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A note on the Glauberman correspondence of p-blocks of finite p-solvable groups

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Abstract. We show that a p-block B of a p-solvable group and the Glauberman correspondent of B are Morita equivalent.

Key words: finite groups, Glauberman correspondence, modular representations.

Let G and S be finite groups. Let $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ be a p-modular system and assume $\mathcal{K}$ is an algebraically closed field. If S acts on G and $(|G|, |S|) = 1$, then it is well-known that there exists a one-to-one map called the Glauberman-Isaacs correspondence

$$\pi(G, S) : \mathrm{Irr}_S(G) \to \mathrm{Irr}(C_G(S))$$

where $\mathrm{Irr}_S(G)$ is the set of all S-invariant ordinary irreducible characters of G and $\mathrm{Irr}(C_G(S))$ is the set of all ordinary irreducible characters of $C_G(S)$ ([Gl], [I1], [I2, Section 13]). Let $\mathrm{Bl}(G)$ be the set of p-blocks of G. A p-block B of G means a block ideal of $\mathcal{O}G$ or $\mathcal{F}G$. Let $\widetilde{\mathrm{Bl}}_S(G)$ be the set of S-invariant p-blocks B of G such that a defect group of B is centralized by S. By [Wa, Theorem 1] and [H, Theorem 1] the correspondence $\pi(G, S)$ induces a one-to-one map

$$\widetilde{\pi}(G, S) : \widetilde{\mathrm{Bl}}_S(G) \to \mathrm{Bl}(C_G(S)).$$

In fact the character correspondence $\pi(G, S)$ gives the perfect isometry between $B \in \widetilde{\mathrm{Bl}}_S(G)$ and $B^* \in \mathrm{Bl}(C_G(S))$ where $B^* = \widetilde{\pi}(G, S)(B)$. B^* is called the Glauberman-Isaacs correspondent of B. B and B^* have a common defect group and when $|G|$ is odd, B and B^* have the same Cartan matrix ([H, Theorem 1]). Now we are interested in relations between mod B and mod B^* where mod B is the category of finite generated B-modules. Let b and b^* be the Brauer correspondent of B and B^* respectively. Recently Koshitani-Michler [KM] showed that if S is solvable (the Glauberman

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correspondence case), b and b^* are Morita equivalent over $\mathcal{F}$, that is, mod b and mod b^* are equivalent over $\mathcal{F}$ ([KM, Theorem 2.12, Theorem 3.4]). In particular if a defect group of B is a normal subgroup of G, then B and B^* are Morita equivalent. In this paper, we will show that when S is solvable and G is p-solvable, B and B^* are Morita equivalent, by using a lemma in [KM].

Theorem 1 *Let S be a finite solvable group and G a finite p-solvable group. Suppose S acts on G and $(|G|,|S|)=1$. Let D be a p-subgroup of $C_G(S)$. If B is an S-invariant block of $\mathcal{O}G$ with defect group D, then B and the Glauberman correspondent B^* are Morita equivalent.*

We review the Clifford extensions. Let K be a normal subgroup of G and let θ be a G-invariant irreducible character of K. We denote by e_θ the primitive idempotent of the center of $\mathcal{K}K$ which corresponds to θ. Then

$$\mathcal{K}Ge_\theta = \mathcal{K}Ke_\theta C_{\mathcal{K}Ge_\theta}(\mathcal{K}Ke_\theta) \simeq \mathrm{Mat}_{\theta(1)}(\mathcal{K}) \otimes_{\mathcal{K}} \mathcal{K}^{(\alpha)}[G/K]$$

for some $\alpha \in Z^2(G/K,\mathcal{K}^\times)$ where $\mathrm{Mat}_{\theta(1)}(\mathcal{K})$ is the $(\theta(1),\theta(1))$-matrix algebra over $\mathcal{K}$ and $\mathcal{K}^{(\alpha)}[G/K]$ is the twisted group algebra of G/K over $\mathcal{K}$ with a factor set α. We call α a factor set with respect to (G,K,θ). If K is a p'-group, then we can take $\alpha \in Z^2(G/K,\mathcal{O}^\times)$ where $\mathcal{O}^\times$ is the group of units of $\mathcal{O}$ and α also induces a decomposition

$$\mathcal{O}Ge_\theta = \mathcal{O}Ke_\theta C_{\mathcal{O}Ge_\theta}(\mathcal{O}Ke_\theta) \simeq \mathrm{Mat}_{\theta(1)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}^{(\alpha)}[G/K]$$

(see [NT, Chapter V, Theorem 7.2]).

The proof of [KM, Lemma 3.2] says the following fact implicitly. This plays an essential role in the proof of Theorem 1.

Lemma 2 (Dade-Koshitani-Michler ([KM, Lemma 3.2])) *Suppose S is cyclic of prime order. Let K be a normal subgroup of G such that $G = KC_G(S)$ and K is S-invariant. Let $\theta \in \mathrm{Irr}_S(K)$ such that θ is G-invariant and let $\theta^* \in \mathrm{Irr}(C_K(S))$ be the Glauberman correspondent of θ. Let $\alpha \in Z^2(G/K,\mathcal{K}^\times)$ be a factor set with respect to (G,K,θ) and $\alpha^* \in Z^2(C_G(S)/C_K(S),\mathcal{K}^\times)$ a factor set with respect to $(C_G(S),C_K(S),\theta^*)$. Then we have*

$$\alpha B^2(G/K,\mathcal{K}^\times) = \alpha^* B^2(C_G(S)/C_K(S),\mathcal{K}^\times)$$

via the isomorphism $G/K \simeq C_G(S)/C_K(S)$.

Remark θ^* is $C_G(S)$-invariant by [I2, Theorem 13.1 (c)] (cf. [Wo, Lemma 2.5]).

Proof of Theorem 1. Since S is solvable, there exists a composition series

$$S = S_n \rhd S_{n-1} \rhd \cdots \rhd S_1 \rhd S_0 = 1$$

of S such that S_i/S_{i-1} is cyclic of prime order, and then we have

$$\pi(G,S) = \pi(C_G(S_{n-1}), S/S_{n-1}) \circ \cdots \circ \pi(C_G(S_1), S_2/S_1) \circ \pi(G, S_1)$$

by [I2, Theorem 13.1]. Thus we may assume that S is cyclic of prime order.

Now we prove the theorem by induction on $|G|$. Put $K = O_{p'}(G)$. Let Θ be the set of all irreducible characters of K which are covered by B. Since B is S-invariant, S acts on Θ. Moreover G acts on Θ transitively and

$$\begin{aligned}(\theta^g)^s(k) &= \theta^g(k^{s^{-1}}) = \theta((k^{s^{-1}})^{g^{-1}}) = \theta((k^{(g^s)^{-1}})^{s^{-1}}) \\ &= \theta^s(k^{(g^s)^{-1}}) = (\theta^s)^{g^s}(k)\end{aligned}$$

for all $\theta \in \Theta$, $g \in G$, $s \in S$ and $k \in K$. Therefore there exists an S-invariant irreducible character θ of K which is covered by B by a lemma of Glauberman [I2, Lemma 13.8]. Let T be the inertial subgroup of θ in G. Since θ is S-invariant, S stabilizes T. Let $\widetilde{B}$ be the Clifford correspondent of B. Since B is S-invariant, $\widetilde{B}$ is S-invariant. Put $\widetilde{B}^* = \widetilde{\pi}(T,S)(\widetilde{B})$. By [Wo, Lemma 2.5], B^* covers θ^* and $C_T(S)$ is the inertial subgroup of θ^* in $C_G(S)$ and $B^* = \widetilde{\pi}(G,S)(B)$ is the Clifford correspondent of $\widetilde{B}^*$. If $G \gneq T$, by induction, $\widetilde{B}$ and $\widetilde{B}^*$ are Morita equivalent. Since the Clifford correspondence also induces a Morita equivalent, B and B^* are Morita equivalent.

Now we may assume that $G = T$, that is, θ is G-invariant. Then B is a unique block of G which covers θ and D is a Sylow p-subgroup of G by [N, Theorem 10.20]. Since G is p-solvable, there exists a Hall p'-subgroup H of G by [Go, Chapter 6, Theorem 3.5]. Let $\mathfrak{H}$ be the set of all Hall p'-subgroups of G. Then S and G act on $\mathfrak{H}$. Moreover G acts on $\mathfrak{H}$ transitively by [Go, Chapter 6, Theorem 3.6] and there exists an S-invariant Hall p'-subgroup H of G by a lemma of Glauberman [I2, Lemma 13.8]. Since $G = DH$ and $D \leq C_G(S)$, we have $[G,S] = [H,S]$. Since $[g,s]^x = (gx)^{-1}(gx)^s(x^{-1}x^s)^{-1} \in [G,S]$ for all $g, x \in G$ and $s \in S$, $[G,S]$ is a normal subgroup of G. Hence we have $K = O_{p'}(G) \geq [H,S] = [G,S]$ and $G = C_G(S)[G,S] = C_G(S)K$ by [I1, p.629].

Let $\alpha \in Z^2(G/K, \mathcal{O}^\times)$ be a factor set with respect to (G, K, θ) and let $\alpha^* \in Z^2(C_G(S)/C_K(S), \mathcal{O}^\times)$ be a factor set with respect to $(C_G(S), C_K(S), \theta^*)$. By Lemma 2 we have

$$\alpha B^2(G/K, \mathcal{K}^\times) = \alpha^* B^2(C_G(S)/C_K(S), \mathcal{K}^\times),$$

that is, there exists a 1-cochain $\gamma \in C^1(G/K, \mathcal{K}^\times)$ such that

$$\alpha(\overline{g_1}, \overline{g_2})\alpha^*(\overline{g_1}, \overline{g_2})^{-1} = \gamma(\overline{g_1})\gamma(\overline{g_2})\gamma(\overline{g_1 g_2})^{-1}$$

for all $\overline{g_1}, \overline{g_2} \in G/K$. Then we have

$$\gamma(\overline{g})^{|G/K|} = \prod_{\overline{x} \in G/K} \alpha(\overline{g}, \overline{x})\alpha^*(\overline{g}, \overline{x})^{-1} \in \mathcal{O}^\times$$

and hence $\gamma(\overline{g}) \in \mathcal{O}^\times$ for all $\overline{g} \in G/K$. Therefore we have

$$\alpha B^2(G/K, \mathcal{O}^\times) = \alpha^* B^2(C_G(S)/C_K(S), \mathcal{O}^\times).$$

In particular $\mathcal{O}^{(\alpha)}[G/K]$ and $\mathcal{O}^{(\alpha)}[C_G(S)/C_K(S)]$ are isomorphic. Since θ is covered by the unique block B, we have

$$B \simeq \mathrm{Mat}_{\theta(1)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}^{(\alpha)}[G/K]$$

and hence we have also

$$B^* \simeq \mathrm{Mat}_{\theta^*(1)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}^{(\alpha^*)}[C_G(S)/C_K(S)].$$

Hence B and B^* are Morita equivalent. □

The case where D is abelian in Theorem 1 is obtained in [KM, Corollary 3.5].

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