

The joint approximate point spectrum of an operator

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Abstract. A new proof of a result, due to Xia, concerning the joint approximate point spectrum of an operator is given. This result is then applied to obtain certain spectral properties for operators, such as p -hyponormal and log-hyponormal operators, which have the identical approximate and joint approximate point spectra.

Key words: approximate and joint approximate point spectra, p - and log-hyponormal operators, invariant subspace.

1. Introduction

Let T be a bounded linear operator on a Hilbert space H . A complex number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum $\sigma_a(T)$ of the operator T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \rightarrow 0$. If in addition, $(T^* - \bar{\lambda})x_n \rightarrow 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T . The boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of the operator T is known to be a subset of $\sigma_a(T)$. Although, in general, one has $\sigma_{ja}(T) \subset \sigma_a(T)$, there are many classes of operators T for which

$$\sigma_{ja}(T) = \sigma_a(T). \quad (1)$$

For example, if T is either normal or hyponormal, then T satisfies (1). More generally, (1) holds if T is semi-hyponormal [15], p -hyponormal [7] or log-hyponormal [14], [4, Corollary 4.5]. In [10], Duggal introduced a class $K(p)$ of operators which contains the p -hyponormal operators and showed [10, Theorem 4] that operators T in the class $K(p)$ satisfy (1).

In this paper we give a new proof of a result, due to Xia [15], concerning the joint approximate point spectrum of an operator. The result is then applied to obtain certain spectral properties for operators T for which (1) is satisfied. All operators considered in this paper are assumed to be bounded linear operators on the Hilbert space H . This paper may be considered, for

the cases of p -hyponormal and log-hyponormal operators, as a continuation of the paper [5].

2. The Joint Approximate Point Spectrum

In this section we give a new proof of a result, due to Xia [15], on the joint approximate point spectrum of an operator. The result implies that if $T = U|T|$ is the polar decomposition of the operator T and $\lambda \in \sigma_{ja}(T)$ with joint “approximate eigenvectors” $\{x_n\}$, then $|\lambda| \in \sigma(|T|) \cap \sigma(|T^*|)$ with approximate eigenvectors $\{x_n\}$. If in addition, $\lambda = |\lambda|e^{i\theta} \neq 0$, then $e^{i\theta} \in \sigma_{ja}(U)$ with joint approximate eigenvectors $\{x_n\}$. Applications of this result will be given in the remaining sections.

Theorem 1 *Let $T = U|T|$ be the polar decomposition of the operator T , $\lambda \in \mathbb{C}$, and $\{x_n\}$ be a sequence of vectors. If $(T - \lambda)x_n \rightarrow 0$ and $(T^* - \bar{\lambda})x_n \rightarrow 0$, then $(|T| - |\lambda|x_n) \rightarrow 0$ and $(|T^*| - |\lambda|x_n) \rightarrow 0$. If in addition, $\lambda = |\lambda|e^{i\theta} \neq 0$, then $(U - e^{i\theta})x_n \rightarrow 0$ and $(U^* - e^{-i\theta})x_n \rightarrow 0$.*

Proof. Since $\|Tx\| = \||T|x\|$ for any vector x , $Tx_n \rightarrow 0$ if and only if $|T|x_n \rightarrow 0$. Similarly, $T^*x_n \rightarrow 0$ if and only if $|T^*|x_n \rightarrow 0$. This proves the theorem for the case $\lambda = 0$. Now, assume $\lambda = |\lambda|e^{i\theta} \neq 0$. This implies the positive operators $|T| + |\lambda|$ and $|T^*| + |\lambda|$ are invertible. The first result follows since

$$(|T| + |\lambda|)(|T| - |\lambda|) = T^*(T - \lambda) + \lambda(T^* - \bar{\lambda}) \tag{2}$$

and

$$(|T^*| + |\lambda|)(|T^*| - |\lambda|) = T(T^* - \bar{\lambda}) + \bar{\lambda}(T - \lambda). \tag{3}$$

The second result follows since

$$|\lambda|(U - e^{i\theta}) = (T - \lambda) - U(|T| - |\lambda|) \tag{4}$$

and

$$|\lambda|(U^* - e^{-i\theta}) = (T^* - \bar{\lambda}) - U^*(|T^*| - |\lambda|). \tag{5}$$

The proof is complete. □

Although the proof of Theorem 1 is entirely elementary, except in [5], equations (2)–(5) do not seem to have been observed. Corollary 2 below was previously established by Chō [6, Lemma 3] under the added assump-

tion that U is unitary. More recently, Duggal [10, Theorem 4], employing the more elaborate Berberian's extension technique, proved the equivalence between parts (a) and (b) of Corollary 2.

Corollary 2 *Let $T = U|T|$ be the polar decomposition of the operator T , $\lambda = |\lambda|e^{i\theta} \neq 0$, and $\{x_n\}$ be a sequence of vectors. The following assertions are equivalent.*

- (a) $(T - \lambda)x_n \rightarrow 0$ and $(T^* - \bar{\lambda})x_n \rightarrow 0$.
- (b) $(|T| - |\lambda|)x_n \rightarrow 0$ and $(U - e^{i\theta})x_n \rightarrow 0$.
- (c) $(|T^*| - |\lambda|)x_n \rightarrow 0$ and $(U^* - e^{-i\theta})x_n \rightarrow 0$.

Proof. Both (b) and (c) follow from (a) by Theorem 1. That (b) implies (a) follows from equations (4) and (2) in the proof of Theorem 1. Similarly, (c) implies (a) follows from equations (5) and (3). \square

3. Operators Having Identical Approximate and Joint Approximate Point Spectra

In this section we apply Theorem 1 to obtain several spectral properties for operators whose approximate and joint approximate point spectra are identical. In particular, the results obtained here apply to p -hyponormal and log-hyponormal operators, and to operators in the class $K(p)$.

Theorem 3 below is a direct consequence of Theorem 1. We therefore omit its proof. With the added assumption that the polar factor U is unitary, part (a) of Theorem 3 was shown to hold for hyponormal operators by Putnam [12, Theorem 1], and for p -hyponormal operators by Chō, Huruya and Itoh [8, Theorem 2]. On the other hand, Duggal [10, Theorem 4] obtained part (a) in its generality for operators in the class $K(p)$.

Theorem 3 *Let $T = U|T|$ be the polar decomposition of the operator T with $\sigma_{ja}(T) = \sigma_a(T)$.*

- (a) *If $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma(|T|) \cap \sigma(|T^*|)$. In particular, if $\lambda \in \partial\sigma(T)$, then $|\lambda| \in \sigma(|T|) \cap \sigma(|T^*|)$.*
- (b) *If $\lambda = |\lambda|e^{i\theta} \neq 0$ is such that $\lambda \in \sigma_a(T)$, then $e^{i\theta} \in \sigma_{ja}(U)$.*

For hyponormal operators $T = U|T|$ with unitary polar factor U , Putnam [12, Theorem 3] proved that if $z = |z|e^{i\theta} \neq 0$ is such that $z \in \sigma(T)$, then $e^{i\theta} \in \sigma(U)$. Our next application is to generalize Putnam's result.

Theorem 4 Let $T = U|T|$ be the polar decomposition of the operator T with $\sigma_{ja}(T) = \sigma_a(T)$. If $z = |z|e^{i\theta} \neq 0$ is such that $z \in \sigma(T)$, then $e^{i\theta} \in \sigma_{ja}(U)$.

Proof. Let $\lambda = re^{i\theta}$, where $r = \sup\{|w| : |w|e^{i\theta} \in \sigma(T)\}$. Then $\lambda \in \partial\sigma(T) \subset \sigma_a(T)$. The result follows from Theorem 3. \square

For $r > 0$, let

$$C_r = \{z : |z| = r\}$$

be the circle in \mathbb{C} with center 0 and radius r .

Corollary 5 Let $T = U|T|$ be the polar decomposition of the operator T with $\sigma_{ja}(T) = \sigma_a(T)$. If $\sigma(U)$ does not contain the unit circle C_1 , then $\sigma(T)$ does not contain the circle C_r for any $r > 0$.

Proof. The assumption implies there is a $\theta \in \mathbb{R}$ for which $e^{i\theta} \notin \sigma(U)$. Therefore, $e^{i\theta} \notin \sigma_{ja}(U)$ and the result follows from Theorem 4. \square

The idea of extending from a point $z \in \sigma(T)$ radially to a point $\lambda \in \sigma_a(T)$ in Theorem 4 is due to Putnam [12]. In the next application, extending from a point $z \in \sigma(T)$ circularly to a point $\lambda \in \sigma_a(T)$ will be employed. For that purpose, let

$$A_r(\alpha, \beta) = \{re^{i\theta} : \alpha \leq \theta \leq \beta\}$$

be the arc on the circle C_r with endpoints $re^{i\alpha}$ and $re^{i\beta}$.

Let $T = U|T|$ be a p -hyponormal operator. Does it follow that $z \in \sigma(T)$ implies $|z| \in \sigma(|T|)$? Putnam [12] gave an example which shows that even if the polar factor U is unitary, the answer to the question is in the negative. The next application shows that if $z \neq 0$ and $\sigma(T)$ does not contain the circle $C_{|z|}$ then the answer to the question becomes affirmative.

Theorem 6 Let $T = U|T|$ be an operator with $\sigma_{ja}(T) = \sigma_a(T)$, and let $z = |z|e^{i\alpha} \neq 0$ be such that $z \in \sigma(T)$. If $\sigma(T)$ does not contain the circle $C_{|z|}$, then $|z| \in \sigma(|T|)$ and $C_{|z|} \cap \sigma_a(T) \neq \emptyset$.

Proof. Let $r = |z|$ and let $\lambda = re^{i\beta}$, where $\beta = \sup\{\gamma : A_r(\alpha, \gamma) \subset \sigma(T)\}$. The assumption that $\sigma(T)$ does not contain the circle C_r implies $\lambda \in \partial\sigma(T) \subset \sigma_a(T)$, and the result follows from Theorem 3. \square

Corollary 7 *Let $T = U|T|$ be an operator with $\sigma_{ja}(T) = \sigma_a(T)$, and let $z = |z|e^{i\alpha} \neq 0$ be such that $z \in \sigma(T)$. If either $\sigma(U)$ does not contain the unit circle C_1 or there is a $\theta \in \mathbb{R}$ for which $\{re^{i\theta} : r > 0\} \cap \sigma(T) = \emptyset$, then $|z| \in \sigma(|T|)$ and $C_{|z|} \cap \sigma_a(T) \neq \emptyset$.*

Proof. The assumption that $\{re^{i\theta} : r > 0\} \cap \sigma(T) = \emptyset$ implies that $\sigma(T)$ does not contain the circle C_r for any $r > 0$. By Corollary 5, the same implication follows if $\sigma(U)$ does not contain the unit circle C_1 . The result follows from Theorem 6. \square

4. p -Hyponormal and log-Hyponormal Operators

An operator T is said to be p -hyponormal, $p > 0$, if $(T^*T)^p \geq (TT^*)^p$. A p -hyponormal operator is called hyponormal if $p = 1$, semi-hyponormal if $p = 1/2$. It is a consequence of the Löwner-Heinz inequality that if T is p -hyponormal, then it is q -hyponormal for any $0 < q \leq p$. An invertible operator T is said to be log-hyponormal if $\log(T^*T) \geq \log(TT^*)$. Log-hyponormal operators were independently introduced in [3] and [13]. It was shown in [13], through an example, that neither the class of p -hyponormal operators nor the class of log-hyponormal contains the other.

Let $T = U|T|$ be the polar decomposition of the operator T . Following [1] (see also [2]) we define $\tilde{T} = |T|^{1/2}U|T|^{1/2}$. The operator \tilde{T} plays an important role in the study of spectral properties of the p - or log-hyponormal operator T . We need two spectral properties concerning T and \tilde{T} which serve to motivate our further development of this section. The first property is well-known. It shows that the spectra of T and \tilde{T} are identical. The second, which does not seem to have been recognized before, shows that the approximate point spectra of T and \tilde{T} are identical as well.

Lemma 8 *For an operator $T = U|T|$, $\sigma(T) = \sigma(\tilde{T})$.*

Lemma 9 *For an operator $T = U|T|$, $\sigma_a(T) = \sigma_a(\tilde{T})$.*

Proof. Let $\lambda \neq 0$ be such that $(T - \lambda)x_n \rightarrow 0$ for some sequence $\{x_n\}$ of unit vectors. Since $\|(T - \lambda)x_n\| \geq \|\|Tx_n\| - |\lambda|\|$, we may assume the sequence $\{\|Tx_n\|\} = \{\||T|x_n\|\}$ is bounded away from 0. Since $\||T|x_n\| \leq \||T|^{1/2}\| \||T|^{1/2}x_n\|$, the sequence $\{\||T|^{1/2}x_n\|\}$ is bounded away from 0. Thus,

$$(\tilde{T} - \lambda)|T|^{1/2}x_n = |T|^{1/2}(T - \lambda)x_n \rightarrow 0.$$

If $Tx_n \rightarrow 0$, then $|T|x_n \rightarrow 0$. Consequently, $|T|^{1/2}x_n \rightarrow 0$. Thus, $\tilde{T}x_n \rightarrow 0$. Therefore, $\sigma_a(T) \subset \sigma_a(\tilde{T})$. Now, let $\lambda \neq 0$ be such that $(\tilde{T} - \lambda)x_n \rightarrow 0$ for some sequence $\{x_n\}$ of unit vectors. Again, we may assume the sequence $\{\|\tilde{T}x_n\|\}$ is bounded away from 0. Consequently, the sequence $\{\|U|T|^{1/2}x_n\|\}$ is bounded away from 0. Thus,

$$(T - \lambda)U|T|^{1/2}x_n = U|T|^{1/2}(\tilde{T} - \lambda)x_n \rightarrow 0.$$

Suppose $\tilde{T}x_n = |T|^{1/2}U|T|^{1/2}x_n \rightarrow 0$. There are two cases to consider. Case 1, the sequence $\{\|U|T|^{1/2}x_n\|\}$ is bounded away from 0. In this case we have

$$T(U|T|^{1/2}x_n) = U|T|^{1/2}\tilde{T}x_n \rightarrow 0.$$

Case 2, the sequence $\{\|U|T|^{1/2}x_n\|\}$ is not bounded away from 0. Passing to a subsequence if necessary, we may assume $U|T|^{1/2}x_n \rightarrow 0$. Consequently, $U^*U|T|^{1/2}x_n = |T|^{1/2}x_n \rightarrow 0$, and hence $Tx_n \rightarrow 0$. Therefore, $\sigma_a(\tilde{T}) \subset \sigma_a(T)$. \square

If T is p -hyponormal, does it follow that $|T|$ and $|\tilde{T}|$ have identical spectra? Lemmas 8 and 9 notwithstanding, the answer to the question is in the negative [11]. In the remainder of this section, we give conditions under which $|T|$ and $|\tilde{T}|$ will have identical spectra.

Lemma 10 ([3]) *If $T = U|T|$ is either p -hyponormal or log-hyponormal, then $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. Consequently, \tilde{T} is semi-hyponormal.*

Theorem 11 ([9, 12, 14, 15]) *If $T = U|T|$ is either p -hyponormal or log-hyponormal, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \rightarrow \mathbb{R}$ is defined by $\rho(z) = |z|$.*

The above theorem was proven for hyponormal operators by Putnam [12, Theorem 7], for semi-hyponormal operators by Xia [15], and for p -hyponormal operators by Chō and Itoh [9, Theorem 4]. As for log-hyponormal operators, Tanahashi observed in the proof of [14, Theorem 7] that if $T = U|T|$ is log-hyponormal, then, replacing T by cT for sufficiently large $c > 0$ if necessary, one may assume that $\log|T|$ is both positive and invertible. Consequently, there is a semi-hyponormal operator S for which $|S| = \log|T|$ and $T = Ue^{|S|}$. It is then seen that the above theorem, in the case of log-hyponormal T , follows from Tanahashi's [14, Lemma 6].

Lemma 12 *Let $T = U|T|$ be an operator which satisfies $\sigma_{ja}(T) = \sigma_a(T)$ and $\sigma(|T|) \subset \rho(\sigma(T))$. If $\sigma(T)$ does not contain the circle C_r for any $r > 0$, then $\rho(\sigma_a(T)) = \sigma(|T|)$.*

Proof. Let $r \in \sigma(|T|)$ be such that $r \neq 0$. The assumption implies there is a $z \in \sigma(T)$ for which $|z| = r$. Since $\sigma(T)$ does not contain $C_{|z|}$, there is a $\lambda \in \partial\sigma(T) \subset \sigma_a(T)$ for which $|\lambda| = r$ by Theorem 6. If $0 \in \sigma(|T|)$, then there is a sequence $\{x_n\}$ of unit vectors for which $\|Tx_n\| = \||T|x_n\| \rightarrow 0$. Therefore, $\sigma(|T|) \subset \rho(\sigma_a(T))$. On the other hand, $\rho(\sigma_a(T)) \subset \sigma(|T|)$ follows from Theorem 3. \square

Corollary 13 *Let $T = U|T|$ be an operator which satisfies $\sigma_{ja}(T) = \sigma_a(T)$ and $\sigma(|T|) \subset \rho(\sigma(T))$. If either $\sigma(U)$ does not contain the unit circle C_1 , or there is a $\theta \in \mathbb{R}$ for which $\{re^{i\theta} : r > 0\} \cap \sigma(T) = \emptyset$, then $\rho(\sigma_a(T)) = \sigma(|T|)$.*

Proof. Either the assumption on $\sigma(U)$ or the assumption on $\sigma(T)$ implies that $\sigma(T)$ does not contain the circle C_r for any $r > 0$. The result follows from Lemma 12. \square

Theorem 14 *Let $T = U|T|$ be either p -hyponormal or log-hyponormal. If either*

- (a) $\sigma(T)$ does not contain the circle C_r for any $r > 0$,
 - (b) $\sigma(U)$ does not contain the unit circle C_1 , or
 - (c) there is a $\theta \in \mathbb{R}$ for which $\{re^{i\theta} : r > 0\} \cap \sigma(T) = \emptyset$,
- then $\rho(\sigma_a(T)) = \sigma(|T|)$.

Theorem 15 *Let $T = U|T|$ be either p -hyponormal or log-hyponormal. If either*

- (a) $\sigma(T)$ does not contain the circle C_r for any $r > 0$,
- (b) $\sigma(U)$ does not contain the unit circle C_1 , or
- (c) there is a $\theta \in \mathbb{R}$ for which $\{re^{i\theta} : r > 0\} \cap \sigma(T) = \emptyset$,

then $\sigma(|T|) = \sigma(|\tilde{T}|)$, and for each $r \in \sigma(|T|)$ there is a sequence $\{x_n\}$ of unit vectors for which $(|T| - r)x_n \rightarrow 0$ and $(|\tilde{T}| - r)x_n \rightarrow 0$.

Proof. Theorem 14, Lemma 9 and Theorem 3 imply

$$\sigma(|T|) = \rho(\sigma_a(T)) = \rho(\sigma_a(\tilde{T})) \subset \sigma(|\tilde{T}|). \tag{6}$$

Since $|\tilde{T}| \geq |T|$, it follows that if $0 \in \sigma(|\tilde{T}|)$ and $\{x_n\}$ is a sequence of unit vectors for which $|\tilde{T}|x_n \rightarrow 0$, then $0 \in \sigma(|T|)$ and $|T|x_n \rightarrow 0$. Let $r \neq 0$ be

such that $r \in \sigma(|\tilde{T}|)$. Theorem 11 and Lemma 8 imply there is a $z \in \sigma(T)$ for which $|z| = r$. Since $\sigma(T)$ does not contain the circle C_r , it follows from Theorem 6 and Lemma 9 that there is a $\lambda \in \sigma_a(T)$ for which $|\lambda| = r$. Consequently, Theorem 3 implies $r \in \sigma(|T|)$, and hence $\sigma(|\tilde{T}|) \subset \sigma(|T|)$. Thus, $\sigma(|\tilde{T}|) = \sigma(|T|)$. Again, it follows from (6) that there is a $\lambda = re^{i\theta} \in \sigma_a(\tilde{T})$ and a sequence $\{x_n\}$ of unit vectors for which $(\tilde{T} - \lambda)x_n \rightarrow 0$. Then, $(\tilde{T}^* - \bar{\lambda})x_n \rightarrow 0$ by [7] and [14]. It follows from Theorem 1 that $(|\tilde{T}| - r)x_n \rightarrow 0$ and $(|\tilde{T}^*| - r)x_n \rightarrow 0$. Since

$$|\tilde{T}| - r \geq |T| - r \geq |\tilde{T}^*| - r,$$

we have

$$(|\tilde{T}| - r) - (|\tilde{T}^*| - r) \geq |T| - |\tilde{T}^*| \geq 0.$$

Therefore, $(|T| - |\tilde{T}^*|)x_n \rightarrow 0$, and consequently,

$$(|T| - r)x_n = (|T| - |\tilde{T}^*|)x_n + (|\tilde{T}^*| - r)x_n \rightarrow 0.$$

The proof is complete. □

5. Invariant Subspace

In this final section we consider conditions which are sufficient for an operator T satisfying $\sigma_a(T) = \sigma_{ja}(T)$ to possess a nontrivial invariant subspace. A complex number λ is in the compression spectrum $\sigma_c(T)$ of an operator T if the range of $T - \lambda$ is not dense in H . It is known that $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$ for any operator T . Moreover, if $\lambda \in \sigma_c(T)$ and $T \neq \lambda$, then it is readily verified that the closure of the range of $T - \lambda$ is a nontrivial invariant subspace of T .

Theorem 16 *Let $T = U|T|$ be an operator with $\sigma_a(T) = \sigma_{ja}(T)$. If there is a $\lambda \in \sigma(T)$ for which $|\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|)$, then T has a nontrivial invariant subspace.*

Proof. By Theorem 3, $\lambda \notin \sigma_a(T)$. Therefore, $\lambda \in \sigma_c(T)$, and hence T has a nontrivial invariant subspace. □

Theorem 17 *Let $T = U|T|$ be either p -hyponormal or log-hyponormal. If $\sigma(|T|) \neq \sigma(|\tilde{T}|)$, then T has a nontrivial invariant subspace.*

Proof. Let $r \in \sigma(|T|)$. Theorem 11 and Lemma 8 imply there is a $z \in$

$\sigma(T) = \sigma(\tilde{T})$ for which $|z| = r$. If $r \notin \sigma(|\tilde{T}|)$, then $z \notin \sigma_a(\tilde{T}) = \sigma_a(T)$ by Theorem 3 and Lemma 9. Consequently, $z \in \sigma_c(T)$, and hence T has a nontrivial invariant subspace. Similarly, if there is an $r \in \sigma(|\tilde{T}|)$ and $r \notin \sigma(|T|)$, then T has a nontrivial invariant subspace. \square

Theorem 18 *Let $T = U|T|$ be an operator satisfying $\sigma_a(T) = \sigma_{ja}(T)$ and $\sigma(|T|) \subset \rho(\sigma(T))$. If $\sigma(|T|)$ is not connected, then T has a nontrivial invariant subspace.*

Proof. Without loss of generality, we may assume $\sigma(T) = \sigma_a(T)$. This assumption and Theorem 3 imply $\sigma(|T|) = \rho(\sigma(T))$. If $\sigma(|T|)$ is not connected, then $\sigma(T)$ is not connected, and hence T has a nontrivial invariant subspace. \square

Corollary 19 *Let $T = U|T|$ be either p -hyponormal or log-hyponormal. If $\sigma(|T|)$ is not connected, then T has a nontrivial invariant subspace.*

The above corollary was proven for hyponormal operators by Putnam [12, Theorem 10], and for p -hyponormal operators by Chō, Huruya and Itoh [8, Theorem 4]. In both cases they proved the corollary under the assumption that $\sigma(|T^*|)$ is not connected. It is easy to verify that the reduction $\sigma(T) = \sigma_a(T)$ in the proof of Theorem 18 assures that $\sigma(|T|) = \sigma(|T^*|)$.

The proof of Corollary 19 relies on Theorem 11. It is not known whether Theorem 11 holds for operators T in Duggal's class $K(p)$. Thus, we prove a generalization of Corollary 19 which does apply to the class $K(p)$ as well. This generalization is based on the fact that both p -hyponormal and log-hyponormal operators are normaloid. Recall that an operator T is normaloid if $\|T\| = r(T)$, the spectral radius of T . Operators in $K(p)$ are normaloid as shown in [10].

Theorem 20 *Let $T = U|T|$ be a normaloid operator which satisfies $\sigma_a(T) = \sigma_{ja}(T)$ and $\inf\{|z| : z \in \sigma(T)\} \leq \inf\{r : r \in \sigma(|T|)\}$. If $\sigma(|T|)$ is not connected, then T has a nontrivial invariant subspace.*

Proof. Again, we may assume $\sigma(T) = \sigma_a(T)$. Let $a = \inf\{|z| : z \in \sigma(T)\}$ and $b = \|T\| = \||T|\|$. Since T is normaloid, there is a $z_2 \in \sigma(T)$ such that $|z_2| = b$. Let $z_1 \in \sigma(T)$ be such that $|z_1| = a$. Since $\sigma(|T|)$ is not connected, there is an $r \notin \sigma(|T|)$ such that $a < r < b$. Theorem 3 implies $C_r \cap \sigma(T) = \emptyset$. Therefore, $\sigma(T)$ is not connected, whence T has a nontrivial

invariant subspace. □

Theorem 20 is a generalization of Corollary 19. Indeed, in view of Theorem 11, the inequality in Theorem 20 clearly holds for p -hyponormal and log-hyponormal operators.

Corollary 21 *Let $T = U|T|$ be a noninvertible normaloid operator with $\sigma_a(T) = \sigma_{ja}(T)$. If $\sigma(|T|)$ is not connected, then T has a nontrivial invariant subspace.*

Proof. Since $0 \in \sigma(T)$, the inequality in Theorem 20 automatically holds for T . □

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