show that the choice of ϵ in the definition of $G_{q,r}^{\alpha,k}(f)$ does not affect $|f; E_{p,q}^{\alpha,k}(\Omega)|$ up to the equivalence.

Proposition 1 *Let $0 < \epsilon_1, \epsilon_2 < 1$ and let G_1 and G_2 be the functions $G_{q,r}^{\alpha,k}(f)$ defined with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$ respectively. Then $\|G_1\|_{p,\Omega} \approx \|G_2\|_{p,\Omega}$.*

Proof. We may assume $\Omega \neq \mathbb{R}^n$ and $\epsilon_1 < \epsilon_2$. Then obviously $\|G_1\|_{p,\Omega} \leqq \|G_2\|_{p,\Omega}$. In order to prove the reverse estimate $\|G_2\|_{p,\Omega} \leqq c\|G_1\|_{p,\Omega}$, it is sufficient to prove that the pointwise inequality

$$G_2(x) \leqq c_\eta M_\eta(G_1)(x), \quad x \in \Omega,$$

with $c_\eta = c(n, k, \alpha, q, r, \epsilon_1, \epsilon_2, \eta)$, holds for every $\eta > 0$. We fix an $x \in \Omega$. We shall simply write $v(R) = v_r^k(f, R)$ and $Q^* = Q(x, \epsilon_2 \rho_\Omega(x))$. We have

$$G_2(x) \leqq c\,G_1(x) + c\left(\int_{\epsilon_1\rho_\Omega(x)}^{\epsilon_2\rho_\Omega(x)} \left(t^{-\alpha}v(Q(x,t))\right)^q \frac{dt}{t}\right)^{1/q}.$$

The second term on the right hand side can be majorized by $c\,\ell(Q^*)^{-\alpha}v(Q^*)$. Hence it is sufficien to show the estimate

$$\ell(Q^*)^{-\alpha}v(Q^*) \leqq c_\eta M_\eta(G_1)(x). \tag{2.1}$$

Since $\epsilon_2 < 1$, we have

$$\rho_\Omega(y) \approx \ell(Q^*) \quad \text{for all} \quad y \in Q^*. \tag{2.2}$$

Take a sufficiently large positive integer N and decompose Q^* into N^n congruent cubes R_j. We choose N so large that

$$2R_j \subset Q(y, 2^{-1}\epsilon_1\rho_\Omega(y)) \quad \text{for all} \quad y \in R_j. \tag{2.3}$$

Notice that N can be chosen depending only on ϵ_1 and ϵ_2. We number the cubes R_j so that $R_j \cap R_{j+1} \neq \emptyset$.

For each R_j, we take $\pi_j \in \Pi_k^1(f, r, R_j)$. By Lemma 1 (1) and Lemma

2 (3), we have

$$\begin{aligned}\|\pi_i - \pi_{i+1}\|_{\infty,R_j} &\leqq c_N \|\pi_i - \pi_{i+1}\|_{\infty,R_i} \\ &\leqq c_N \left(v(2R_i) + v(2R_{i+1})\right).\end{aligned}$$

Summing over i's, we obtain

$$\|\pi_1 - \pi_j\|_{\infty,R_j} \leqq c_N \sum_i v(2R_i).$$

Thus

$$\begin{aligned}\|f - \pi_1\|_{r,R_j} &\leqq c\|f - \pi_j\|_{r,R_j} + c\|\pi_j - \pi_1\|_{r,R_j} \\ &\leqq c|R_j|^{1/r} v(R_j) + c_N |R_j|^{1/r} \sum_i v(2R_i) \\ &\leqq c_N |Q^*|^{1/r} \sum_i v(2R_i).\end{aligned}$$

Summing over j's, we obtain

$$v(Q^*) \leqq |Q^*|^{-1/r} \|f - \pi_1\|_{r,Q^*} \leqq c_N \sum_{i=1}^{N^n} v(2R_i). \tag{2.4}$$

On the other hand, from (2.3) and (2.2), using Lemma 1 (2), we see that $v(2R_j) \leqq c_N v(Q(y,t))$ for all $y \in R_j$ and for $2^{-1}\epsilon_1 \rho_\Omega(y) < t < \epsilon_1 \rho_\Omega(y)$, and hence

$$v(2R_j) \leqq c_N \ell(Q^*)^\alpha \inf_{R_j} G_1 \leqq c_{N,\eta} \ell(Q^*)^\alpha M_\eta(G_1)(x) \tag{2.5}$$

for every $\eta > 0$. Now (2.1) follows from (2.4) and (2.5). Proposition 1 is proved. □

We shall introduce a variant of $G^{\alpha,k}_{q,r}(f)$. Let f be a function on Ω. Taking real numbers a and b satisfying $1 < a < b$, we define

$$g^{\alpha,k}_{q,r}(f)(x) = \left(\sum_{\substack{\mathcal{D} \ni Q \ni x \\ bQ \subset \Omega}} \left(\ell(Q)^{-\alpha} v^k_r(f, aQ) \right)^q \right)^{1/q}, \quad x \in \Omega.$$

By the same argument as in the proof of Proposition 1, we can prove the following proposition.

Proposition 2 *Let $1 < a_1 < b_1 < \infty$ and $1 < a_2 < b_2 < \infty$ and let $g_i(f)$, $i = 1, 2$, be the functions $g_{q,r}^{\alpha,k}(f)$ defined with $(a, b) = (a_i, b_i)$. Then*

$$\|g_1(f)\|_{p,\Omega} \approx \|g_2(f)\|_{p,\Omega} \approx \|G_{q,r}^{\alpha,k}(f)\|_{p,\Omega}.$$

In the sequel, if no comment is made on the choices of the parameters ϵ and a and b, then it should be understood that $G_{q,r}^{\alpha,k}(f)$ and $g_{q,r}^{\alpha,k}(f)$ are defined with $\epsilon = 1/2$ and with $a = 2$ and $b = 3$.

With the aid of Proposition 2, the next proposition follows from the fact that $g_{q,r}^{\alpha,k}(f)(x)$ is nonincreasing with respect to q.

Proposition 3 *If $q_1 > q_2$, then $|f; E_{p,q_1}^{\alpha,k}(\Omega)| \leqq c|f; E_{p,q_2}^{\alpha,k}(\Omega)|$.*

We recall the sharp maximal function of DeVore and Sharpley [DS]. Let Q be a cube and h a function on Q. Following [DS], we define

$$h_{k,\alpha,r}^{\#Q}(x) = \sup\{\ell(R)^{-\alpha} v_r^k(f, R) \mid R : \text{cube},\ x \in R \subset Q\}, \quad x \in Q.$$

The following lemma is implicitly given in [DS].

Lemma 3 *Let h be a function on a cube $Q \subset \mathbb{R}^n$. Let $1 \leqq A < \infty$ and $\pi \in \Pi_k^A(h, r, Q)$. Then:*

(1) *If $1/p > \alpha/n$ and $1/q = 1/p - \alpha/n$, then*

$$\|h - \pi\|_{q,Q} \leqq cA\|h_{k,\alpha,r}^{\#Q}\|_{p,Q};$$

(2) *If $1/p < \alpha/n$, then*

$$\|h - \pi\|_{\infty,Q} \leqq cA\ell(Q)^{\alpha - n/p}\|h_{k,\alpha,r}^{\#Q}\|_{p,Q}.$$

Here $c = c(n, k, \alpha, p, r)$.

Note on the proof. If $\alpha > 0$, the claim (1) can be proved by the argument of [DS; Proof of Theorem 4.3]. If $\alpha = 0$, it can be proved by modifying the argument of [FS2; Proof of Theorem 5] (cf. also [DS; Theorem 6.8]). The claim (2) can he proved by the argument of [DS; pp. 23–25]; cf. also [*ibid.*; Theorem 9.1 and Section 12].

The above lemma and Lemma 2 (2) imply the following.

Corollary *Let h be a function on a cube $Q \subset \mathbb{R}^n$. Then:*

(1) *If* $1/p > \alpha/n$ *and* $1/q = 1/p - \alpha/n$, *then*

$$\|h\|_{q,Q} \leqq c\|h^{\#Q}_{k,\alpha,r}\|_{p,Q} + c|Q|^{1/q-1/r}\|h\|_{r,Q};$$

(2) *If* $1/p < \alpha/n$, *then*

$$\|h\|_{\infty,Q} \leqq c\,\ell(Q)^{\alpha-n/p}\|h^{\#Q}_{k,\alpha,r}\|_{p,Q} + c|Q|^{-1/r}\|h\|_{r,Q}.$$

Here $c = c(n,k,\alpha,p,r)$.

The next proposition shows that the choice of r in (1.3)–(1.2) does not affect $|f; E^{\alpha,k}_{p,q}(\Omega)|$ up to the equivalence.

Proposition 4 *Let either* $q = \infty$ *or* $0 < p,\ q,\ \alpha < \infty$. *Also let* $0 < r_1,\ r_2 \leqq \infty$ *and suppose* (1.2) *holds for both* $r = r_1$ *and* $r = r_2$. *Then*

$$\|G^{\alpha,k}_{q,r_1}(f)\|_{p,\Omega} \approx \|G^{\alpha,k}_{q,r_2}(f)\|_{p,\Omega}.$$

This proposition for the case $0 < p,\ q,\ \alpha < \infty$ is implicitly proved in [S; Proof of Theorem 1]. The case $q = \infty$ is proved in [DS; Theorem 4.3 and its proof]. In fact, [DS; *loc. cit.*] contains arguments which are valid only in the case $\alpha > 0$ or in the case $r < \infty$ (the proof of the inequalities (4.10) and (4.10)′ in [*ibid.*; p. 25] and part of the proof of Theorem 4.3). But, using our Lemma 3, where (1) is valid for $\alpha = 0$ as well and (2) gives estimates of L^∞-norms, we can easily modify the argument of [DS] to cover the cases $\alpha = 0$ or $r = \infty$. Detailed proof of Proposition 4 is left to the reader.

The next proposition and the corollary to follow it show that the case $k > \alpha$ is not much different from the case $k = [\alpha]$ (we use $[\alpha]$ to denote the integer which satisfies $[\alpha] \leqq \alpha < [\alpha] + 1$).

Proposition 5 *Let* f *be a function on* $\mathbb{R}^n$ *and let* $0 \leqq \alpha < k$. *Then:*

(1) *There exists a polynomial* π *in* $\mathcal{P}_k$ *such that*

$$G^{\alpha,[\alpha]}_{q,r}(f-\pi)(x) \leqq c(n,k,\alpha,q,r)G^{\alpha,k}_{q,r}(f)(x)$$

for all q *and* r *and all* $x \in \mathbb{R}^n$. (*The polynomial* π *does not depend on* q *and* r.)

(2) *If there exists an* r *such that*

$$\lim_{j\to\infty} j^{-[\alpha]-1-n/r}\|f\|_{r,R_j} = 0,$$

where $R_j = [-j,j]^n$, *then the* π *of* (1) *can be taken to be* 0.

Corollary *If $0 \leqq \alpha < k$ and $0 < s \leqq \infty$, then for functions f on $\mathbb{R}^n$ holds*

$$\|G_{q,r}^{\alpha,k}(f)\|_{p,\mathbb{R}^n} + \|f\|_{s,\mathbb{R}^n} \approx \|G_{q,r}^{\alpha,[\alpha]}(f)\|_{p,\mathbb{R}^n} + \|f\|_{s,\mathbb{R}^n}.$$

Proof. We shall give only the outline of the proof. First, following the argument of [DS; Proofs of Lemmas 2.3 and 4.4], we can prove that for $\pi_Q \in \Pi_k^1(f, r, Q)$ holds

$$\begin{aligned} &G_{q,r}^{\alpha,[\alpha]}\left((f - \pi_Q)|Q^\circ\right)(x) \\ &\qquad \leqq c(n,k,\alpha,q,r)G_{q,r}^{\alpha,k}(f)(x) \quad \text{for} \quad x \in Q^\circ, \end{aligned} \tag{2.6}$$

where Q° denotes the interior of Q. Notice that the left hand side of (2.6) remains unchanged if we replace π_Q by

$$\tilde{\pi}_Q(x) = \sum_{\alpha < |\nu| \leqq k} \frac{\partial^\nu \pi_Q(0)}{\nu!} x^\nu.$$

Secondly, for each q and r, we can obtain the inequality of (1) with π possibly depending on q and r by taking the limit of (2.6). To be precise, fix q and r and suppose $G_{q,r}^{\alpha,k}(f)(x_0) < \infty$ for at least one $x_0 \in \mathbb{R}^n$. For $R_j = [-j, j]^n$, take $\pi_{R_j} \in \Pi_k^1(f, r, R_j)$. Then for $|\nu| > \alpha$ the limit

$$\lim_{j \to \infty} \partial^\nu \pi_{R_j} = a_\nu$$

exists uniformly on compact sets, and, by taking limit of (2.6) with $Q = R_j$ and with π_Q replaced by $\tilde{\pi}_{R_j}$, we can prove that the polynomial

$$\pi(x) = \lim_{j \to \infty} \tilde{\pi}_{R_j}(x) = \sum_{\alpha < |\nu| \leqq k} \frac{a_\nu(0)}{\nu!} x^\nu \tag{2.7}$$

satisfies the inequality of (1). Thirdly, by observing that the polynomial P for which

$$G_{q,r}^{\alpha,[\alpha]}(f - P)(x) < \infty \quad \text{for some} \quad (q, r, x)$$

(if there exists any such P) is unique mod $\mathcal{P}_k$, we can easily see that the π of (1) can be taken independent of q and r. Fourthly, the claim (2) easily follows from the formula (2.7). Finally, the corollary immediately follows from the proposition since, by (2) of the proposition, $\|f\|_{s,\mathbb{R}^n} < \infty$ implies that we can take $\pi = 0$ in the inequality of (1) of the proposition. □

As the final subject of this section, we shall see the relations between the function $G^{\alpha,k}_{\infty,r}(f)$ and the sharp and flat maximal functions of DeVore and Sharpley [DS]. For functions f on Ω and for $1 \leqq b < \infty$, we define

$$f^{\#(b)}_{k,\alpha,r}(x) = \sup\{\ell(R)^{-\alpha} v^k_r(f,R) \mid R : \text{cube},\ R \ni x,\ bR \subset \Omega\}, \qquad x \in \Omega.$$

The maximal functions $f^{\#}_{\alpha,r}$ and $f^{\flat}_{\alpha,r}$ of [DS; p. 22] coincide with $f^{\#(1)}_{k,\alpha,r}$ with $k \leqq \alpha < k+1$ (*i.e.*, $k = [\alpha]$) and $k < \alpha \leqq k+1$ respectively.

The following proposition holds.

Proposition 6 (1) *We have*

$$c^{-1} f^{\#(5)}_{k,\alpha,r}(x) \leqq G^{\alpha,k}_{\infty,r}(f)(x) \leqq f^{\#(2)}_{k,\alpha,r}(x), \qquad x \in \Omega.$$

(2) *For* $1 < b < \infty$, *we have*

$$\|f^{\#(b)}_{k,\alpha,r}\|_{p,\Omega} \approx \|G^{\alpha,k}_{\infty,r}(f)\|_{p,\Omega}. \tag{2.8}$$

(3) *If* $1/r + \alpha/n > 1/p$, *then* (2.8) *holds for* $b = 1$ *as well.*

Proof. The claim (1) is easy to prove. The claim (2) can be proved by the same method as in the proof of Proposition 1. The claim (3) follows if we prove the estimate

$$\|f^{\#(1)}_{k,\alpha,r}\|_{p,\Omega} \leqq c \|f^{\#(b)}_{k,\alpha,r}\|_{p,\Omega} \tag{2.9}$$

for $b > 1$ and $1/r + \alpha/n > 1/p$. In fact (2.9), combined with the obvious inequality $\|f^{\#(b)}_{k,\alpha,r}\|_{p,\Omega} \leqq \|f^{\#(1)}_{k,\alpha,r}\|_{p,\Omega}$ and with (2), implies the desired result. To prove (2.9), observe first that the claim of Proposition 4 for $q = \infty$ holds if we replace $G^{\alpha,k}_{q,r_i} = G^{\alpha,k}_{\infty,r_i}$ by $f^{\#(b)}_{k,\alpha,r_i}$ $(1 \leqq b < \infty)$. Hence, in order to prove (2.9) for $1/r + \alpha/n > 1/p$, it is sufficient to prove it for sufficiently small r. If $r \leqq 1$, then by following the argument in [J1; Lemma 2.3 and Corollary in §3], we can prove that

$$f^{\#(1)}_{k,\alpha,r} \leqq c M_r(f^{\#(b)}_{k,\alpha,r})(x) \quad \text{for all} \quad x \in \Omega;$$

this inequality implies (2.9) if in addition $r < p$. Details are left to the reader. □

3. General results for extension

In this section, we prove a general result concerning the extension which holds for arbitrary Ω and then prove Theorem 1.

We first recall the Whitney decomposition of open subsets of $\mathbb{R}^n$. For $\Omega \neq \mathbb{R}^n$, let $\mathcal{G}(\Omega)$ be the set of maximal dyadic cubes Q satisfying $3Q \subset \Omega$. The following lemma is well known.

Lemma 4 *Let $\Omega \neq \mathbb{R}^n$.*

(1) *The interiors of the cubes in $\mathcal{G}(\Omega)$ are disjoint and the union of all the cubes in $\mathcal{G}(\Omega)$ is equal to Ω.*

(2) *If $0 < a < 3$, then the cubes aQ with $Q \in \mathcal{G}(\Omega)$ have bounded overlaps.*

(3) *If two cubes Q_1 and Q_2 in $\mathcal{G}(\Omega)$ have nonempty intersection, then $2^{-1} \leqq \ell(Q_1)/\ell(Q_2) \leqq 2$.*

(4) *There exists a family of C^∞ functions $\{\phi_Q^\Omega \mid Q \in \mathcal{G}(\Omega)\}$ such that $\operatorname{supp} \phi_Q^\Omega \subset 2Q$, $0 \leqq \phi_Q^\Omega(x) \leqq 1$, $\sum_{Q \in \mathcal{G}(\Omega)} \phi_Q^\Omega(x) = 1$ for all $x \in \Omega$, and $|\partial_x^\nu \phi_Q^\Omega(x)| \leqq c_\nu \ell(Q)^{-|\nu|}$ for each multi-index ν.*

We next recall the vector maximal inequality of Fefferman and Stein [FS1].

Lemma 5 *Let $\mathcal{A}$ be a countable set of cubes and suppose a nonnegative real number a_Q is associated with each $Q \in \mathcal{A}$. If $0 < p, q < \infty$ and if $\infty > \lambda > \max\{n/p, n/q\}$, then*

$$\left\| \left(\sum_{Q \in \mathcal{A}} a_Q{}^q \left(\frac{\ell(Q)}{\ell(Q) + |x - x_Q|} \right)^{\lambda q} \right)^{1/q} \right\|_{p, \mathbb{R}^n} \leqq c \left\| \left(\sum_{Q \in \mathcal{A}} a_Q{}^q \chi_Q \right)^{1/q} \right\|_{p, \mathbb{R}^n}$$

with $c = c(n, \lambda, p, q)$.

In fact, this lemma follows from the inequality of [FS1; Theorem 1 (1)] once one observes that

$$M_s(a_Q \chi_Q)(x) \approx a_Q \left(\frac{\ell(Q)}{\ell(Q) + |x - x_Q|} \right)^{n/s}, \quad 0 < s < \infty.$$

Now let $\Omega \neq \mathbb{R}^n$ and f a function on Ω. Let $1 \leqq A < \infty$ and fix a k (nonnegative integer) and an r ($0 < r \leqq \infty$). For each $Q \in \mathcal{G}(\Omega)$, take a

$\pi_Q \in \Pi_k^A(f, r, Q)$. We define

$$F_1 = \sum_{Q \in \mathcal{G}(\Omega)} \pi_Q \phi_Q^\Omega$$

and

$$(f - F_1)^\sim(x) = \begin{cases} f(x) - F_1(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

Then we have the following proposition.

Proposition 7 *Let F_1 be as given above. Suppose either* (1.6) *or* (1.7) *holds. Also suppose* (1.2) *holds. Then*

$$|(f - F_1)^\sim; E_{p,q}^{\alpha,k}(\mathbb{R}^n)| \leqq c|f; E_{p,q}^{\alpha,k}(\Omega)|$$

with $c = c(n, k, \alpha, p, q, r, A)$.

Proof. In this proof, c denotes various positive constants which depend only on n, k, α, p, q, r, A, and other parameters (if any) indicated as subscripts, and $\bar{c}$ denotes various positive constants which depend only on n.

We devide the set $\mathcal{D}$ of all dyadic cubes into $\mathcal{D}_1 = \{R \in \mathcal{D} \mid 3R \subset \Omega\}$ and $\mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1$. We write

$$g(h) = g_{q,r}^{\alpha,k}(h) \qquad \text{and}$$

$$g_i(h)(x) = \left(\sum_{x \in R \in \mathcal{D}_i} \left(\ell(R)^{-\alpha} v_r^k(h, 2R) \right)^q \right)^{1/q}, \quad i = 1,\ 2.$$

Obviously $g(h) \leqq c\, g_1(h) + c\, g_2(h)$. Hence the desired inequality follows if we prove

$$\|g_i\left((f - F_1)^\sim\right)\|_{p,\mathbb{R}^n} \leqq c\|g(f)\|_{p,\Omega} \tag{3.1$_i$}$$

for $i = 1$ and 2 (by Proposition 2).

In the sequel we shall use the following simple notations:

$$v_R = v_r^k(f, 2R), \qquad f^\# = f_{k,\alpha,r}^{\#(3/2)}.$$

By Propositions 6, 3, and 2, we have

$$\|f^\#\|_{p,\Omega} \approx |f; E_{p,\infty}^{\alpha,k}(\Omega)| \leqq c|f; E_{p,q}^{\alpha,k}(\Omega)| \approx \|g(f)\|_{p,\Omega}. \tag{3.2}$$

Proof of (3.1_1). The function $g_1((f-F_1)^{\sim})$ vanishes outside Ω, and on Ω we have

$$g_1((f-F_1)^{\sim}) \leqq c\, g_1(f) + c\, g_1(F_1) = c\, g(f) + c\, g_1(F_1).$$

For $g_1(F_1)$, we shall prove that

$$g_1(F_1)(x) \leqq c_\eta M_\eta(f^{\#})(x), \quad x \in \Omega, \tag{3.3}$$

for every $\eta > 0$, which, combined with the inequality just above and with (3.2), implies (3.1_1).

We shall make a further reduction. Since every dyadic cube R satisfying $3R \subset \Omega$ is included in a cube $K \in \mathcal{G}(\Omega)$, and since each fixed $x \in \Omega$ is contained in at most $\bar{c}$ cubes K in $\mathcal{G}(\Omega)$, we have

$$g_1(F_1)(x) \leqq c \sup_{x \in K \in \mathcal{G}(\Omega)} \left(\sum_{x \in R \subset K} (\ell(R)^{-\alpha} v_r^k(F_1, 2R))^q \right)^{1/q}.$$

Hence, inorder to prove (3.3), it is sufficient to prove that

$$\left(\sum_{x \in R \subset K} (\ell(R)^{-\alpha} v_r^k(F_1, 2R))^q \right)^{1/q} \leqq c_\eta M_\eta(f^{\#})(x), \quad x \in K, \tag{3.4}$$

for every fixed $K \in \mathcal{G}(\Omega)$ and for every $\eta > 0$.

We shall prove (3.4). Fix an x and a K such that $x \in K \in \mathcal{G}(\Omega)$. Set

$$\mathcal{A} = \{Q \in \mathcal{G}(\Omega) \mid 2Q \cap 2K \neq \emptyset\} \qquad \text{and}$$
$$\mathcal{B} = \{T \in \mathcal{G}(\Omega) \mid T \cap 2Q \neq \emptyset \ \text{ for some } \ Q \in \mathcal{A}\}.$$

We have $\#\mathcal{A} \leqq \bar{c}$ and $\#\mathcal{B} \leqq \bar{c}$ and

$$\ell(Q) \approx \ell(T) \approx \ell(K) \quad \text{for} \ \ Q \in \mathcal{A} \ \text{ and } \ T \in \mathcal{B}. \tag{3.5}$$

For each $Q \in \mathcal{A}$, there exist cubes $T_j \in \mathcal{B}$ $(j = 0, 1, \ldots, m)$ such that $T_0 = Q$, $T_m = K$, and $T_j \cap T_{j+1} \neq \emptyset$. Then, by Lemma 2 (3),

$$\|\pi_{T_j} - \pi_{T_{j+1}}\|_{\infty, 2Q} \leqq c \|\pi_{T_j} - \pi_{T_{j+1}}\|_{\infty, T_j} \leqq c(v_{T_j} + v_{T_{j+1}})$$

(the first inequality holds because $2Q \subset \bar{c}T_j$). Summing over j's, we have

$$\|\pi_Q - \pi_K\|_{\infty, 2Q} \leqq c \sum_{T \in \mathcal{B}} v_T$$

and hence

$$\|\partial^\nu(\pi_Q - \pi_K)\|_{\infty,2Q} \leqq c\,\ell(Q)^{-|\nu|} \sum_{T\in\mathcal{B}} v_T$$

for all multi-index ν. From this estimate and from (3.5), we obtain

$$\begin{aligned}\|\nabla^{k+1}F_1\|_{\infty,2K} &= \left\|\nabla^{k+1}\left(\sum_{Q\in\mathcal{A}}(\pi_Q - \pi_K)\phi_Q^\Omega\right)\right\|_{\infty,2K}\\ &\leqq c\,\ell(K)^{-k-1}\sum_{T\in\mathcal{B}} v_T,\end{aligned}$$

where $\nabla^{k+1}F = (\partial^\nu F)_{|\nu|=k+1}$. Hence, if R is a dyadic cube with $R \subset K$, then by approximating F_1 by its Taylor polynomial of order k expanded about the center of R we obtain

$$v_r^k(F_1, 2R) \leqq v_\infty^k(F_1, 2R) \leqq c(\ell(R)\ell(K)^{-1})^{k+1}\sum_{T\in\mathcal{B}} v_T$$

and hence, using (3.5) again,

$$\begin{aligned}\ell(R)^{-\alpha}v_r^k(F_1, 2R) &\leqq c(\ell(R)\ell(K)^{-1})^{k+1-\alpha}\sum_{T\in\mathcal{B}}\ell(T)^{-\alpha}v_T\\ &\leqq c(\ell(R)\ell(K)^{-1})^{k+1-\alpha}\sum_{T\in\mathcal{B}}\inf_T f^\#\\ &\leqq c_\eta(\ell(R)\ell(K)^{-1})^{k+1-\alpha}M_\eta(f^\#)(x)\end{aligned}$$

for every $\eta > 0$ (the last inequality follows from the fact that $T \in \mathcal{B}$ satisfy $\bar{c}T \supset K \ni x$ and that $\#\mathcal{B} \leqq \bar{c}$). Taking ℓ^q-quasinorm over R's, we obtain (3.4). Thus (3.1_1) is proved.

Proof of (3.1_2). We shall prove the estimate

$$\|g_2\left((f - F_1)^\sim\right)\|_{p,\mathbb{R}^n} \leqq c\|f^\#\|_{p,\Omega}, \tag{3.6}$$

which is stronger than (3.1_2) (see (3.2)).

Suppose $R \in \mathcal{D}_2$. We set

$$\mathcal{A}(R) = \{Q \in \mathcal{G}(\Omega) \mid 2Q \cap 2R \neq \emptyset\}.$$

There exists an absolute positive constant a such that all $Q \in \mathcal{A}(R)$ satisfy

$Q \subset aR$. We have

$$|2R|^{1/r} v_r^k \left((f - F_1)^\sim, 2R\right) \leqq \left(\int_{2R\cap\Omega} |f(x) - F_1(x)|^r dx\right)^{1/r}$$
$$\leqq c\left(\sum_{Q\in\mathcal{A}(R)} \int_{2Q} |f(x) - \pi_Q(x)|^r dx\right)^{1/r},$$

from which we obtain

$$\ell(R)^{-\alpha} v_r^k \left((f - F_1)^\sim, 2R\right) \leqq c \left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} |Q| {v_Q}^r\right)^{1/r} \ell(R)^{-n/r-\alpha}. \tag{3.7}$$

We shall consider two cases separately.

First, suppose (1.7) and (1.2) hold. We set $s = \min\{q, r\}$. Then from (3.7) we obtain

$$g_2\left((f - F_1)^\sim\right)(x) \leqq c \left(\sum_{x\in R\in\mathcal{D}_2} \left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} |Q| {v_Q}^r\right)^{q/r} \ell(R)^{-(n/r+\alpha)q}\right)^{1/q}$$
$$\leqq c \left(\sum_{x\in R\in\mathcal{D}_2} \left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} |Q| {v_Q}^r\right)^{s/r} \ell(R)^{-(n/r+\alpha)s}\right)^{1/s}$$
$$\leqq c \left(\sum_{x\in R\in\mathcal{D}_2} \sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} |Q|^{s/r} {v_Q}^s \ell(R)^{-(n/r+\alpha)s}\right)^{1/s}.$$

We estimate the last term by taking the sum over R first and using the fact that the cubes R with $x \in R$ and $Q \subset aR$ satisfy $\bar{c}\,\ell(R) \geqq \ell(Q) + |x - x_Q|$; the result is

$$g_2\left((f-F_1)^{\sim}\right)(x) \leqq c\left(\sum_{Q\in\mathcal{G}(\Omega)} \left(\ell(Q)^{-\alpha}v_Q\right)^s \left(\frac{\ell(Q)}{\ell(Q)+|x-x_Q|}\right)^{(n/r+\alpha)s}\right)^{1/s}.$$

Now we use the vector maximal inequality of Fefferman and Stein (see Lemma 5) to obtain

$$\begin{aligned}\|g_2\left((f-F_1)^{\sim}\right)\|_{p,\mathbb{R}^n} &\leqq c\left\|\left(\sum_{Q\in\mathcal{G}(\Omega)} \left(\ell(Q)^{-\alpha}v_Q\right)^s \chi_Q\right)^{1/s}\right\|_{p,\mathbb{R}^n}\\ &\leqq c\|f^{\#}\|_{p,\Omega},\end{aligned}$$

where the last inequality holds because the cubes in $\mathcal{G}(\Omega)$ are essentially disjoint. Thus (3.6) is proved under (1.7) and (1.2).

Next suppose (1.6) and (1.2) hold. We set $1/r+\alpha/n=1/\sigma$. Then the right hand side of (3.7) is written as

$$c|R|^{-1/\sigma}\left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} \left(|Q|^{1/\sigma}\ell(Q)^{-\alpha}v_Q\right)^r\right)^{1/r},$$

which is majorized by

$$\begin{aligned}&c|R|^{-1/\sigma}\left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} \left(|Q|^{1/\sigma}\ell(Q)^{-\alpha}v_Q\right)^\sigma\right)^{1/\sigma}\\ &\qquad\leqq c|R|^{-1/\sigma}\left(\sum_{\substack{Q\in\mathcal{G}(\Omega)\\ Q\subset aR}} |Q|\left(\inf_Q f^{\#}\right)^\sigma\right)^{1/\sigma}\\ &\qquad\leqq c|R|^{-1/\sigma}\left(\int_{\Omega\cap aR} (f^{\#})^\sigma\right)^{1/\sigma}.\end{aligned}$$

Hence, taking sup over R's, we obtain

$$g_2\left((f-F_1)^{\sim}\right)(x) \leqq cM_\sigma(f^{\#})(x).$$

Since $\sigma < p$, this inequality implies (3.6). Thus (3.1_2) is proved. This completes the proof of Proposition 7. □

We shall now prove Theorem 1.

Proof of Theorem 1. Take an r satisfying $1/r+\alpha/n > \max\{1/p, 1/q_0\}$. Let f be a function on Ω. Let F_1 be the function as treated in Proposition 7. Since Ω is an extension domain for $E^{\alpha,k}_{p,q_0}$, there exists a function F_1^* on $\mathbb{R}^n$ which is an extension of F_1 and which satisfies

$$|F_1^*; E^{\alpha,k}_{p,q_0}(\mathbb{R}^n)| \leqq A|F_1; E^{\alpha,k}_{p,q_0}(\Omega)|.$$

Notice that in the proof of Proposition 7 we actually proved that

$$|F_1; E^{\alpha,k}_{p,q_0}(\Omega)| \leqq c|f; E^{\alpha,k}_{p,\infty}(\Omega)|$$

(see (3.3) and (3.2)). Hence

$$|F_1^*; E^{\alpha,k}_{p,q_0}(\mathbb{R}^n)| \leqq cA|f; E^{\alpha,k}_{p,\infty}(\Omega)|. \tag{3.8}$$

Now, if $q_0 < q \leqq \infty$, then (3.8) combined with Proposition 3 implies that

$$|F_1^*; E^{\alpha,k}_{p,q}(\mathbb{R}^n)| \leqq cA|f; E^{\alpha,k}_{p,q}(\Omega)|,$$

from which and from Proposition 7, we see that $f^* = (f-F_1)^{\sim}+F_1^*$, which is an extension of f, satisfies

$$|f^*; E^{\alpha,k}_{p,q}(\mathbb{R}^n)| \leqq cA|f; E^{\alpha,k}_{p,q}(\Omega)|.$$

Theorem 1 is proved. □

4. Extension for (ϵ, δ)-domains

The purpose of this section is to prove Theorem 2.

Throughout this section, we assume $\Omega \neq \mathbb{R}^n$ and Ω is an (ϵ,δ)-domain with $0 < \epsilon \leqq 1$ and $0 < \delta \leqq \infty$. Let $\hat{\Omega}$ denote the interior of Ω^c and let $\delta_1 = \min\{\delta, \operatorname{diam}\Omega\}$.

The following properties of the (ϵ,δ)-domain are known; see [J1], [J2; §2], or [M; §3].

Lemma 6 (1) *If* Q, $S \in \mathcal{G}(\Omega)$, $0 < A < \infty$, $\ell(Q) \leqq \ell(S)$, $\operatorname{dis}(Q,S) \leqq A\ell(S)$, *and* $\operatorname{dis}(Q,S) < \epsilon\delta$, *then there exist cubes* $T_j \in \mathcal{G}(\Omega)$ $(j=0,1,\dots,N)$ *such that* $T_0 = Q$, $T_N = S$, $T_j \cap T_{j+1} \neq \emptyset$, $c(n,\epsilon,A)T_j \supset Q$, $T_j \subset c(n,\epsilon,A)S$, *and*

$$N \leqq c(n,\epsilon)\log\left(\ell(S)/\ell(Q)\right) + c(n,\epsilon,A).$$

(2) *There exists a positive constant β which depends only on n and ϵ and with which the following holds: For each $Q \in \mathcal{G}(\hat{\Omega})$ with $\ell(Q) < \beta\delta_1$, there exists a cube $\check{Q}$ in $\mathcal{G}(\Omega)$ such that $\ell(\check{Q}) = \ell(Q)$ and $\operatorname{dis}(Q,\check{Q}) \leqq c(n,\epsilon)\ell(Q)$.*

(3) *For each cube R with $\ell(R) < \delta$, there exists a cube S in $\mathcal{G}(\Omega)\cup\mathcal{G}(\hat{\Omega})$ such that $S \cap R \neq \emptyset$ and $\ell(S) \geqq c(n,\epsilon)\ell(R)$.*

(4) *The boundary of Ω has measure 0.*

(5) *If $\delta = \infty > \operatorname{diam}\Omega$ and if Q_0 is a cube in $\mathcal{G}(\Omega)$ which has the maximum sidelength, then $\ell(Q_0) \geqq c(n,\epsilon)\operatorname{diam}\Omega$.*

In the sequel, we fix a mapping $Q \mapsto \check{Q}$ as mentioned in (2) of the above lemma and also fix, in the case of (5), a cube Q_0 as mentoioned there.

We shall prove Theorem 2 by using the extension method of [J1], [C], [M], and [S]. We shall recall the method.

Let f be a function on Ω. We take a k (nonnegative integer) and an r ($0 < r \leqq \infty$) and a real number A satisfying $1 \leqq A < \infty$. For each $Q \in \mathcal{G}(\Omega)$, we take $\pi_Q \in \Pi_k^A(f,r,Q)$. We take d such that $0 < d \leqq \beta^*\delta_1$, where β^* is a positive constant which depends only on n and ϵ and which is sufficiently small (in particular, it is not larger than the β of Lemma 6 (2)), and we define the function F_2 on $\mathbb{R}^n$ by $F_2 = f$ on Ω and

$$F_2 = \sum_{\substack{Q\in\mathcal{G}(\hat{\Omega})\\ \ell(Q)<d}} \pi_{\check{Q}}\phi_Q^{\hat{\Omega}} \quad \text{on} \quad \Omega^c.$$

Notice that d can be equal to ∞ if $\delta_1 = \infty$. If $\delta = \infty$ and $\delta_1 = \operatorname{diam}\Omega < \infty$, then we define the function F_3 on $\mathbb{R}^n$ by $F_3 = f$ on Ω and

$$F_3 = \sum_{\substack{Q\in\mathcal{G}(\hat{\Omega})\\ \ell(Q)<\beta\delta_1}} \pi_{\check{Q}}\phi_Q^{\hat{\Omega}} + \sum_{\substack{Q\in\mathcal{G}(\hat{\Omega})\\ \ell(Q)\geqq\beta\delta_1}} \pi_{Q_0}\phi_Q^{\hat{\Omega}} \quad \text{on} \quad \Omega^c.$$

In the papers mentioned above, it is shown that F_2 or F_3 give the desired extension if we choose r appropriately. The choice of r is crucial to our problem of the linearity of the extension operator. The reason is this: There exists a linear mapping $S_Q : L^r(Q) \to \mathcal{P}_k$ satisfying

$$S_Q(f) \in \Pi_k^A(f,r,Q) \quad \text{for all} \quad f \in L^r(Q), \tag{4.1}$$

with A independent of f, if and only if $r \geqq 1$. If $1 \leqq r \leqq \infty$, an example of the linear mapping $S_Q : L^r(Q) \to \mathcal{P}_k$ satisfying (4.1) with $A = c(n,k,r)$ is defined by

$$\int_Q (f - S_Q(f))\, P dx = 0 \quad \text{for all} \quad P \in \mathcal{P}_k.$$

In [J1], [C], and [S], it is shown that the choice $r = \min\{1,p,q\}$ is allowed in order that F_2 or F_3 satisfy the estimate for $|\cdot\,; E^{\alpha,k}_{p,q}|$. In [M], it is shown that r satisfying (1.2) works well in the case $q = \infty$ and $\alpha > 0$. We shall show that these results can be extended as in the following proposition.

Proposition 8 *Let $\Omega \neq \mathbb{R}^n$ be an (ϵ,δ)-domain, $0 < \epsilon \leqq 1$, $0 < \delta \leqq \infty$, and let F_2 and F_3 be defined as above. Suppose either* (1.6) *or* (1.7) *holds. Also suppose* (1.2) *holds. Then:*

(1) *We have*

$$|F_2; E^{\alpha,k}_{p,q}(\mathbb{R}^n)| \leqq c_\epsilon \left(|f; E^{\alpha,k}_{p,q}(\Omega)| + d^{-\alpha} \|f\|_{p,\Omega} \right)$$

(here $d^{-\alpha} = 0$ if $d = \infty$).

(2) *If $d < \infty$, then*

$$\|F_2\|_{p,\mathbb{R}^n} \leqq c_\epsilon \left(d^\alpha |f; E^{\alpha,k}_{p,\infty}(\Omega)| + \|f\|_{p,\Omega} \right).$$

(3) *If $\delta = \infty > \operatorname{diam}\Omega$, then*

$$|F_3; E^{\alpha,k}_{p,q}(\mathbb{R}^n)| \leqq c_\epsilon |f; E^{\alpha,k}_{p,q}(\Omega)|.$$

Here $c_\epsilon = c(n,k,r,A,p,q,\alpha,\epsilon)$.

Before going into the proof of this proposition, we shall see that Theorem 2 follows from it. In fact, under the assumption of Theorem 2 we can take $r = 1$ and, as mentioned above, we can define the operators $f \mapsto F_2$ and $f \mapsto F_3$ as linear operators. Then the operator T_2 of Theorem 2 (2) can be given by $T_2 f = F_2$ with $d < \infty$. If $\delta = \infty = \operatorname{diam}\Omega$, the operator T_1 of Theorem 2 (1) can be given by $T_1 f = F_2$ with $d = \infty$; if $\delta = \infty > \operatorname{diam}\Omega$, it is given by $T_1 f = F_3$.

Proof of Proposition 8. In this proof, various positive constants shall be denoted by the letters c and $\bar{c}$. These are used as follows: c denotes various positive constants which depend only on n, k, r, A, α, p, q, and on other parameters (if any) indicated as subscripts; $\bar{c}$ denotes various positive constants which depend only on n and other parameters (if any) indicated as subscripts. In what follows, we shall introduce two positive real numbers t and s. Since t and s can be chosen depending only on α, p, q, r, and n, we shall omit to indicate the dependence of the constant c on t or s.

We use the same abbreviations as in the proof of Proposition 7:

$$g(f) = g_{q,r}^{\alpha,k}(f), \quad f^{\#} = f_{k,\alpha,r}^{\#(3/2)}, \quad v_R = v_r^k(f, 2R).$$

Recall that (3.2) holds. We shall begin with (2).

Proof of (2). Suppose $d < \infty$. Take a positive real number t such that $1/r + \alpha/n > 1/t > 1/p$. Let $Q \in \mathcal{G}(\hat{\Omega})$ with $\ell(Q) < d$. Using Lemma 2 (2) and the corollary to Lemma 3, we see that

$$\begin{aligned} \|\pi_{\check{Q}}\|_{\infty,\check{Q}} &\leqq c|\check{Q}|^{-1/r}\|f\|_{r,\check{Q}} \\ &\leqq c\left(|\check{Q}|^{\alpha/n-1/t}\|f^{\#}\|_{t,\check{Q}} + |\check{Q}|^{-1/t}\|f\|_{t,\check{Q}}\right). \end{aligned}$$

Since $2Q \subset \bar{c}_\epsilon \check{Q}$, we have

$$\|\pi_{\check{Q}}\|_{\infty,2Q} \leqq c_\epsilon \|\pi_{\check{Q}}\|_{\infty,\check{Q}}.$$

Combining these inequalities and using the fact $2Q \subset \bar{c}_\epsilon \check{Q}$ again, we see that

$$|F_2(x)| \leqq c_\epsilon(d^\alpha M_t(f^{\#})(x) + M_t(f)(x)) \tag{4.2}$$

for $x \in \Omega^c$. Obviously this inequality also holds a.e. on Ω. The claim (2) follows from (4.2) and (3.2).

Proof of (1). Let t be the same as above and take a real number s such that $0 < s \leqq \min\{1, q, r\}$ and $s < p$. We decompose the set $\mathcal{D}$ of all dyadic cubes into the following three subsets:

$$\begin{aligned} \mathcal{D}_1 &= \{R \in \mathcal{D} \mid 3R \subset \Omega\}, \\ \mathcal{D}_2 &= \{R \in \mathcal{D} \mid 3R \subset \hat{\Omega}\}, \\ \mathcal{D}_3 &= \mathcal{D} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2). \end{aligned}$$

For functions F on $\mathbb{R}^n$, we write

$$\bar{g}(F) = g_{q,s}^{\alpha,k}(F) \qquad \text{and}$$

$$\bar{g}_i(F) = \left(\sum_{R \in \mathcal{D}_i} \left(\ell(R)^{-\alpha} v_s^k(F, 2R) \right)^q \chi_R \right)^{1/q}, \quad i = 1,\, 2,\, 3.$$

Obviously $\bar{g}(F) \leqq c\,(\bar{g}_1(F) + \bar{g}_2(F) + \bar{g}_3(F))$. Hence, in order to prove (1), it is sufficient to prove the estimate

$$\|\bar{g}_i(F_2)\|_{p,\mathbb{R}^n} \leqq c_\epsilon \left(\|g(f)\|_{p,\Omega} + d^{-\alpha} \|f\|_{p,\Omega} \right) \tag{4.3$_i$}$$

for $i = 1,\, 2,\, 3$ (by virtue of Propositions 2 and 4).

Proof of (4.3$_1$). The function $\bar{g}_1(F_2)$ vanishes outside Ω, and on Ω we have $\bar{g}_1(F_2) = g_{q,s}^{\alpha,k}(f) \leqq g(f)$ (since $s \leqq r$). Hence (4.3$_1$) is obvious.

Proof of (4.3$_2$). We shall prove the pointwise estimate

$$\bar{g}_2(F_2) \leqq c_\epsilon (M_t(f^\#) + d^{-\alpha} M_t(f)),$$

which, combined with (3.2), implies (4.3$_2$). For $K \in \mathcal{G}(\hat{\Omega})$, we set

$$\bar{g}_{2,K}(F_2) = \left(\sum_{\substack{R \in \mathcal{D} \\ R \subset K}} \left(\ell(R)^{-\alpha} v_s^k(F_2, 2R) \right)^q \chi_R \right)^{1/q}.$$

Then, by the same reason as in the proof of Proposition 7 (see the argument between (3.3) and (3.4)), the pointwise estimate for $\bar{g}_2(F_2)$ mentioned above follows if we prove the estimate

$$\bar{g}_{2,K}(F_2) \leqq c_\epsilon (M_t(f^\#) + d^{-\alpha} M_t(f)) \quad \text{on} \quad K \tag{4.4}$$

for each $K \in \mathcal{G}(\hat{\Omega})$.

We shall prove (4.4). Fix an x and a K such that $x \in K \in \mathcal{G}(\hat{\Omega})$. Set

$$\mathcal{A} = \{ Q \in \mathcal{G}(\hat{\Omega}) \mid 2Q \cap 2K \neq \emptyset \}.$$

Notice that $\#\mathcal{A} \leqq \bar{c}$ and that $\ell(Q) \approx \ell(K)$ for all $Q \in \mathcal{A}$. We shall consider three cases separately.

Case (i) $\max\{\ell(Q) \mid Q \in \mathcal{A}\} < d$. For each $Q \in \mathcal{A}$, we have

$$\operatorname{dis}(\check{Q}, \check{K}) \leqq \bar{c}_\epsilon \ell(K) < \bar{c}_\epsilon d \leqq \bar{c}_\epsilon \beta^* \delta_1 \leqq \epsilon\delta$$

(the constant β^* is chosen so small that the last inequality hold), and hence, by Lemma 6 (1), there exist cubes $T_j \in \mathcal{G}(\Omega)$ ($j = 0,\, 1, \ldots, m$) such that $T_0 = \check{Q}$, $T_m = \check{K}$, $T_j \cap T_{j+1} \neq \emptyset$, and $m \leqq \bar{c}_\epsilon$. These T_j necessarily satisfy

$\bar{c}_\epsilon T_j \supset Q \cup K$ and $\bar{c}_\epsilon^{-1} \leqq \ell(T_j)/\ell(K) \leqq \bar{c}_\epsilon$. Let $\mathcal{B}$ denote the set of all T_j's arising from all $Q \in \mathcal{A}$. We have $\#\mathcal{B} \leqq \bar{c}_\epsilon$ since $\#\{T_j\} \leqq \bar{c}_\epsilon$ for each $Q \in \mathcal{A}$ and $\#\mathcal{A} \leqq \bar{c}$. Using the $\{T_j\}$ and $\mathcal{B}$, we can deduce the estimate of $\bar{g}_{2,K}(F_2)$ in the same way as in the proof of Proposition 7 (see Proof of (3.1_1)); *i.e.*, we first obtain

$$\|\pi_{\check{Q}} - \pi_{\check{K}}\|_{\infty,2Q} \leqq c_\epsilon \sum_{T\in\mathcal{B}} v_T$$

for all $Q \in \mathcal{A}$, from which we can deduce

$$\|\nabla^{k+1}F_2\|_{\infty,2K} \leqq c_\epsilon \ell(K)^{-k-1} \sum_{T\in\mathcal{B}} v_T$$

and then

$$\begin{aligned} \ell(R)^{-\alpha} v_s^k(F_2, 2R) &\leqq \ell(R)^{-\alpha} v_\infty^k(F_2, 2R) \\ &\leqq c_\epsilon \left(\frac{\ell(R)}{\ell(K)}\right)^{k+1-\alpha} \sum_{T\in\mathcal{B}} \ell(T)^{-\alpha} v_T \end{aligned}$$

for all dyadic cubes R with $R \subset K$, and finally we obtain the estimate

$$\bar{g}_{2,K}(F_2)(x) \leqq c_{\epsilon,\eta} M_\eta(f^\#)(x)$$

for every $\eta > 0$, which a fortiori implies (4.4).

Case (ii) $\min\{\ell(Q) \mid Q \in \mathcal{A}\} \geqq d$. In this case (4.4) is obvious since $F_2 = 0$ on $2K$ and $\bar{g}_{2,K}(F_2) = 0$.

Case (iii) $\min\{\ell(Q) \mid Q \in \mathcal{A}\} < d \leqq \max\{\ell(Q) \mid Q \in \mathcal{A}\}$. This case occurs only when $d < \infty$. Note that $\ell(K) \approx d$ in this case. As we showed in the proof of (2), the estimate

$$\|\pi_{\check{Q}}\|_{\infty,2Q} \leqq c_\epsilon \left(|\check{Q}|^{\alpha/n-1/t}\|f^\#\|_{t,\check{Q}} + |\check{Q}|^{-1/t}\|f\|_{t,\check{Q}}\right)$$

holds for all $Q \in \mathcal{G}(\hat{\Omega})$ with $\ell(Q) < d$. Using this estimate and using the fact that the cubes Q in $\mathcal{A}$ with $\ell(Q) < d$ satisfy $\ell(Q) \approx \ell(K)$ and $\bar{c}_\epsilon \check{Q} \supset \bar{c}Q \supset K \ni x$, we obtain

$$\|\nabla^{k+1}F_2\|_{\infty,2K} \leqq c_\epsilon \ell(K)^{-k-1}(d^\alpha M_t(f^\#)(x) + M_t(f)(x)).$$

From this estimate, we see that

$$\begin{aligned}\ell(R)^{-\alpha}v_s^k(F_2,2R)\\ \leqq c_\epsilon\left(\frac{\ell(R)}{\ell(K)}\right)^{k+1-\alpha}(M_t(f^{\#})(x)+d^{-\alpha}M_t(f)(x))\end{aligned}$$

for all dyadic cubes R included in K. Taking ℓ^q-quasinorm over R's, we obtain (4.4). Thus (4.3_2) is proved.

Proof of (4.3_3). This is the main part of the proof of (1). For $R\in\mathcal{D}_3$, we set

$$\mathcal{A}(R)=\{Q\in\mathcal{G}(\Omega)\cup\mathcal{G}(\hat{\Omega})\mid 2Q\cap 2R\neq\emptyset\}.$$

As is easily seen, there exists an absolute positive constant a such that $\ell(Q)\leqq a\ell(R)$ for all $Q\in\mathcal{A}(R)$ and all $R\in\mathcal{D}_3$. We decompose the set $\mathcal{D}_3$ into the following two subsets:

$$\begin{aligned}\mathcal{D}_{3,1}&=\{R\in\mathcal{D}_3\mid\ell(R)<d/a\},\\ \mathcal{D}_{3,2}&=\{R\in\mathcal{D}_3\mid\ell(R)\geqq d/a\}.\end{aligned}$$

For functions F on $\mathbb{R}^n$, we set

$$\bar{g}_{3,j}(F)=\left(\sum_{R\in\mathcal{D}_{3,j}}\left(\ell(R)^{-\alpha}v_s^k(F,2R)\right)^q\chi_R\right)^{1/q},\quad j=1,\,2.$$

Obviously $\bar{g}_3(F)\leqq c\,(\bar{g}_{3,1}(F)+\bar{g}_{3,2}(F))$. We shall prove

$$\|\bar{g}_{3,j}(F_2)\|_{p,\mathbb{R}^n}\leqq c_\epsilon(\|f^{\#}\|_{p,\Omega}+d^{-\alpha}\|f\|_{p,\Omega})\tag{4.5_j}$$

for $j=1,\,2$, which, combined with (3.2), implies (4.3_3).

Proof of (4.5_1). Let F_1 be the function defined in Section 3. We set $F_2^*=F_2-(f-F_1)^{\sim}$, which can be written as

$$F_2^*=\sum_{Q\in\mathcal{G}(\Omega)}\pi_Q\phi_Q^{\Omega}+\sum_{\substack{Q\in\mathcal{G}(\hat{\Omega})\\ \ell(Q)<d}}\pi_{\check{Q}}\phi_Q^{\hat{\Omega}}.$$

We have

$$\bar{g}_{3,1}(F_2)\leqq c\,\bar{g}_{3,1}(F_2^*)+c\,\bar{g}_{3,1}\left((f-F_1)^{\sim}\right).$$

In the proof of Proposition 7, we proved

$$\|\bar{g}_{3,1}\left((f-F_1)^{\sim}\right)\|_{p,\mathbb{R}^n} \leqq c\|f^{\#}\|_{p,\Omega} \tag{4.6}$$

(see (3.6)). Here we shall prove

$$\|\bar{g}_{3,1}(F_2^*)\|_{p,\mathbb{R}^n} \leqq c_\epsilon\|f^{\#}\|_{p,\Omega}. \tag{4.7}$$

If this is done, then (4.5_1) follows from (4.6) and (4.7).

Suppose $R \in \mathcal{D}_{3,1}$. For all $Q \in \mathcal{A}(R)$, we have $\ell(Q) \leqq a\ell(R) < d$. For $Q \in \mathcal{A}(R)$, we define $\tilde{Q}$ as $\tilde{Q} = Q$ if $Q \in \mathcal{G}(\Omega)$ and $\tilde{Q} = \check{Q}$ if $Q \in \mathcal{G}(\hat{\Omega})$. We take a cube S in $\mathcal{A}(R)$ which has the maximum sidelength. Notice that $\ell(R) \leqq \bar{c}_\epsilon \ell(S)$ by Lemma 6 (3).

Let $Q \in \mathcal{A}(R)$. We have

$$\operatorname{dis}(\tilde{Q}, \tilde{S}) \leqq \bar{c}_\epsilon \ell(R) \leqq \bar{c}_\epsilon \ell(S) \qquad \text{and}$$
$$\operatorname{dis}(\tilde{Q}, \tilde{S}) \leqq \bar{c}_\epsilon \ell(R) < \bar{c}_\epsilon d/a \leqq \bar{c}_\epsilon \beta^* \delta_1/a \leqq \epsilon\delta$$

(the constant β^* is chosen so small that the last inequality hold). Thus, by Lemma 6 (1), there exist cubes $T_j \in \mathcal{G}(\Omega)$ $(j = 0, 1, \ldots, N)$ such that $T_0 = \tilde{Q}$, $T_N = \tilde{S}$, $T_j \cap T_{j+1} \neq \emptyset$, $\bar{c}_\epsilon T_j \supset \tilde{Q}$, and $T_j \subset \bar{c}_\epsilon \tilde{S}$. Since $Q \subset \bar{c}_\epsilon \tilde{Q}$ and $\tilde{S} \subset \bar{c}_\epsilon R$, we have $Q \subset \bar{c}_\epsilon T_j$ and $T_j \subset \bar{c}_\epsilon R$. In the same way as in the proof of Proposition 7 (see Proof of (3.1_1)), we have

$$\|\pi_{\tilde{Q}} - \pi_{\tilde{S}}\|_{\infty,2Q} \leqq c_\epsilon \sum_{j=0}^{N} v_{T_j} \leqq c_\epsilon \sum_{T} v_T,$$

where the last sum is taken over the cubes T satisfying

$$T \in \mathcal{G}(\Omega), \quad Q \subset \bar{c}_\epsilon T, \quad \text{and} \quad T \subset \bar{c}_\epsilon R. \tag{4.8}$$

Since the boundary of Ω has measure 0 (Lemma 6 (4)), we have

$$F_2^* - \pi_{\tilde{S}} = \sum_{Q \in \mathcal{A}(R)} (\pi_{\tilde{Q}} - \pi_{\tilde{S}})\phi_Q \qquad \text{a.e. on } 2R,$$

where ϕ_Q stands for ϕ_Q^{Ω} or $\phi_Q^{\hat{\Omega}}$ according as $Q \in \mathcal{G}(\Omega)$ or $Q \in \mathcal{G}(\hat{\Omega})$. Hence, using the fact that the cubes $2Q$ for $Q \in \mathcal{A}(R)$ have bounded overlaps and using the estimate of $\|\pi_{\tilde{Q}} - \pi_{\tilde{S}}\|_{\infty,2Q}$ given above, we see that

$$\|F_2^* - \pi_{\tilde{S}}\|_{s,2R}^s \leqq c \sum_{Q \in \mathcal{A}(R)} |Q|\, \|\pi_{\tilde{Q}} - \pi_{\tilde{S}}\|_{\infty,2Q}^s$$

$$\leqq c_\epsilon \sum_{Q \in \mathcal{A}(R)} |Q| \Bigg(\sum_{T:\,(4.8)} v_T \Bigg)^s$$
$$\leqq c_\epsilon \sum_{Q \in \mathcal{A}(R)} |Q| \sum_{T:\,(4.8)} {v_T}^s$$
$$\leqq c_\epsilon \sum_{\substack{T \in \mathcal{G}(\Omega) \\ T \subset \bar{c}_\epsilon R}} |T| {v_T}^s.$$

Hence

$$\ell(R)^{-\alpha} v_s^k(F_2^*, 2R) \leqq c_\epsilon \left(\sum_{\substack{T \in \mathcal{G}(\Omega) \\ T \subset \bar{c}_\epsilon R}} |T| {v_T}^s \right)^{1/s} \ell(R)^{-n/s-\alpha}. \tag{4.9}$$

We shall deduce (4.7) from (4.9) by the same method as in the proof of Proposition 7 (see Proof of (3.1_2)). In the case of (1.7): From (4.9), we have

$$\begin{aligned} \bar{g}_{3,1}(F_2^*)(x) &\leqq c_\epsilon \left(\sum_{x \in R \in \mathcal{D}_{3.1}} \left(\sum_{\substack{T \in \mathcal{G}(\Omega) \\ T \subset \bar{c}_\epsilon R}} |T| {v_T}^s \right)^{q/s} \ell(R)^{-(n/s+\alpha)q} \right)^{1/q} \\ &\leqq c_\epsilon \left(\sum_{x \in R \in \mathcal{D}_{3.1}} \sum_{\substack{T \in \mathcal{G}(\Omega) \\ T \subset \bar{c}_\epsilon R}} |T| {v_T}^s \ell(R)^{-(n/s+\alpha)s} \right)^{1/s} \\ &\leqq c_\epsilon \left(\sum_{T \in \mathcal{G}(\Omega)} \left(\ell(T)^{-\alpha} v_T\right)^s \left(\frac{\ell(T)}{\ell(T) + |x - x_T|} \right)^{(n/s+\alpha)s} \right)^{1/s}, \end{aligned}$$

and hence, using Lemma 5, we obtain

$$\|\bar{g}_{3,1}(F_2^*)\|_{p,\mathbb{R}^n} \leqq c_\epsilon \left\| \left(\sum_{T \in \mathcal{G}(\Omega)} \left(\ell(T)^{-\alpha} v_T\right)^s \chi_T \right)^{1/s} \right\|_{p,\mathbb{R}^n} \leqq c_\epsilon \|f^\#\|_{p,\Omega}.$$

In the case of (1.6): With σ given by $1/\sigma = 1/s + \alpha/n$, the right hand side of (4.9) can be written as

$$c_\epsilon |R|^{-1/\sigma} \left(\sum_{\substack{T\in\mathcal{G}(\Omega)\\ T\subset \bar{c}_\epsilon R}} \left(|T|^{1/\sigma} \ell(T)^{-\alpha} v_T \right)^s \right)^{1/s}$$

and this is majorized by

$$\begin{aligned} c_\epsilon |R|^{-1/\sigma} & \left(\sum_{\substack{T\in\mathcal{G}(\Omega)\\ T\subset \bar{c}_\epsilon R}} \left(|T|^{1/\sigma} \ell(T)^{-\alpha} v_T \right)^\sigma \right)^{1/\sigma} \\ & \leqq c_\epsilon |R|^{-1/\sigma} \left(\sum_{\substack{T\in\mathcal{G}(\Omega)\\ T\subset \bar{c}_\epsilon R}} |T| \left(\inf_T f^{\#} \right)^\sigma \right)^{1/\sigma} \\ & \leqq c_\epsilon |R|^{-1/\sigma} \left(\int_{\Omega\cap \bar{c}_\epsilon R} (f^{\#})^\sigma \right)^{1/\sigma} ; \end{aligned}$$

hence

$$\bar{g}_{3,1}(F_2^*)(x) = \sup_{x\in R\in\mathcal{D}_{3.1}} \{\ell(R)^{-\alpha} v_s^k(F_2^*, 2R)\} \leqq c_\epsilon M_\sigma(f^{\#})(x),$$

from which follows (4.7). Thus (4.5_1) is proved.

Proof of (4.5_2). We may assume $d < \infty$. For R with $x \in R$, we have

$$\begin{aligned} \ell(R)^{-\alpha} v_s^k(F_2, 2R) & \leqq \ell(R)^{-\alpha} |2R|^{-1/s} \|F_2\|_{s,2R} \\ & \leqq c\,\ell(R)^{-\alpha} M_s(F_2)(x). \end{aligned}$$

Taking ℓ^q-quasinorm over R's satisfying $x \in R \in \mathcal{D}_{3,2}$, we have

$$\bar{g}_{3,2}(F_2)(x) \leqq c d^{-\alpha} M_s(F_2)(x).$$

Combining this estimate with (4.2), we obtain

$$\bar{g}_{3,2}(F_2)(x) \leqq c_\epsilon \left(M_s(M_t(f^{\#}))(x) + d^{-\alpha} M_s\left(M_t(f)\right)(x) \right),$$

from which follows (4.5_2). Now (4.3_3) is proved and the proof of (1) is complete.

Proof of (3). Here we shall be brief since (3) can be proved by only slightly modifying the proof of (1). We set $F_3^* = F_3 - (f - F_1)^\sim$ with F_1 given in Section 3. Since the estimate for $(f - F_1)^\sim$ is already given in Proposition 7, it is sufficient to estimate F_3^*. We shall prove

$$|F_3^*; E_{p,q}^{\alpha,k}(\mathbb{R}^n)| \leqq c_\epsilon |f; E_{p,\infty}^{\alpha,k}(\Omega)|, \tag{4.10}$$

which is sufficient for our purpose (by Proposition 3).

For $Q \in \mathcal{G}(\Omega) \cup \mathcal{G}(\hat{\Omega})$, we define $\tilde{Q}$ as follows: $\tilde{Q} = Q$ if $Q \in \mathcal{G}(\Omega)$; $\tilde{Q} = \check{Q}$ if $Q \in \mathcal{G}(\hat{\Omega})$ and $\ell(Q) < \beta\delta_1$; $\tilde{Q} = Q_0$ if $Q \in \mathcal{G}(\hat{\Omega})$ and $\ell(Q) \geqq \beta\delta_1$. The function F_3^* can be written as

$$F_3^* = \sum_{Q \in \mathcal{G}(\Omega) \cup \mathcal{G}(\hat{\Omega})} \pi_{\tilde{Q}} \phi_Q,$$

where ϕ_Q stands for ϕ_Q^Ω or $\phi_Q^{\hat{\Omega}}$ according as $Q \in \mathcal{G}(\Omega)$ or $Q \in G(\hat{\Omega})$.

The basic task amounts to the estimate of the difference $\pi_{\tilde{Q}} - \pi_{\tilde{S}}$ for Q, $S \in \mathcal{G}(\Omega) \cup \mathcal{G}(\hat{\Omega})$ with, say, $\ell(Q) \leqq \ell(S)$ and $\mathrm{dis}(Q, S) \leqq \bar{c}_\epsilon \ell(S)$. If $\tilde{Q} = \tilde{S} = Q_0$, there is no problem. If $\tilde{Q} \neq Q_0$ or $\tilde{S} \neq Q_0$, then we see that there exist cubes $T_j \in \mathcal{G}(\Omega)$ $(j = 0, 1, \ldots, N)$ such that $T_0 = \tilde{Q}$, $T_N = \tilde{S}$, $T_j \cap T_{j+1} \neq \emptyset$, $\bar{c}_\epsilon T_j \supset Q$, and $T_j \subset \bar{c}_\epsilon S$, and using these T_j we can estimate $\|\pi_{\tilde{Q}} - \pi_{\tilde{S}}\|_{\infty,2Q}$ in the same way as in the proof of (1).

Let s be the same as in the proof of (1). Using the estimate of $\|\pi_{\tilde{Q}} - \pi_{\tilde{S}}\|_{\infty,2Q}$ as explained above and following the argument in the proof of (1), we can prove the inequality

$$\|g_{q,s}^{\alpha,k}(F_3^*)\|_{p,\mathbb{R}^n} \leqq c_\epsilon \|f^\#\|_{p,\Omega},$$

which, combined with Propositions 2 and 4 and with (3.2), implies (4.10). This completes the proof of (3). Thus Proposition 8 and Theorem 2 are proved. □

References

[C] Christ M., *The extension problem for certain function spaces involving fractional orders of differentiability.* Ark. Mat. **22** (1984), 63–81.

[DS] DeVore R.A. and Sharpley R.C., *Maximal Functions Measuring Smoothness.* Mem. Amer. Math. Soc. **47**, No. 293, Amer. Math. Soc., Providence, 1984.

[FS1] Fefferman C. and Stein E.M., *Some maximal inequalities.* Amer. J. Math. **93** (1971), 107–115.

[FS2] Fefferman C. and Stein E.M., *H^p spaces of several variables.* Acta. Math. **129** (1972), 137–193.

[J1] Jones P., *Extension theorems for BMO*. Indiana Univ. Math. J. **29** (1980), 41–66.

[J2] Jones P., *Quasiconformal mappings and extendability of functions in Sobolev spaces.* Acta Math. **147** (1981), 71–88.

[M] Miyachi A., *Extension theorems for the function spaces of DeVore and Sharpley.* Math. Japonica **38** (1993), 1033–1049.

[S] Seeger A., *A note on Triebel-Lizorkin spaces.* Approximation and Function Spaces, Banach Center Publ. 22, PWN-Polish Sci. Publ., Warsaw, 1989, 391–400.

Dept. of Mathematics
Tokyo Woman's Christian University
Zempukuji, Suginami-ku, Tokyo 167-8585
Japan
E-mail: miyachi@twcu.ac.jp