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Basis properties and complements of complex exponential systems

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Abstract. In this note, we show that some families of complex exponentials are either Riesz sequences or not basic sequences in $L^2[-\pi, \pi]$. Besides, we show that every incomplete complex exponential system satisfying some condition can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

Key words: basis, Riesz basis, Riesz sequence, complete and minimal sequence.

1. Introduction

Let $\lambda = \{\lambda_n\}$, $-\infty < n < \infty$, be a sequence of distinct complex numbers, then a system $e(\lambda) \equiv \{e^{i\lambda_n t}\}$ is said to be a *basis* for $L^2[-\pi, \pi]$ if any function $f(t)$ in $L^2[-\pi, \pi]$ has a unique expansion

$$f(t) = \sum_n c_n e^{i\lambda_n t} \text{ (in the mean)}$$

for some sequence $\{c_n\}$. Also, $e(\lambda)$ is said to be a *basic sequence* if it is a basis of the closure of the space spanned by the distinct elements $e^{i\lambda_n t}$. Next $e(\lambda)$ is said to be a *Riesz basis* if there exists an isomorphism

$$T\colon L^2[-\pi, \pi] \longrightarrow L^2[-\pi, \pi]$$

and

$$T(e^{int}) = e^{i\lambda_n t}$$

for any n. Moreover, $e(\lambda)$ is said to be a *Riesz sequence* if it is a Riesz basis of the closure of the space spanned by the distinct elements $e^{i\lambda_n t}$. $e(\lambda)$ is said to be *complete* in $L^2[-\pi, \pi]$ if the linear subspace spanned by the distinct elements $e^{i\lambda_n t}$ is dense in $L^2[-\pi, \pi]$. And $e(\lambda)$ is said to be *minimal* in $L^2[-\pi, \pi]$ if each element of $e(\lambda)$ lies outside the closed linear span of the others. Obviously, we see that if $e(\lambda)$ is a Riesz basis, then it is a basis and if it is a basis, then it is complete and minimal. We say the system $\{e^{i\lambda_n t}\}$

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has *excess* N if it remains complete and becomes minimal when N terms $e^{i\lambda_n t}$ are removed and we define

$$E(\lambda) = N.$$

Conversely we define the excess

$$E(\lambda) = -N$$

if it becomes complete and minimal when N terms

$$e^{i\mu_1 t}, \dots, e^{i\mu_N t}$$

are adjoined. By convention we define $E(\lambda) = \infty$ if arbitrarily many terms can be removed without losing completeness and $E(\lambda) = -\infty$ if arbitrarily many terms can be adjoined without getting completeness. It is obvious that $\{e^{i\lambda_n t}\}$ is to be complete and minimal if and only if $E(\lambda) = 0$.

We refer to N. Levinson [L], R.M. Young [Y4] and R.M. Redheffer [R] on the theory of nonharmonic Fourier series which we take up in this note.

R.M. Young showed in the proof of [Y2, Theorem 2] that if

$$\lambda_n = \begin{cases} n - \frac{1}{4}, & n > 0, \\ n + \frac{1}{4}, & n < 0, \end{cases} \tag{1.1}$$

then $e(\lambda)$ was not a basis. Besides he showed in [Y3, Theorem 2] that if

$$\mu_n = \begin{cases} n + \frac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n - \frac{1}{4}, & n < 0, \end{cases} \tag{1.2}$$

then $e(\mu)$ was not also a basis. In this note, we first show that if

$$\lambda_n = \begin{cases} n - \alpha, & n > 0, \\ n + \alpha, & n < 0, \end{cases}$$

and

$$\mu_n = \begin{cases} n + \alpha, & n > 0, \\ 0, & n = 0, \\ n - \alpha, & n < 0, \end{cases}$$

then $e(\lambda)$ and $e(\mu)$ are either Riesz sequences or not basic sequences in $L^2[-\pi, \pi]$ for $0 < \alpha < 1$.

Next let

$$\lambda_n = \begin{cases} n - \alpha_n, & n > 0, \\ n + \alpha_n, & n < 0 \end{cases}$$

for $0 < \alpha_n < 1$, then we consider whether $e(\lambda)$ is a Riesz basis or not, and moreover it is a basis or not. One of the problems is whether $e(\lambda)$ is a basis or not in the case of which

$$\varepsilon_n \to 0 \quad \text{as } n \to \pm\infty \text{ and } \sum_n |\varepsilon_n| = \infty$$

for $\alpha_n = 3/4 + \varepsilon_n$.

In this note, we need the following "stability results".

Theorem A (see [Y4, p. 161, Corollary]) *If $e(\lambda)$ is a Riesz basis for $L^2[-\pi, \pi]$, then there is a positive constant L with the property that $e(\mu)$ is also a Riesz basis for $L^2[-\pi, \pi]$ whenever $|\lambda_n - \mu_n| \leq L$ for every n.*

Theorem B (see [Y4, p. 165, Prob. 2]) *Let $e(\lambda)$ be a basis for $L^2[-\pi, \pi]$ and suppose that $\sup_n |\operatorname{Im} \lambda_n| < \infty$. If $\mu = \{\mu_n\}$ satisfies*

$$\sum_n |\lambda_n - \mu_n| < \infty,$$

then, $e(\mu)$ is also a basis for $L^2[-\pi, \pi]$.

The following result follows from Theorem B immediately. We see also Lemma II.4.11 of S.A. Avdonin and S.A. Ivanov [AI] about the same result.

Corollary 1.1 *We suppose that $\sup_n |\operatorname{Im} \lambda_n| < \infty$ and $e(\lambda)$ is a basis. If we replace finitely many points λ_n by the same number of points $\mu_n \notin \{\lambda_n\}$, $\mu_n \neq \mu_m$, $n \neq m$, then the basis property of $e(\lambda)$ is not violated. Consequently the same applies to any Riesz basis.*

Remark 1.1 Theorem A holds even if "Riesz sequence" that excess is finite is taken. So far as we know, it is unknown whether Theorem A holds or not if a Riesz basis is replaced with a basis. However, it is also unknown whether such a basis which is conditional exists or not.

In §4, we show that every incomplete complex exponential system satisfying some condition can be complemented up to a complete and minimal system of complex exponentials. It is unknown, so far as we know, whether every incomplete complex exponential system can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$ or not. This problem is originated in [Y1, Remark]. On the other hand, K. Seip has shown in [S, Theorem 2.8] that there exists a Riesz sequence of complex exponentials which cannot be complemented up to a Riesz basis. He has given a sequence

$$e(\lambda) = \{e^{\pm i(n+\sqrt{n})t}\}_{n>1}$$

as an example of such a Riesz sequence.

And he raised the next question personally:

Question *Can every Riesz sequence of complex exponentials be complemented up to a complete and minimal system of complex exponentials?*

In this section, we show that it is possible for some families of comlex exponential systems which include many Riesz sequnces of $E(\lambda) = -\infty$. Let $e(\lambda)$ be a complex exponential system which has the excess $E(\lambda) = -\infty$. Our method is to construct a sequence $\mu = \{\mu_n\}$ such that $\lambda \subset \mu$ and the system $e(\mu)$ has a finite excess. If we can construct such a sequence μ, then we see that the system $e(\lambda)$ can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$. For this purpose, we use the next theorem:

Theorem C ([R, Theorem 47]) *For $-\infty < n < \infty$, let $\lambda \equiv \{\lambda_n\}$ be a sequence of complex numbers satisfying $|\lambda_n - n| \le h$ where h is a positive constant. Then $E(\lambda)$ satisfies*

$$-\Big(4h + \frac{1}{2}\Big) < E(\lambda) \le 4h + \frac{1}{2}.$$

2. Basis properties of complex exponential systems

We first consider the system $e(\lambda)$,

$$\lambda_n = \begin{cases} n - \alpha, & n > 0, \\ n + \alpha, & n < 0, \end{cases} \tag{2.1}$$

for $0 < \alpha < 1$. We see from Kadec's 1/4-theorem(M.I. Kadec, 1964; see [Y4, p. 36]) that $e(\lambda)$ is a Riesz sequence for $0 < \alpha < 1/4$. It has been shown in

[Y2, Theorem 2] that $e(\lambda)$ is not a basis for $L^2[-\pi, \pi]$ for $\alpha = 1/4$. Besides, it has been also known that $e(\lambda)$ is a Riesz basis for $L^2[-\pi, \pi]$ for $1/4 < \alpha < 3/4$ by using the isometric isomorphism

$$\phi(t) \longmapsto e^{it/2}\phi(t)$$

on $L^2[-\pi, \pi]$ and Kadec's 1/4-theorem.

Proposition 2.1 *Let $\lambda = \{\lambda_n\}$ be a sequence given by* (2.1)*.*
(i) *Let $\alpha = 3/4$. If we remove any element λ_{n_0} in λ, then $e(\lambda')$ for $\lambda' = \lambda - \{\lambda_{n_0}\}$ is complete and minimal, but it is not a basis for $L^2[-\pi, \pi]$;*
(ii) *$e(\lambda')$ of* (i) *is a Riesz basis for $L^2[-\pi, \pi]$ for $3/4 < \alpha < 1$.*

Proof. We see that $e(\lambda')$ is complete and minimal by [N, Theorem 1.1]. But if we write

$$n - \frac{3}{4} = (n-1) + \frac{1}{4}, \quad -n + \frac{3}{4} = -(n-1) - \frac{1}{4}$$

for $n \geq 1$, then we see that $e(\lambda')$ is not a basis for $L^2[-\pi, \pi]$ because $e(\mu)$, where the μ_n are given by (1.2), is not a basis. This prove (i).

Now if we write

$$n - \alpha = (n-1) + (1-\alpha), \quad -n + \alpha = -(n-1) - (1-\alpha)$$

for $n \geq 1$, then (ii) is trivial by Kadec's 1/4-theorem. □

Next we consider the system $e(\mu)$,

$$\mu_n = \begin{cases} n+\alpha, & n > 0, \\ 0, & n = 0, \\ n-\alpha, & n < 0, \end{cases} \tag{2.2}$$

for $0 < \alpha < 1$.

It has already been known by Kadec's 1/4-theorem that $e(\mu)$ is a Riesz basis for $0 < \alpha < 1/4$. It has also been shown in [Y3, Theorem 2] that $e(\mu)$ is not a basis for $L^2[-\pi, \pi]$ for $\alpha = 1/4$.

Proposition 2.2 *Let $\mu = \{\mu_n\}$ be a sequence given by* (2.2)*.*
(i) *$e(\mu)$ is not a basic sequence for $\alpha = 3/4$;*
(ii) *$e(\mu)$ is a Riesz sequence for $1/4 < \alpha < 3/4$ or $3/4 < \alpha < 1$.*

Proof. If we write

$$n + \frac{3}{4} = (n+1) - \frac{1}{4}, \quad -n - \frac{3}{4} = -(n+1) + \frac{1}{4}$$

for $n \geq 1$, then (i) is an immediate consequence from the fact that $e(\lambda)$, where the λ_n are given by (1.1), is not a basis for $L^2[-\pi, \pi]$. Next we write

$$n + \alpha = (n+1) - (1-\alpha), \quad -n - \alpha = -(n+1) + (1-\alpha) \tag{2.3}$$

for $n \geq 1$. We see that $e(\mu)$ is a Riesz sequence from (2.3) and the known result which $e(\lambda)$ given by (2.1) is a Riesz basis for $1/4 < \alpha < 3/4$. Moreover, it is a Riesz sequence by (2.3) and Kadec's 1/4-theorem for $3/4 < \alpha < 1$. □

From the above results, we have obtained the following results:

Corollary 2.1 *Let $e(\gamma)$ be a system given by* (2.1) *or* (2.2), *then $e(\gamma)$ is either a Riesz sequence or not a basic sequence in $L^2[-\pi, \pi]$.*

Now we next consider the system $e(\lambda)$,

$$\lambda_n = \begin{cases} n - \alpha_n, & n > 0, \\ n + \alpha_n, & n < 0, \end{cases} \tag{2.4}$$

for $0 < \alpha_n < 1$. The cases of $\sup_n \alpha_n < 1/4$ and $1/4 < \inf_n \alpha_n \leq \sup_n \alpha_n < 3/4$, $3/4 < \inf_n \alpha_n$ are trivial by Kadec's 1/4-theorem, and so we deal with the case which the numbers α_n behave the neighborhood of 1/4 or 3/4.

Theorem 2.1 *Let $\alpha_n = 3/4 + \varepsilon_n$ or $\alpha_n = 1/4 + \varepsilon_n$. Then we obtain the following results for $e(\lambda)$ given by* (2.4):

(1) *If $\varepsilon_n \to 0$ as $n \to \pm\infty$, then $e(\lambda)$ is not a Riesz basis for $L^2[-\pi, \pi]$.*

(2) *Furthermore, if $\sum_n |\varepsilon_n| < \infty$, then $e(\lambda)$ is not a basis for $L^2[-\pi, \pi]$.*

Proof. First we consider the case of $\alpha_n = 3/4 + \varepsilon_n$ in (2.4). Then

$$\lambda_n = \begin{cases} n - \dfrac{3}{4} - \varepsilon_n, & n > 0, \\ n + \dfrac{3}{4} + \varepsilon_n, & n < 0. \end{cases}$$

Now, if we take

$$\gamma_n = \begin{cases} n - \dfrac{3}{4}, & n > 0, \\ n + \dfrac{3}{4}, & n < 0, \end{cases}$$

$e(\gamma)$ is not a basis for $L^2[-\pi, \pi]$ by (i) of Proposition 2.1.

We suppose $\varepsilon_n \to 0$ as $n \to \pm\infty$. We refer to [RY, p. 107, Corollary] about the next arguement. If $e(\lambda)$ is a Riesz basis for $L^2[-\pi, \pi]$, then there exists a positive constant L by Theorem A such that if

$$|\lambda_n - \delta_n| \le L \text{ for } \forall n,$$

$e(\delta)$ is also a Riesz basis for $L^2[-\pi, \pi]$. By the hypothesis, we can choose a positive integer n_0 such that

$$|\lambda_n - \gamma_n| = |\varepsilon_n| \le L \text{ for } \forall |n| \ge n_0.$$

Hence,

$$\{e^{i\lambda_n t}\}_{|n|<n_0} \bigcup \{e^{i\gamma_n t}\}_{|n|\ge n_0}$$

is a Riesz basis for $L^2[-\pi, \pi]$. Consequently, by Corollary 1.1, $e(\gamma)$ is also a Riesz basis for $L^2[-\pi, \pi]$. This contradicts, hence $e(\lambda)$ is not a Riesz basis for $L^2[-\pi, \pi]$.

Next we suppose $\sum_n |\varepsilon_n| < \infty$. If $e(\lambda)$ is a basis for $L^2[-\pi, \pi]$, then $e(\gamma)$ is also a basis for $L^2[-\pi, \pi]$ by Theorem B. This contradicts, hence $e(\lambda)$ is not a basis.

Second we consider the case of $\alpha_n = 1/4 + \varepsilon_n$ in (2.4). Then

$$\lambda_n = \begin{cases} n - \dfrac{1}{4} - \varepsilon_n, & n > 0, \\ n + \dfrac{1}{4} + \varepsilon_n, & n < 0. \end{cases}$$

We suppose that $e(\lambda)$ is a Riesz basis for $L^2[-\pi, \pi]$. Considering the isometiric isomorphism

$$\phi(t) \longmapsto e^{it/2}\phi(t),$$

it follows that $e(\lambda^{(1)})$ is also a Riesz basis for $L^2[-\pi, \pi]$, where

$$\lambda_n^{(1)} = \begin{cases} n + \dfrac{1}{4} - \varepsilon_n, & n > 0, \\ n + \dfrac{3}{4} + \varepsilon_n, & n < 0. \end{cases}$$

Moreover, we rewrite

$$\lambda_n^{(1)} = (n+1) - \frac{1}{4} + \varepsilon_n, \quad n < 0,$$

and if we substitute 0 for $\lambda_{-1}^{(1)}$, we see that $e(\lambda^{(2)})$ is also a Riesz basis for $L^2[-\pi, \pi]$, where

$$\lambda_n^{(2)} = \begin{cases} n + \frac{1}{4} - \varepsilon_n, & n > 0, \\ 0, & n = 0, \\ n - \frac{1}{4} + \varepsilon_{n-1}, & n < 0, \end{cases}$$

by Corollary 1.1. By the way, we know that $e(\mu)$ is not a basis for the sequence $\{\mu_n\}$ given by (1.2). Following the arguement of the proof of the case $\alpha_n = 3/4 + \varepsilon_n$ if $\varepsilon_n \to 0$ as $n \to \pm\infty$, we see $e(\mu)$ is a Riesz basis for $L^2[-\pi, \pi]$. This contradicts, hence $e(\lambda)$ is not a Riesz basis for $L^2[-\pi, \pi]$. Next we suppose $\sum_n |\varepsilon_n| < \infty$. If $e(\lambda)$ is a basis, then $e(\lambda^{(2)})$ is also a basis. Now, by the same arguement as the one used in the case $\alpha_n = 3/4 + \varepsilon_n$, it follows that $e(\mu)$ is also a basis. This contradicts too, hence $e(\lambda)$ is not a basis. □

3. Some problems

From the examination until now, we have some problems. We suppose that

$$\varepsilon_n \to 0 \text{ as } n \to \pm\infty \quad \text{and} \quad \sum_n |\varepsilon_n| = \infty$$

for $\alpha_n = 3/4 + \varepsilon_n$ in (2.4). Then, we have some questions, i.e., does $e(\lambda)$ become a basis for $L^2[-\pi, \pi]$ or a basic sequence?

We write

$$\lambda_n = \begin{cases} n - \alpha_n = (n-1) + \frac{1}{4} - \varepsilon_n, & n > 0, \\ n + \alpha_n = (n+1) - \frac{1}{4} + \varepsilon_n, & n < 0. \end{cases}$$

And let $\lambda' = \{\lambda'_n\}$,

$$\lambda'_n = \begin{cases} n + \dfrac{1}{4} - \varepsilon_n, & n > 0, \\ 0, & n = 0, \\ -\lambda'_{-n}, & n < 0, \end{cases} \tag{3.1}$$

then, we have $E(\lambda - \{\lambda_1\}) = E(\lambda')$ and the basis property of $e(\lambda - \{\lambda_1\})$ is same as the one of $e(\lambda')$ by Corollary 1.1.

1. The case that $e(\lambda')$ is complete.
   As

   $$E(\lambda - \{\lambda_1\}) = E(\lambda') \geq 0,$$

   we have $E(\lambda) \geq 1$, hence $e(\lambda)$ is not a basis for $L^2[-\pi, \pi]$. This can happen if

   $$\sum_n \frac{|\varepsilon_n|}{|n| + 1} < \infty \tag{3.2}$$

   by [R, p. 45].
2. The case that $e(\lambda')$ is not complete.
   Redheffer and Young have given the next example of the sequence $\{\varepsilon_n\}$ which does not satisfy (3.2):

Theorem D (see [RY, Theorem 3]) *Let*

$$\mu_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ n + \dfrac{1}{4} + \dfrac{\beta}{\log n}, & n \geq 2 \\ -\mu_{-n}, & n < 0, \end{cases} \tag{3.3}$$

then $e(\mu)$ *is complete in* $L^2[-\pi, \pi]$ *if* $0 \leq \beta \leq 1/4$ *and not if* $\beta > 1/4$.

More precisely, by [R, Theorem 47] and [FNR], we have $E(\mu) = 0$ for $0 \leq \beta \leq 1/4$ and $E(\mu) = -1$ for $\beta > 1/4$.

Problem 3.1 We raise the next problems:

(i) Is the system $e(\mu)$ in Theorem D basis for $0 < \beta \leq 1/4$?

(ii) Is the system $e(\mu)$ in Theorem D basic sequence for $\beta > 1/4$?

Moreover we have the problem which is equivalent to the above problem (ii):

Let

$$\gamma_1 = \frac{1}{4}, \quad \gamma_{-1} = -\gamma_1$$

and

$$\gamma_n = \begin{cases} n - \dfrac{3}{4} + \dfrac{\beta}{\log n}, & n \geq 2, \\ -\gamma_{-n}, & n \leq -2, \end{cases}$$

then is the system $e(\gamma)$ basis for $L^2[-\pi, \pi]$ for $\beta > 1/4$?

In (3.3), if we replace "$n + 1/4$" with "$n + 1/4 - \varepsilon$", where ε is any small positive number, then the above problems are trivial by Kadec's 1/4-theorem.

4. Complements of complex exponential systems

In this section, we show that every incomplete complex exponential system satisfying some condition can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$. We have the following result.

Theorem 4.1 *Let $\{\delta_n\}$ be a real sequence such that*

$$1 \leq \delta_1, \quad \delta_n \leq \delta_{n+1}$$

and

$$\lim_{n\to\infty} \delta_n = \infty.$$

If $\lambda \equiv \{\lambda_n\}$ is a sequence where

$$\lambda_0 = 0,\ \lambda_n = n + \delta_n,\ \lambda_{-n} = -\lambda_n\ (n = 1, 2, \ldots),$$

then the system $e(\lambda) \equiv \{e^{i\lambda_n t}\}$ has the excess $E(\lambda) = -\infty$ in $L^2[-\pi, \pi]$ and $e(\lambda)$ can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

Proof. We may choose $\mu = \{\mu_n\}$ such that $\lambda \subset \mu$ and $e(\mu) \equiv \{e^{i\mu_n t}\}$ is complete and it has a finite excess in $L^2[-\pi, \pi]$.

Firstly, we choose a positive integer k_1 such that

$$k_1 \leq \delta_1 < k_1 + 1.$$

Then we take

$$\mu_0 = 0,\ \mu_1 = 1,\ \ldots,\ \mu_{k_1+1} = k_1 + 1.$$

Moreover, we define

$$\mu_{k_1+2} = \begin{cases} k_1 + 2, & \delta_1 = k_1, \\ \lambda_1, & \delta_1 \neq k_1. \end{cases}$$

Generally we choose a positive integer k_j for $j \geq 2$ such that

$$k_j \leq \delta_j < k_j + 1.$$

If $k_j = k_{j-1} + \ell$ $(1 \leq \ell \leq k_j - k_{j-1})$, we take

$$\begin{aligned} \mu_{k_j+j} &= k_j + j \\ \mu_{k_j+(j-1)} &= k_j + (j-1) \\ &\vdots \\ \mu_{k_j+(j+1-\ell)} &= k_j + (j+1-\ell), \end{aligned}$$

and

$$\mu_{k_j+(j+1)} = \begin{cases} k_j + (j+1), & \delta_j = k_j, \\ \lambda_j, & \delta_j \neq k_j. \end{cases}$$

Next if $k_{j-1} = k_j$, we take

$$\mu_{k_j+(j+1)} = \mu_{k_{j-1}+(j+1)} = \begin{cases} k_j + (j+1), & \delta_j = k_j, \\ \lambda_j, & \delta_j \neq k_j. \end{cases}$$

Finally let $\mu_{-n} = -\mu_n$. Thus we choose the sequence $\mu = \{\mu_n\}$.

For $t > 1$, we denote by $n(t)$ and $n_1(t)$ the number of integers n inside the interval $|x| \leq t$ and the number of points μ_n inside the interval $|x| \leq t$, respectively. From the definition of the sequence $\{\mu_n\}$, we have

$$n_1(t) \geq n(t),$$

and hence, we see by [Y4, pp. 99∼100, Theorem 3, 4] that $e(\mu)$ is complete in $L^2[-\pi, \pi]$, i.e. $E(\mu) \geq 0$. Besides, since $k_j \leq \delta_j < k_j + 1$, $\lambda_j = j + \delta_j$, we have

$$k_j + j \leq \lambda_j < k_j + (j+1).$$

Therefore we see that

$$n - 1 \le \mu_n \le n \quad \text{for } \forall n \ge 1$$

hold. Since $\mu_{-n} = -\mu_n$, the inequalities

$$|\mu_n - n| \le 1 \quad \text{for } \forall n$$

hold. Applying Theorem C, we conclude that

$$E(\mu) \le 4,$$

consequently

$$0 \le E(\mu) \le 4.$$

Hence we can reduce $e(\mu)$ to a complete and minimal system. Thus $e(\lambda)$ has the excess $E(\lambda) = -\infty$ in $L^2[-\pi, \pi]$ and it can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$. □

Remark 4.1 The author does not know whether the system $e(\lambda)$ in Theorem 4.1 is always a Riesz sequence or not. But some examples of Riesz sequences for $L^2[-\pi, \pi]$ seen so far satisfiy the condition in Theorem 4.1 as shown by the examples in the next section.

5. Examples and remark

The first example is given in [S, Theorem 2.8] as an example of a Riesz sequence of complex exponentials which it cannot be complemented up to a Riesz basis of complex exponentials.

Example 5.1 Let $\lambda = \{\pm(n + \sqrt{n})\}_{n>1}$ and $e(\lambda) \equiv \{e^{\pm i(n+\sqrt{n})t}\}_{n>1}$.

If we take $\delta_n = \sqrt{n}$ in Theorem 4.1, then we see that the system $e(\lambda)$ can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

Next we deal with the next example. We may refer to [Y4, p. 136, Theorem 5 and p. 138, Theorem 6].

Example 5.2 Let $\lambda = \{\lambda_n\}$ be a sequence of real numbers such that

$$\lambda_{n+1} - \lambda_n \ge \gamma > 1 \ (n = 0, 1, 2, \ldots),$$
$$\lambda_{-n} = -\lambda_n \ (n = 0, 1, 2, \ldots).$$

Then $e(\lambda) \equiv \{e^{i\lambda_n t}\}$ is a Riesz sequence which has the excess $E(\lambda) = -\infty$ in $L^2[-\pi, \pi]$. Now we can write

$$\lambda_{n+1} - \lambda_n = 1 + \varepsilon_n, \quad \varepsilon_n \geq \varepsilon > 0 \ (n = 0, 1, 2, \ldots).$$

So we have

$$\lambda_n = n + \sum_{k=0}^{n-1} \varepsilon_k, \quad n \geq 1.$$

If we take

$$\delta_n = \sum_{k=0}^{n-1} \varepsilon_k, \quad n \geq 1,$$

there exists a positive integer n_0 such that $\delta_{n_0} \geq 1$. We see by Theorem 4.1 that $e(\lambda') \equiv \{e^{i\lambda_n t}\}_{|n| \geq n_0}$ can be complemented up to a complete and minimal system of complex exponentials. Consequently, by [L, p. 7, Theorem 6], the system $e(\lambda)$ can also be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

Remark 5.1 Let $n^+(r)$ denote the largest number of points from λ to be found in an interval of length of r (see [S, p. 133]) and we define

$$D^+(\lambda) = \lim_{r \to \infty} \frac{n^+(r)}{r}.$$

Then K. Seip has proved in [S, Theorem 2.2] that if $e(\lambda)$ is a Riesz sequnce, we have $D^+(\lambda) \leq 1$. Moreover, he has proved in [S, Theorem 2.4] that if λ satisfies $D^+(\lambda) < 1$, $e(\lambda)$ can be complemented up to a Riesz basis of complex exponentials in $L^2[-\pi, \pi]$. Consequently, the problem is whether every Riesz sequnce $e(\lambda)$ satisfying $D^+(\lambda) = 1$ can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$. If, in Theorem 4.1,

$$\lim_{n \to \infty} \frac{\delta_n}{n} = 0,$$

then we obtain $D^+(\lambda) = 1$.

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References

[AI] Avdonin S.A. and Ivanov S.A., *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*. Cambridge University Press, Cambridge, 1995.

[FNR] Fujii N., Nakamura A. and Redheffer R.M., *On the Excess of Sets of Complex Exponentials*. Proc. Amer. Math. Soc. **127** (1999), 1815–1818.

[L] Levinson N., *Gap and Density Theorems*. Amer. Math. Soc. Colloquium Publications, **26**, 1940.

[N] Nakamura A., *On the excess of a sequence of exponentials with perturbations at some subsequences of integers*. Hokkaido Math. J. (2) **29** (2000), 303–313.

[R] Redheffer R.M., *Completeness of Sets of Complex Exponentials*. Advances in Mathematics **24** (1977), 1–62.

[RY] Redheffer R.M. and Young R.M., *Completeness and Basis Properties of Complex Exponentials*. Trans. Amer. Math. Soc. **277** (1983), 93–111.

[S] Seip K., *On the Connection between Exponential Bases and Certain Related Sequence in* $L^2[-\pi, \pi]$. Journal of Functional Analysis **130** (1995), 131–160.

[Y1] Young R.M., *On complete biorthogonal systems*. Proc. Amer. Math. Soc. (3) **83** (1981), 537–540.

[Y2] Young R.M., *On a Theorem of Ingham on Nonharmonic Fourier Series*. Proc. Amer. Math. Soc. **92** (1984), 549–553.

[Y3] Young R.M., *On the Stability of Exponential Bases in* $L^2[-\pi, \pi]$. Proc. Amer. Math. Soc. **100** (1987), 117–122.

[Y4] Young R.M., *An Introduction to Nonharmonic Fourier Series*. revised first edition, Academic Press, 2001.

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