Hokkaido Mathematical Journal Vol. 39 (2010) p. 389–403

The unique ergodicity of equicontinuous laminations

Shigenori MATSUMOTO

(Received December 14, 2009; Revised May 24, 2010)

Abstract. We prove that a transversely equicontinuous minimal lamination on a locally compact metric space Z has a transversely invariant nontrivial Radon measure. Moreover if the space Z is compact, then the tranversely invariant Radon measure is shown to be unique up to a scaling.

Key words: lamination, foliation, transversely invariant measure, unique ergodicity.

1. Introduction

Let $\mathcal{L}$ be a p-dimensional lamination on a locally compact metric space Z. Let X be a transversal of $\mathcal{L}$. See Notes A for these fundamental concepts. *We assume throughout that $\mathcal{L}$ is minimal, i.e. all the leaves of $\mathcal{L}$ are dense in Z.* Notice that any leaf of $\mathcal{L}$ intersects X.

Given a leafwise curve c joining two points x and y on X, a holonomy map along c is defined as usual to be a local homeomorphism γ from an open neighbourhood $\mathrm{Dom}(\gamma)$ of x onto an open neighbourhood $\mathrm{Range}(\gamma)$ of y. We say that $\mathcal{L}$ is *transversely equicontinuous with respect to a transversal X* if the family of all the corresponding holonomy maps is equicontinuous.

Theorem 1.1 *Let $\mathcal{L}$ be a minimal lamination on a locally compact metric space Z, transversely equicontinuous with respect to a transversal X. Then there is a nontrivial Radon measure on X which is left invariant by any holonomy map. If further Z is compact, then the invariant measure is unique up to a scaling.*

The existence of invariant measure was already shown by R. Sackesteder in [S] for a pseudogroup acting on a compact metric space. But the compactness condition for a transversal is too strong to obtain a corresponding result for laminations or foliations in general (even on compact spaces or

---

2000 Mathematics Subject Classification : Primary 53C12; secondary 37C85.

The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540096.

manifolds). See Notes B.

In Section 2 we therefore include a slightly general theorem applicable to laminations; the proof closely follows an argument in Lemme 4.4 in [DKN], which is meant for codimension one foliations. In Section 3 we show the uniqueness for a compact lamination.

The author expresses his hearty gratitude to the referees, whose valuable comments are indeed helpful for the improvement of the presentation of the paper.

2. The existence

Let Y be a Hausdorff space. By a local homeomorphism, we mean a homeomorphism γ from an open subset $\mathrm{Dom}(\gamma)$ of Y onto an open subset $\mathrm{Range}(\gamma)$. A set Γ of local homeomophisms of Y is called a *pseudo*group*, if it satisfies the following conditions.

(1) If $\gamma \in \Gamma$ and U is an open subset of $\mathrm{Dom}(\gamma)$, then the restriction $\gamma|_U$ is in Γ.
(2) The identity id_Y belongs to Γ.
(3) If $\gamma, \gamma' \in \Gamma$ and $\mathrm{Dom}(\gamma') = \mathrm{Range}(\gamma)$, then the composite $\gamma' \circ \gamma$ is in Γ.
(4) If $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$.

The set Γ is called a *pseudogroup* if it satisfies in addition the following condition.

(5) If $\gamma : U \to V$ is a local homeomorphism of X, and if for any point $x \in U$ there is a neighbouhood U_x of x in U such that $\gamma|_{U_x} \in \Gamma$, then $\gamma \in \Gamma$.

Thus for example the set of all the holonomy maps with respect to a transversal given in section 1 forms a pseudo*group while the pseudogroup they generate might be bigger, and even if the former is equicontinuous, the latter is not equicontinuous in general. See Notes C. Also by the same reason some part of the argument in Section 3 fails in the pseudogroup setting. These force us to adopt the framework of pseudo*groups, and in fact this has already been done in the literature e.g. Definition 2.1.3 in [HH], as is pointed out by a referee.

Let X be a locally compact metric space and Γ a pseudo*group of local homeomorphsims of X. We assume that the action is minimal, i.e. the Γ-orbit of any point is dense in X, and that the action is equicontinuous, i.e. for any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if $\gamma \in \Gamma$, $x, x' \in \mathrm{Dom}(\gamma)$ and $d(x, x') < \delta(\varepsilon)$, then we have $d(\gamma x, \gamma x') \le \varepsilon$.

Denote by $C_c(X)$ the space of real valued continuous functions ζ whose support $\mathrm{supp}\zeta$ is compact. A Radon measure μ on X is called Γ-invariant if whenever $\zeta \in C_c(X)$ and $\gamma \in \Gamma$ satisfy $\mathrm{supp}\zeta \subset \mathrm{Dom}(\gamma)$, we have $\mu(\zeta \circ \gamma^{-1}) = \mu(\zeta)$ for the composite $\zeta \circ \gamma^{-1} \in C_c(X)$. In fact if μ is Γ-invariant, we get a bit more: if $\zeta \in C_c(X)$ vanishes outside $\mathrm{Dom}(\gamma)$, we have $\mu(\zeta \circ \gamma^{-1}) = \mu(\zeta)$, as the dominated convergence theorem shows. This will be used in the proof of Lemma 3.8.

Let X_0 be a relatively compact open subset of X, and denote by Γ_0 the restriction of Γ to X_0 i.e.

$$\Gamma_0 = \{\gamma \in \Gamma \mid \mathrm{Dom}(\gamma) \cup \mathrm{Range}(\gamma) \subset X_0\}.$$

The purpose of this section is to show the following theorem.

Theorem 2.1 *There exists a nontrivial finite Γ_0-invariant Radon measure μ on X_0.*

The minimality assumption shows then the existence of Γ-invariant measure on X and the proof of the existence part of Theorem 1.1 will be complete. Also if $\hat{\Gamma}$ denotes the pseudogroup that Γ generates, then the measure is invariant by $\hat{\Gamma}$. In fact, if $\hat{\Gamma}'$ denotes the set of all the *measure preserving* local homeomorphisms, then $\hat{\Gamma}'$ contains Γ and satisfies the conditions (1)$\sim$(5) above. Therefore $\hat{\Gamma}$, being the minimal set containing Γ and satisfying (1)$\sim$(5), is contained in $\hat{\Gamma}'$.

Let us define

$$C_c(X)_{\ge 0} = \{\zeta \in C_c(X) \mid \zeta \ge 0\} \quad \text{and}$$
$$C_c(X)_{>0} = \{\zeta \in C_c(X)_{\ge 0} \mid \zeta(x) > 0,\ \exists x \in X\}.$$

For any $\psi \in C_c(X)$ and $\gamma \in \Gamma$, extend the function $\psi \circ \gamma^{-1}$ to the whole X so as to vanish outside $\mathrm{Range}(\gamma)$ and still denote it by $\psi \circ \gamma^{-1}$. It may no longer be continuous. For any $\zeta \in C_c(X)_{\ge 0}$ and $\psi \in C_c(X)_{>0}$, define $(\zeta : \psi)$ by

$$(\zeta : \psi) = \inf\left\{ \sum_{i=1}^{n} c_i \,\middle|\, \zeta \le \sum_{i=1}^{n} c_i \psi \circ \gamma_i^{-1},\ c_i > 0,\ \gamma_i \in \Gamma,\ n \in \mathbb{N} \right\}.$$

Notice that the minimality of Γ implies that $(\zeta : \psi) < \infty$ and $(\zeta : \psi) = 0$ if and only if $\zeta = 0$.

Fix once and for all a function $\chi \in C_c(X)_{>0}$ such that $\chi = 1$ on X_0, and define a map $L_\psi : C_c(X)_{\ge 0} \to \mathbb{R}$ by

$$L_\psi(\zeta) = (\zeta : \psi)/(\chi : \psi).$$

It is routine to show the following properties of L_ψ.

$$L_\psi(c\zeta) = cL_\psi(\zeta),\ \forall c \ge 0, \tag{2.1}$$

$$L_\psi(\zeta_1 + \zeta_2) \le L_\psi(\zeta_1) + L_\psi(\zeta_2), \tag{2.2}$$

$$\zeta_1 \le \zeta_2 \Rightarrow L_\psi(\zeta_1) \le L_\psi(\zeta_2), \tag{2.3}$$

$$\mathrm{supp}\zeta \subset \mathrm{Dom}(\gamma) \Rightarrow L_\psi(\zeta \circ \gamma^{-1}) = L_\psi(\zeta), \tag{2.4}$$

$$\zeta \in C_c(X)_{>0} \Rightarrow L_\psi(\zeta) \ge 1/(\chi : \zeta). \tag{2.5}$$

The following lemma plays a key role in this section. As pointed out by a referee, a simplified version of it can be found as Lemma 7.4.1 (p. 189) in [SK].

Lemma 2.2 *If $\eta > 0$ and $\xi, \xi' \in C_c(X)_{\ge 0}$ satisfy $\xi + \xi' = \chi$, then there is $\delta > 0$ such that if $\psi \in C_c(X)_{>0}$,* $\mathrm{diam}(\mathrm{supp}\psi) < \delta$ *and $\zeta \in C_c(X_0)_{\ge 0}$ we have*

$$L_\psi(\xi\zeta) + L_\psi(\xi'\zeta) \le (1 + 2\eta)L_\psi(\zeta).$$

Proof. Given $\eta > 0$, there is $\varepsilon > 0$ such that if $x, x' \in X_0$ and $d(x, x') \le \varepsilon$, then $|\xi(x) - \xi(x')| \le \eta$. Also this implies $|\xi'(x) - \xi'(x')| \le \eta$. Choose $\delta = \delta(\epsilon)$, the modulus of the equicontinuity. Let ψ be as in the lemma and assume

$$\zeta \le \sum_i c_i \psi \circ \gamma_i^{-1}. \tag{2.6}$$

Notice that if we restrict γ_i in (2.6) to $\mathrm{Dom}(\gamma_i) \cap \mathrm{supp}\psi \cap \gamma_i^{-1}(\mathrm{supp}\zeta)$, still the inequality (2.6) holds. Hence if we choose x_i from $\mathrm{Range}(\gamma_i) \subset \mathrm{supp}(\zeta) \subset X_0$, then for any $x \in \mathrm{Range}(\gamma_i)$, we have

$$|\xi(x) - \xi(x_i)| \leq \eta \quad \text{and} \quad |\xi'(x) - \xi'(x_i)| \leq \eta.$$

Moreover the following inequality

$$\xi(x)\psi \circ \gamma_i^{-1}(x) \leq (\xi(x_i) + \eta)\psi \circ \gamma_i^{-1}(x)$$

holds for any $x \in X$, since if $x \notin \mathrm{Range}(\gamma_i)$ the both hand sides are 0. Then we have

$$\begin{aligned} \zeta(x)\xi(x) &\leq \sum_i c_i \xi(x)\psi \circ \gamma_i^{-1}(x) \\ &\leq \sum_i c_i(\xi(x_i) + \eta)\psi \circ \gamma_i^{-1}(x). \end{aligned}$$

This shows

$$(\zeta\xi : \psi) \leq \sum_i c_i(\xi(x_i) + \eta).$$

We have a similar inequality for ξ'. Since $x_i \in X_0$ and thus $\xi(x_i)+\xi'(x_i) = 1$, we have

$$(\zeta\xi : \psi) + (\zeta\xi' : \psi) \leq (2\eta + 1)\sum_i c_i.$$

The lemma follows from this. □

Continuing the proof of Theorem 1.1, let us first restrict the operator L_ψ to $C_c(X_0)_{\geq 0}$, and then extend it to $C_c(X_0)$ (still denoted by the same letter L_ψ) by just putting

$$L_\psi(\zeta) = L_\psi(\zeta_+) - L_\psi(\zeta_-),$$

where ζ_+ (resp. ζ_-) is the positive (resp. negative) part of ζ.

Then we have:

$$|L_\psi(\zeta)| \leq \|\zeta\|_\infty, \quad \forall \zeta \in C_c(X_0). \tag{2.7}$$

In fact if $\zeta \geq 0$, then $\zeta \leq \|\zeta\|_\infty \chi$, and thus $L_\psi(\zeta) \leq \|\zeta\|_\infty$, the general case following easily from this.

Let us identify L_ψ with the following point of a compact Hausdorff space:

$$L_\psi = \{L_\psi(\zeta)\}_\zeta \in \prod_{\zeta \in C_c(X_0)} \big[-\|\zeta\|_\infty, \|\zeta\|_\infty\big].$$

Let $\{\psi_n\}$ be a sequence in $C_c(X)_{>0}$ such that $\mathrm{diam}(\mathrm{supp}\psi_n) \to 0$. Choose an operator $L \in \bigcap_m \mathrm{Cl}\{L_{\psi_n} \mid n \geq m\}$. This means that for any finite number of elements $\zeta_\nu \in C_c(X_0)$ and any $\varepsilon > 0$, there is a sequence $n_i \to \infty$ such that $|L(\zeta_\nu) - L_{\psi_{n_i}}(\zeta_\nu)| < \varepsilon$. Now we have the following properties of the map $L : C_c(X_0) \to \mathbb{R}$.

$$L(c\zeta) = cL(\zeta), \quad \forall c \in \mathbb{R}, \tag{2.8}$$

$$L(\zeta_1 + \zeta_2) \leq L_(\zeta_1) + L(\zeta_2), \quad \forall \zeta_1,\ \zeta_2 \geq 0, \tag{2.9}$$

$$\zeta_1 \leq \zeta_2 \Rightarrow L(\zeta_1) \leq L(\zeta_2), \tag{2.10}$$

$$\mathrm{supp}\zeta \subset \mathrm{Dom}(\gamma),\ \gamma \in \Gamma_0 \Rightarrow L(\zeta \circ \gamma^{-1}) = L(\zeta), \tag{2.11}$$

$$\zeta \in C_c(X_0)_{>0} \Rightarrow L(\zeta) \geq 1/(\chi : \zeta), \tag{2.12}$$

$$|L(\zeta)| \leq \|\zeta\|_\infty. \tag{2.13}$$

Moreover by Lemma 2.2 and (2.9), we have

Lemma 2.3 *If $\zeta \in C_c(X_0)_{\geq 0}$ and ξ, $\xi' \in C_c(X)_{\geq 0}$ satisfy $\xi + \xi' = \chi$, then*

$$L(\xi\zeta) + L(\xi'\zeta) = L(\zeta).$$

From this one can derive the linearity of L. First of all notice that

$$\zeta, \zeta' \in C_c(X_0)_{\geq 0} \Rightarrow |L(\zeta) - L(\zeta')| \leq \|\zeta - \zeta'\|_\infty. \tag{2.14}$$

In fact we have

$$L(\zeta') = L(\zeta + \zeta' - \zeta) \leq L(\zeta + (\zeta' - \zeta)_+) \leq L(\zeta) + L((\zeta' - \zeta)_+)$$
$$\leq L(\zeta) + \|(\zeta' - \zeta)_+\|_\infty \leq L(\zeta) + \|\zeta' - \zeta\|_\infty.$$

Continuing the proof of the linearity, notice that it suffices to show it only for those functions $\zeta_1, \zeta_2 \in C_c(X_0)_{\geq 0}$. Choose $\varepsilon > 0$ small and let

$$\xi_j = (\zeta_j + \varepsilon\chi)/(\zeta_1 + \zeta_2 + 2\varepsilon)$$

for $j = 1, 2$. Then we have $\xi_1 + \xi_2 = \chi$. Now

$$\xi_1(\zeta_1 + \zeta_2) - \zeta_1 = \varepsilon(\zeta_2 - \zeta_1)/(\zeta_1 + \zeta_2 + \varepsilon).$$

Therefore by (2.14), we have

$$|L(\xi_1(\zeta_1 + \zeta_2)) - L(\zeta_1)| \leq \varepsilon.$$

On the other hand by Lemma 2.3,

$$L(\xi_1(\zeta_1 + \zeta_2)) + L(\xi_2(\zeta_1 + \zeta_2))) = L(\zeta_1 + \zeta_2).$$

Since ε is arbitrary, we have obtained

$$L(\zeta_1 + \zeta_2) = L(\zeta_1) + L(\zeta_2),$$

as is required.

Now L, being a positive operator, corresponds to a Radon measure μ. By (2.12), the measure μ is nontrivial, and since (2.13) implies

$$\sup\{L(\zeta) \mid \zeta \in C_c(X_0)_{\geq 0},\ \|\zeta\|_\infty \leq 1\} \leq 1,$$

the measure μ satisfies $\mu(X_0) \leq 1$. Finally (2.11) means the Γ_0-invariance of μ.

3. The uniqueness

In this section Γ is again an equicontinuous and minimal pseudo*group of local homeomorphisms of a locally compact metric space X. The modulus of equicontinuity is also denoted by $\varepsilon \mapsto \delta(\varepsilon)$. Without loosing generality one can assume that $\delta(\varepsilon) < \varepsilon$. Denote by $B_r(x)$ the open r-ball in X centered at $x \in X$.

We make the following additional assumption on the pseudo*group Γ.

Assumption 3.1 There is a relatively compact open subset X_0 of X and $a > 0$ such that if $\gamma \in \Gamma$, $x \in X_0$, $x \in \mathrm{Dom}(\gamma) \subset B_a(x)$ and $\gamma x \in X_0$, then there is $\hat{\gamma} \in \Gamma$ such that $\mathrm{Dom}(\hat{\gamma}) = B_a(x)$ and $\hat{\gamma}|_{\mathrm{Dom}(\gamma)} = \gamma$.

The purpose of this section is to show the following theorem.

Theorem 3.2 *Let Γ be an equicontinuous and minimal pseudo*group on X satisfying Assumption* 3.1*. Then the nontrivial Γ-invariant Radon measure on X is unique up to a scaling.*

First of all let us show that the holonomy pseudo*group Γ on a transversal X of a minimal lamination on a compact space Z, equicontinuous with respect to X satisfies Assumption 3.1, and therefore the latter part of Theorem 1.1 reduces to Theorem 3.2. Choose any relatively compact open subset X_0 of X. On one hand by compactness of Z there is $L > 0$ such that the germ of any element of the restriction Γ_0 to X_0 is a finite composite of the holonomy maps along leaf curves of length $\leq L$ that join two points in X_0. On the other hand there is $a' > 0$ such that each leaf curve of length $\leq L$ starting at $x \in X_0$ and ending at a point in X_0 admits a holonomy map defined on the ball $B_{a'}(x)$. An easy induction shows that Assumption 3.1 is satisfied for $a = \delta(a')$. Notice that $a < a'$ by the assumption.

Before starting the proof of Theorem 3.2, we shall outline what we are going to do. Consider the simpler case where a discrete group Λ acts on a compact space X minimally and equicontinuously. Then the completion G of Λ in the compact-open topology is a compact metrizable group and therfore admits a Haar probability measure m. Let μ be an arbitrary Λ-invariant probability measure on X. Choose any continuous function f on X. Define a function $f_m : X \to \mathbb{R}$ by

$$f_m(x) = \int_G f(gx)m(dg).$$

Then the right invariance of m implies that f_m is constant on X. Define a function $f_\mu : G \to \mathbb{R}$ by

$$f_\mu(g) = \int_X f(gx)\mu(dx).$$

Then since μ is G-invariant, f_μ is again constant, equal to $f_\mu(e) = \mu(f)$.

Integrating $f(gx)$ on $G \times X$ and applying Fubini, we get $\mu(f)$ is equal to f_m, a value which is independent of the particular choice of μ, showing the uniqueness of invariant measures.

In our setting of pseudo*group, things becomes a little messy. We consider a tiny compact ball C in the central part of X_0 and define G as the completion of "the restriction of Γ_0 to C". The space G is not a group, but we can construct a pseudo*group $\Gamma_\sharp$ acting on G, together with a $\Gamma_\sharp$-invariant measure m on G. We cannot take G to be compact since we need the minimality of the action of $\Gamma_\sharp$ on G. However it turns out that things work as well and we can apply Fubini.

Let us embark upon the proof of Theorem 3.2. Choosing a even smaller, one may assume that there is a nonempty open subset X_1 of X_0 such that the a-neighbourhood $B_a(x)$ of any point x of X_1 is contained in X_0 and that if $\gamma \in \Gamma$ and $x' \in X_0$ satisfies $\mathrm{Dom}(\gamma) = B_a(x')$ and $\gamma x' \in X_1$, then the image $\mathrm{Range}(\gamma) = \gamma(B_a(x'))$ is contained in X_0. Choose $b > 0$ so that $b \le \delta(a/3)$, and assume there is $x_0 \in X_1$ such that $C = \mathrm{Cl}(B) \subset X_1$, where $B = B_b(x_0)$.

Let M be the space of continuous maps from C to X_0, with the supremum metric d_∞. Define

$$\Gamma_C = \{\gamma|_C \mid \gamma \in \Gamma,\ C \subset \mathrm{Dom}(\gamma),\ \gamma C \subset X_0\}$$

and let G be the closure of Γ_C in M. Notice that G is not compact, because we do not take the closure in a space bigger than M. This choice of G is adapted for Lemma 3.6.

Lemma 3.3

(1) *G is a locally compact metric space.*
(2) *Any $g \in G$ is a homeomorphism onto a compact subset gC in X_0 and g, as well as the inverse map g^{-1}, is $\delta(\varepsilon)$-continuous.*

Proof. All that needs proof is the $\delta(\varepsilon)$-continuity of g^{-1}. Assume $\gamma_n \in \Gamma_C$ converge to $g \in G$ in the d_∞-metric. If $x, x' \in C$ satisfy $d(x, x') > \varepsilon$, then $d(\gamma_n x, \gamma_n x') \ge \delta(\varepsilon)$ by the equicontinuity of the inverse map γ_n^{-1}. Thus $d(gx, gx') \ge \delta(\varepsilon)$, as is required. □

Recall the notations $B = B_b(x_0)$ and $C = \mathrm{Cl}(B)$.

Lemma 3.4 *If $g_n \to g$ in G, and $y \in gB$, then for all n sufficiently large we have $y \in g_n B$ and $g_n^{-1}y \to g^{-1}y$.*

Proof. Choose an arbitrary point $x \in B$ and $\varepsilon > 0$ such that $\mathrm{Cl}(B_\varepsilon(x)) \subset B$. First let us show that for any $\gamma \in \Gamma_C$,

$$B_{\delta(\varepsilon)}(\gamma x) \subset \gamma \mathrm{Cl}(B_\varepsilon(x)). \tag{3.1}$$

In fact, by the choice of the number b, we have $\gamma(B) \subset \mathrm{Cl}(B_{a/3}(\gamma x_0))$. That is, $\gamma(B) \subset B_a(\gamma x)$, and thus $(\gamma|_B)^{-1}$ admits an extension $\widehat{\gamma^{-1}} \in \Gamma$ defined on $B_a(\gamma x)$. Choose an arbitrary point $y \in B_{\delta(\varepsilon)}(\gamma x)$. Then by the $\delta(\varepsilon)$-continuity of $\widehat{\gamma^{-1}}$, the point $x' = \widehat{\gamma^{-1}}y$ lies in $\mathrm{Cl}(B_\varepsilon(x)) \subset B$. On the other hand $x' = \gamma^{-1}\gamma x' = \widehat{\gamma^{-1}}\gamma x'$. Since $\widehat{\gamma^{-1}}$ is injective, we have $y = \gamma x'$. This finishes the proof of (3.1).

Next let us show that for any $g \in G$, we have

$$B_{\delta(\varepsilon)/2}(gx) \subset g\mathrm{Cl}(B_\varepsilon(x)). \tag{3.2}$$

Again assume $\gamma_n \in \Gamma_C$ converge to $g \in G$. Since $\gamma_n x \to gx$, we have for all n sufficiently large that $B_{\delta(\varepsilon)/2}(gx) \subset B_{\delta(\varepsilon)}(\gamma_n x)$. Thus if $y \in B_{\delta(\varepsilon)/2}(gx)$, then by (3.1) $y = \gamma_n x_n$ for some $x_n \in \mathrm{Cl}(B_\varepsilon(x))$. Passing to a subsequence, assume that $x_n \to x' \in \mathrm{Cl}(B_\varepsilon(x))$. Now in the following inequality

$$d(gx', y) = d(gx', \gamma_n x_n) \leq d(gx', \gamma_n x') + d(\gamma_n x', \gamma_n x_n),$$

both terms of the RHS can be arbitrarily small if n is sufficiently large. That is, $y = gx'$, showing (3.2).

To finish the proof of the lemma, assume $g_n \to g \in G$ and $y \in gB$. By (3.2), for any $\varepsilon > 0$ sufficiently small we have $B_{\delta(\varepsilon)/2}(g_n g^{-1}y) \subset g_n\mathrm{Cl}(B_\varepsilon(g^{-1}y))$. Since $g_n g^{-1}y \to y$, we have $y \in g_n\mathrm{Cl}(B_\varepsilon(g^{-1}y))$ for all n sufficiently large and therefore $g_n^{-1}y \in \mathrm{Cl}(B_\varepsilon(g^{-1}y))$. Since ε is arbitrarily small, this shows the lemma. □

Let Γ_0 be the restriction of the pseudo*group Γ to X_0. We shall construct a pseudo*group $\Gamma_\sharp$ of local homeomorphisms of G. For any $\gamma \in \Gamma_0$, define

$$\mathrm{Dom}(\gamma_\sharp) = \{g \in G \mid gC \subset \mathrm{Dom}(\gamma)\},$$
$$\mathrm{Range}(\gamma_\sharp) = \{g \in G \mid gC \subset \mathrm{Range}(\gamma)\},$$
$$\gamma_\sharp g = \gamma \circ g, \quad \forall g \in \mathrm{Dom}(\gamma_\sharp.)$$

It may happen that for some $\gamma \in \Gamma_0$, $\mathrm{Dom}(\gamma_\sharp) = \mathrm{Range}(\gamma_\sharp) = \emptyset$. In that case $\gamma_\sharp$ is not defined.

Lemma 3.5 *The subsets* $\mathrm{Dom}(\gamma_\sharp)$ *and* $\mathrm{Range}(\gamma_\sharp)$ *are open in* G*, and* $\gamma_\sharp$ *is* $\delta(\varepsilon)$*-continuous with respect to the metric* d_∞*.*

Proof. The easy proof is omitted. □

Denote by $\Gamma_\sharp$ the pseudo*group consisting of all the elements $\gamma_\sharp$ for $\gamma \in \Gamma_0$ and their restrictions to open subsets of the domains. The following lemma does not use the minimality assumption of Γ on X.

Lemma 3.6 *The action of* $\Gamma_\sharp$ *on* G *is minimal.*

Proof. First let us show that for $\gamma_1,\ \gamma_2 \in \Gamma_C$, there is $\gamma_\sharp \in \Gamma_\sharp$ such that $\gamma_1 \in \mathrm{Dom}(\gamma_\sharp)$ and that $\gamma_\sharp(\gamma_1) = \gamma_2$. Since $\gamma_1 C \subset B_a(\gamma_1 x_0)$, there is an element $\gamma' \in \Gamma$ defined on $B_a(\gamma_1 x_0)$ which extends $\gamma_2 \circ \gamma_1^{-1}$. Let $\gamma \in \Gamma_0$ be the restriction of γ' to Γ_0, i.e. the restriction such that $\mathrm{Dom}(\gamma) = B_a(\gamma_1 x_0) \cap X_0 \cap \gamma'^{-1} X_0$. Clearly $\gamma_1 C$ is contained in $\mathrm{Dom}(\gamma)$, showing the claim.

Thus we have shown that $\Gamma_\sharp$-orbit of id_C is nothing but Γ_C and hence dense in G. To finish the proof, we shall show that for any $g \in G$, the $\Gamma_\sharp$-orbit of g visits an arbitrarily small neighbourhood of any element $\gamma_2 \in \Gamma_C$. Let ε be any small number such that the 2ε-neighbourhood of $\gamma_2 C$ is contained in X_0. Take $\gamma_1 \in \Gamma_C$ such that $d_\infty(g, \gamma_1) < \delta(\varepsilon)$. Choosing ε and hence $\delta(\varepsilon)$ even smaller, one may very well assume that gC is contained in $B_a(\gamma_1 x_0)$. Then the element $\gamma \in \Gamma_0$ constructed above (for γ_1 and γ_2) contains gC in its domain, i.e. g is contained in $\mathrm{Dom}(\gamma_\sharp)$, and furthermore $d_\infty(\gamma_\sharp g, \gamma_2) < \varepsilon$. □

Now by Lemmata 3.3, 3.5 and 3.6, one can apply Theorem 2.1 to $(\Gamma_\sharp, G)$ to find a nontrivial $\Gamma_\sharp$-invariant Radon measure m on G. (Notice that even if $\Gamma_\sharp$ is equicontinuous, the pseudogroup it generates may not be equicontinuous. Compare Notes C.) One can assume m is a probability measure since G is in fact a precompact open subset of a bigger space. Now let μ and μ'

be distinct Γ_0-invariant probability measures on X_0. Then their restrictions to B are also distinct, by the minimality of the Γ_0-action. That is, there is a function $\zeta \in C_c(B)$ such that $\mu(\zeta) \neq \mu'(\zeta)$. One may assume further that ζ is nonnegative valued.

Lemma 3.7 *For any $g \in G$, we have*

$$\int_{X_0} \zeta(g^{-1}x)\mu(dx) = \int_{X_0} \zeta(x)\mu(dx).$$

Proof. For $g \in \Gamma_C$, this is just the Γ_0-invariance of μ. For general g, assume $\gamma_n \to g$ for $\gamma_n \in \Gamma_C$. Then by Lemma 3.4, if $x \in gB$, then $x \in \gamma_n B$ for all n sufficiently large and $\gamma_n^{-1}x \to g^{-1}x$. If $x \notin gB$, then $\zeta(g^{-1}x) = 0$. On the other hand $\zeta(\gamma_n^{-1}x) = 0$ for any n sufficiently large. In fact since $\gamma_n \to g$ in d_∞-metric, we have $\gamma_n \mathrm{supp}(\zeta) \to g\mathrm{supp}(\zeta)$ in the Hausdorff metric. Since $x \notin g\mathrm{supp}(\xi)$, it follows that $x \notin \gamma_n \mathrm{supp}(\zeta)$ for all n sufficiently large.

In any case for any $x \in X_0$, we have $\zeta(\gamma_n^{-1}x) \to \zeta(g^{-1}x)$. The lemma follows from the dominated convergence theorem. □

Now recall the space X_1. It is an open subset of X_0 which contains C such that the a-neighbourhood $B_a(x)$ of any point x of X_1 is contained in X_0 and that if $\gamma \in \Gamma$ and $x' \in X_0$ satisfies $\mathrm{Dom}(\gamma) = B_a(x')$ and $\gamma x' \in X_1$, then the image $\mathrm{Range}(\gamma) = \gamma(B_a(x'))$ is contained in X_0.

Lemma 3.8 *The function*

$$Z(x) = \int_G \zeta(g^{-1}x)m(dg)$$

is constant on X_0.

Proof. For $x \in X_0$ define a function $\zeta_x : G \to \mathbb{R}$ by $\zeta_x(g) = \zeta(g^{-1}x)$. Lemma 3.4 and an additional argument as in the proof of Lemma 3.7 shows that ζ_x is a continuous function. Also the function $Z(x)$ is continuous since $\zeta \circ g^{-1}$ for $g \in G$ has the same modulus of continuity. Choose any $x \in X_0$ and $x' \in X_1$ on the same Γ-orbit. The proof is complete once we show $Z(x) = Z(x')$ since the Γ-action on X_0 is minimal and any orbit intersects X_1. By the assumption of X_1, there is $\gamma \in \Gamma_0$ such that $\gamma x = x'$ and $\mathrm{Dom}(\gamma) = B_a(x) \cap X_0$ and $\mathrm{Range}(\gamma) \subset X_0$. Then we have

$$\{g \in G \mid \zeta_x(g) > 0\} \subset \mathrm{Dom}(\gamma_\sharp).$$

In fact if $\zeta_x(g) = \zeta(g^{-1}x) > 0$, then $x \in gB$. On the other hand, $\mathrm{diam}(gB) \le 2a/3$, and thus $gC \subset B_a(x) \cap X_0 = \mathrm{Dom}(\gamma)$, i.e. $g \in \mathrm{Dom}(\gamma_\sharp)$.

By the $\Gamma_\sharp$-invariance of the measure m, we have

$$Z(x) = \int_G \zeta_x(g)m(dg) = \int_G \zeta_x(\gamma_\sharp^{-1}(g))m(dg) = \int_G \zeta_x(\gamma^{-1} \circ g)m(dg)$$
$$= \int_G \zeta(g^{-1}\gamma x)m(dg) = \int_G \zeta_{\gamma x}(g)m(dg) = Z(x'),$$

as is required. □

Now let us finish the proof of Theorem 3.2. By Lemma 3.8, the function Z is constant on X_0, depending only on ζ and m. We have on one hand

$$\int_{X_0}\int_G \zeta(g^{-1}x)m(dg)\mu(dx) = \int_{X_0} Z\mu(dx) = Z.$$

On the other hand by Fubini and by Lemma 3.7

$$Z = \int_G\int_{X_0} \zeta(g^{-1}x)\mu(dx)m(dg) = \int_G \mu(\zeta)m(dg) = \mu(\zeta).$$

Since Z does not depend on the choice of μ, we have $\mu(\zeta) = \mu'(\zeta)$, contrary to the assumption.

Remark 3.9 In fact the argument of this section works under the assumption that the pseudo*group Γ acts transitively (having one dense orbit) on X, which is weaker than the minimality. As is pointed out there, Lemma 3.6 does not use the minimality assumption, and all the other parts are valid under the weaker assumption. However any transitive and equicontinuous pseudo*group satisfying Assumption 3.1 can be shown to be minimal. Assume the contrary, and let Z be a proper minimal set and x a point in X_0 whose orbit is dense in X. Assume the distance of x to Z is bigger than a constant $\varepsilon > 0$, and $\delta(\varepsilon) < a$, where a is a constant given in Assumption 3.1. Then there is a point $x' \in \Gamma x \cap X_0$ and a point $y \in Z$ such that $d(x', y) < \delta(\varepsilon)$. By Assumption 3.1 there is $\gamma \in \Gamma$ such that $\mathrm{Dom}(\gamma) = B_a(x')$ and $\gamma x' = x$. Then we have $d(x, \gamma y) \le \varepsilon$. A contradiction.

4. Notes

A. Let Z be a locally compact metric space, covered by a countable number of open sets E_i. Assume there is a homeomorphism $\varphi_i : E_i \to U_i \times X_i$, where U_i is an open ball in $\mathbb{R}^p$ and X_i is a locally compact metric space. If $E_i \cap E_j \neq \emptyset$, then the transition function $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$ is defined as a homeomorphism from $\varphi_i(E_i \cap E_j)$ onto $\varphi_j(E_i \cap E_j)$. Assume that the transition function is of the form

$$\psi_{ji}(u, x) = (\alpha(u, x), \beta(x)).$$

A subset of Z of the form $\varphi_i^{-1}(U_i \times x)$ is called a *plaque*. A maximal connected countable union of plaques is called a *leaf*. This way M admits a decompositon $\mathcal{L}$ into leaves, which is called a *lamination* of dimension p. A subset of the form $\varphi_i^{-1}(u \times X_i)$ is called a *transversal* of $\mathcal{L}$.

B. Let H be the Lie group of all the orientation preserving affine transformations of the real line. A. Haefliger constructed ([Gh]) a minimal Lie H foliation $\mathcal{F}$ on a closed 5-manifold. The corresponding (global) holonomy group Λ is dense in H. The transverse holonomy groupoid is equivalent to (Λ, H), but it is not equivalent to a groupoid on a compact space X. For if it is, then X must be a closed surface with a (H, H)-structure, which is impossible since H is not unimodular.

C. Consider a linear foliation $\mathcal{F}$ on the 2-torus $S^1 \times S^1$ of irrational slope. The pseudo*group on the transversal $S^1 (= S^1 \times \{0\})$ is generated by an irrational rotation R and is equicontinuous, but the psuodogroup is not, since it contains an element γ given for a small positive number ε as follows. (1) $\mathrm{Dom}(\gamma) = (-\varepsilon, 0) \cup (0, \varepsilon)$. (2) $\gamma|_{(-\varepsilon,0)} = R$ and $\gamma|_{(0,\varepsilon)} = R^{-1}$.

References

[DKN] Deroin B., Kleptsyn V. and Navas A., *Sur la dynamique unidimensionelle en régularité intermédiaire.* Acta Math. **199** (2007), 199–262.

[Gh] Ghys E., *Riemannian foliations: Examples and problems.* In: P. Molino, "Riemannian Foliations" Progress of Mathematics vol. 98, Birkhäuser 1988.

[HH] Hector G. and Hirsch U., "Introduction to the Geometry of foliations, Part A" Aspects of Mathematics, vol. E1, Vieweg 1986.

[S] Sacksteder R., *Foliations and pseudogroups.* Amer. J. Math. **87** (1965), 79–102.

[SK] Segal I. E. and Kunze R. A., "Integrals and Operators", Grundlehren der Math. Wiss. vol. 228, Springer Verlag, 1978.

Department of Mathematics
College of Science and Technology
Nihon University
1-8-14 Kanda, Surugadai
Chiyoda-ku, Tokyo, 101-8308 Japan
E-mail: matsumo@math.cst.nihon-u.ac.jp