Hokkaido Mathematical Journal Vol. 33 (2004) p. 255–273

Tangential index of foliations with curves on surfaces

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(Received April 16, 1997; Revised October 21, 2003)

Abstract. In [Br1] M. Brunella defines an index which represents how a curve and a foliation on a complex surface intersect. In this article we give an alternative proof of the index theorem, calculate the index in some cases and consider the behavior of the indices under blowing-ups.

Key words: index, holomorphic foliation, blowing-up.

A dimension on singular foliation on a complex surface is locally defined by a holomorphic vector field and its singular set is defined as the union of the zero set of the vector fields defining the foliation. For singular foliations there exist several kinds of indices, the Poincaré-Hopf index (more generally the Baum-Bott indices [BB]), [CLS], the Camacho-Sad index [CS], [KS], [LS], [Ln], [Su1], [Br2] and the GSV index [GSV], [LSS], [SS]. The last two of these indices are relative not only to foliations but also to curves in the ambient surface. When we consider these relative indices, we assume that the curve C is invariant by a foliation $\mathcal{F}$; $v(f) \in (f)$, where v is a generator of $\mathcal{F}$, f a defining function of C, and (f) the ideal generated by f. In this article, however, we assume that C is *not* invariant; $v(f) \notin (f)$.

Assuming the condition, $v(f) \notin (f)$, we consider the relation between the foliation $\mathcal{F}$ and the curve C. At general points of C, the leaves of $\mathcal{F}$ are transverse to C, but there exist some points where the tangent space of C coincides with the direction of the leaf. So we define the tangency set as the union of the usual singular sets of $\mathcal{F}$ and C and the set of tangent points of $\mathcal{F}$ to C, and we consider an index at each of these points which represents the degree of tangency and study it.

In Section 1, we recall some basic definitions about singular foliations on surfaces, define our index and prove the index formula. This index formula was in fact proved by M. Brunella [Br1]. Here we give an alternative proof of this formula by the method of localization of the Chern class of suitable (virtual) bundle. Note that the index theorems of the other indices can be

1991 Mathematics Subject Classification : Primary 32L30, 32S65, 14B05, 14H20; Secondary 57R20, 58F18.

proved by this method, for examples, [KS], [LS] for Camacho-Sad index and [LSS] for GSV index. In Section 2, we calculate the index in some simple cases using Puiseux parametrizations. In Section 3 we consider the blowing-ups of the singularity. We describe explicitly how the index behaves under blowing-ups. A work on the generalization to the higher dimensional case of the results of this article is in progress.

The author would like to thank Tatsuo Suwa and the referees for many helpful comments.

1. Definitions and index formula

Let X be a complex manifold of dimension two. A dimension one singular foliation $\mathcal{F}$ on X is determined by a triple $(\{U_\alpha\}, v_\alpha, e_{\alpha\beta})$ such that

(1) $\{U_\alpha\}$ is an open covering of X and, for each α, v_α is a holomorphic vector field on U_α,

(2) for each pair (α, β), $e_{\alpha\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$ which satisfies the cocycle condition, $e_{\alpha\gamma} = e_{\alpha\beta}e_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$,

(3) $v_\beta = e_{\alpha\beta}v_\alpha$ on $U_\alpha \cap U_\beta$.

We denote the zero set of v_α by $S(v_\alpha); S(v_\alpha) = \{p \in U_\alpha \mid v_\alpha(p) = 0\}$. Since $e_{\alpha\beta}$ is non-vanishing, $S(v_\alpha)$ coincides $S(v_\beta)$ on $U_\alpha \cap U_\beta$. So the union $\bigcup_\alpha S(v_\alpha)$ is well-defined and called the singular set of the foliation $\mathcal{F}$. It is denoted by $S(\mathcal{F})$. If the singular set $S(\mathcal{F})$ consists of only isolated points, the foliation $\mathcal{F}$ is said to be reduced. Since the system $\{e_{\alpha\beta}\}$ satisfies the cocycle condition, it defines a line bundle F on X, which is called the tangent bundle of the foliation $\mathcal{F}$. Hereafter we consider only reduced foliations on X.

Let C be an analytic curve on X with defining function f_α on U_α, where $\{U_\alpha\}$ is an open covering of X. A curve is said to be reduced if its defining function on each U_α has no multiple factors. In this article we consider only reduced curves. Put $f_{\alpha\beta} = f_\alpha/f_\beta$. Then $f_{\alpha\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$ from the reducibility of C. Since the system $\{f_{\alpha\beta}\}$ of non-vanishing holomorphic functions satisfies the cocycle condition, $f_{\alpha\gamma} = f_{\alpha\beta}f_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, it defines a line bundle N on X. The restriction of N to the regular part of C is isomorphic to the normal bundle of the regular part of C; $N|_{C-\mathrm{Sing}(C)} \cong N_{C-\mathrm{Sing}(C)}$. Thus we call N a holomorphic extension of the normal bundle of the regular part of C, or simply the *normal bundle of* C.

Definition 1.1 Let v be a vector field on an open set U of X, C an analytic curve on U defined by a holomorphic function f and $p \in U \cap C$. We assume that (each component of) C is not a separatrix at p of v, or equivalently $v(f)$ is not an element of the ideal (f) of $\mathcal{O}_p$, where $\mathcal{O}_p$ is the ring of germs of holomorphic functions defined at p. We say that v is tangent to C at p if $v(f)(p) = 0$, and in this case, we define the tangential index $I_p(v, C)$ of v with C at p to be the intersection number of f and $v(f)$ at p;

$$I_p(v, C) = (f, v(f))_p = \dim_{\mathbb{C}} \mathcal{O}_p / \big(f, v(f)\big).$$

Remark 1.2 This index was defined by C. Camacho, A. Lins Neto and P. Sad [CLS] and A. Lins Neto [Ln] when the curve C is smooth. The general definition in the above is due to M. Brunella [Br1].

Remark 1.3 The local intersection number $(C, D)_p$ at p of two curves C and D defined by holomorphic functions f and g, respectively, is represented by the following integrals (See [GH]) Chapter 5 or [Su1]);

$$(C, D)_p = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{df \wedge dg}{fg} = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{dg}{g},$$

where Γ is the 2-cycle $\{|f| = |g| = \varepsilon\}$ with the orientation $d \arg f \wedge d \arg g > 0$ and L is the link of C at p with the orientation induced by the orientation of C; $L = \{f = 0,\ |g| = \varepsilon\}$ for a sufficiently small $\varepsilon > 0$. We use this representation to prove the index formula, which is stated in Theorem 1.6 below.

Note that, if C is irreducible at p and if $\pi : \Delta \to X$ is a parametrization of C near p, then it can also be computed as

$$(C, D)_p = \operatorname{ord}_0 \pi^* g.$$

For $p \in C \setminus \operatorname{Sing}(C)$, $v(f)(p) = 0$ is equivalent to saying that $v(p) \in T_pC$ and this index $I_p(v, C)$ is zero if the integral curve of v and the curve C are transverse at p. So we can consider that this index represents the degree of tangency between C and v at p.

Similarly this index can be defined for foliations by using generators of foliations on each open set as follows.

Definition 1.4 Let $\mathcal{F} = (\{U_\alpha\}, v_\alpha, e_{\alpha\beta})$ be a foliation and C an analytic curve on X defined by a holomorphic function f_α on U_α for each α. We denote the set of tangent points of $\mathcal{F}$ to C on U_α by T_α; $T_\alpha = \{p \in C \cap U_\alpha \mid v_\alpha(f_\alpha)(p) = 0\}$. We assume that C is not a separatrix of $\mathcal{F}$ at $p \in U_\alpha \cap C$. Then we define the tangential index $I_\alpha(\mathcal{F}, C)$ of $\mathcal{F}$ with C by

$$I_p(\mathcal{F}, C) = I_p(v_\alpha, C),$$

and the tangency set $T(\mathcal{F}, C)$ or simply T of $\mathcal{F}$ to C by the union of T_α.

Proposition 1.5 *$I_p(\mathcal{F}, C)$ and $T(\mathcal{F}, C)$ are independent of the choice of generators.*

Proof. On $U_\alpha \cap U_\beta$,

$$\begin{aligned} v_\alpha(f_\alpha) &= e_{\beta\alpha} v_\beta(f_{\alpha\beta} f_\beta) \\ &= e_{\beta\alpha}\{v_\beta(f_{\alpha\beta}) f_\beta + f_{\alpha\beta} v_\beta(f_\beta)\}, \end{aligned}$$

where $f_{\alpha\beta} = f_\alpha / f_\beta$. Hence we have

$$v_\alpha(f_\alpha) = e_{\beta\alpha} f_{\alpha\beta} v_\beta(f_\beta) \quad \text{on} \quad C \cap U_\alpha \cap U_\beta.$$

Since $e_{\beta\alpha} f_{\alpha\beta}$ is a non-vanishing function, the definition of index does not depend on the choice of generators and we have $T_\alpha \cap U_\beta = T_\beta \cap U_\alpha$. So the union $\bigcup_\alpha T_\alpha$ is well-defined. □

We assume T consists of only isolated points in this article.

We have the following index formula, which is proved by M. Brunella in [Br1] when the ambient space X is compact. Here we give an alternative proof, which uses only the compactness of C, but not of X. We write $\int_C c_1(L)$ as $C \cdot L$ for a line bundle L and a compact curve C.

Theorem 1.6 *Let C be an analytic curve which is not invariant by the foliation $\mathcal{F}$. If C is compact then*

$$\sum_{p \in T} I_p(\mathcal{F}, C) = C^2 - C \cdot \mathcal{F},$$

where F is the tangent bundle of $\mathcal{F}$.

Proof. Let N denote the normal bundle of C and consider the first Chern class $c_1(N \otimes F^*)$ of the line bundle $N \otimes F^*$ on X. Put $\mathcal{F} = (\{U_\alpha\}, v_\alpha, e_{\alpha\beta})$, $T = \{p_1, p_2, \ldots, p_r\}$ and suppose C is defined by f_α on U_α, $T \cap U_{\alpha_i} = \{p_i\}$, and $T \cap U_\alpha = \emptyset$ if $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$ such that $D_{\alpha_i} = \{p \in U_{\alpha_i} \mid \rho_{\alpha_i}(p) = 1\}$ is a closed ball with a sufficiently small radius around p_i for each $p_i \in T$.

Since the line bundle $N \otimes F^*$ is defined by the cocycle $\{f_{\alpha\beta}e_{\beta\alpha}\}$ and on $U_\alpha \cap U_\beta \cap C$

$$\begin{aligned} v_\alpha &= f_{\alpha\beta}e_{\beta\alpha}v_\beta(f_\beta) \\ d\log(f_{\alpha\beta}e_{\beta\alpha}) &= \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)} - \frac{dv_\beta(f_\beta)}{v_\beta(f_\beta)}, \end{aligned}$$

$c_1(N \otimes F^*)$ is represented by the following differential forms;

$$\begin{aligned} c_1(N \otimes F^*)|_{U_\alpha \cap C} &= \frac{\sqrt{-1}}{2\pi}\sum_\beta d\big(\rho_\beta d\log(f_{\beta\alpha}e_{\alpha\beta})\big) \\ &= \frac{\sqrt{-1}}{2\pi}\sum_\beta d\rho_\beta \wedge \left(\frac{dv_\beta(f_\beta)}{v_\beta(f_\beta)} - \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)}\right), \\ &= \frac{\sqrt{-1}}{2\pi}\left(\sum_\beta d\rho_\beta \wedge \frac{dv_\beta(f_\beta)}{v_\beta(f_\beta)} - d\sum_\beta \rho_\beta \wedge \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)}\right) \\ &= \frac{\sqrt{-1}}{2\pi}\sum_\beta d\rho_\beta \wedge \frac{dv_\beta(f_\beta)}{v_\beta(f_\beta)}. \end{aligned}$$

When $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_r\}$, $C \cap U_\alpha$ is non-singular. So we can take a coordinate system (x_α, y_α) on U_α such that $C \cap U_\alpha = \{p \in U_\alpha \mid y_\alpha(p) = 0\}$, or equivalently $f_\alpha = y_\alpha$. Then we can assume $v_\alpha(y_\alpha) = 1$ since $v_\alpha(y_\alpha)$ is non-vanishing. Thus

$$c_1(N \otimes F^*)|_{U_{\alpha_i} \cap C} = d\rho_{\alpha_i} \wedge \frac{dv_{\alpha_i}(f_{\alpha_i})}{v_{\alpha_i}(f_{\alpha_i})}.$$

Note that $c_1(N \otimes F^*)|_{U_\alpha \cap C} = 0$ if $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. Therefore

$$\int_C c_1(N \otimes F^*) = \frac{\sqrt{-1}}{2\pi}\sum_{i=1}^r \int_{U_{\alpha_i} \cap C} d\rho_{\alpha_i} \wedge \frac{dv_{\alpha_i}(f_{\alpha_i})}{v_{\alpha_i}(f_{\alpha_i})}.$$

Omit the suffix i for simplicity,

$$\int_{U_\alpha \cap C} d\rho_\alpha \wedge \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)} = \int_{(U_\alpha - D) \cap C} d\rho_\alpha \wedge \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)}$$

$$
\begin{aligned}
&= \int_{(U_\alpha - D)\cap C} d\Big(\rho_\alpha \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)}\Big) \\
&= \int_{\partial\{(U_\alpha - D)\cap C\}} \rho_\alpha \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)} \\
&= \int_{-L} \rho_\alpha \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)} + \int_{\partial\{(U_\alpha\cap C)\}} \rho_\alpha \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)} \\
&= - \int_{-L} \frac{dv_\alpha(f_\alpha)}{v_\alpha(f_\alpha)},
\end{aligned}
$$

where L is the link of C at p. Hence we get

$$\int_C c_1(N \otimes F^*) = \sum_{p\in T} I_p(\mathcal{F}, C)$$

On the other hand,

$$\int_C c_1(N \otimes F^*) = C^2 - F \cdot C.$$

This completes the proof. □

In [Br1] this formula is used to deduce some inequalities among Chern numbers and in [Ln] to define the degree of a foliation on the complex projective plane.

A singular foliation can be also defined in terms of differential forms as follows. A codimension on singular foliation is a triple $\mathcal{E} = (\{U_\alpha\}, w_\alpha, g_{\alpha\beta})$ such that

(1) $\{U_\alpha\}$ is an open covering of X and, for each α, ω_α is a holomorphic 1-form on U_α,

(2) for each pair (α, β), $g_{\alpha\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$ which satisfies the cocycle condition, $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$,

(3) $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on $U_\alpha \cap U_\beta$.

The singular set of $\mathcal{E}$ is the union of the zeros of ω_α similarly to the case of vector fields. A foliation $\mathcal{E}$ is said to be reduced if the singular set consists of only isolated points. The cocycle $\{g_{\alpha\beta}\}$ defines a line bundle, which is called the conormal bundle of $\mathcal{E}$ and is denoted by E.

The above definition is equivalent to the previous definition, as long as we consider only reduced foliations, in the sense that there is a natural one-to-one correspondence between the reduced dimension one foliations and the reduced codimension one foliations ([Su2]). In fact, let $\mathcal{F} = (\{U_\alpha\}, v_\alpha, e_{\alpha\beta})$ be a reduced dimension one foliation, then the correspondence is given by assigning $\mathcal{F}$ to the reduced codimension one foliation $\mathcal{E} = (\{U_\alpha\}, \omega_\alpha, g_{\alpha\beta})$ with $\langle v_\alpha, \omega_\alpha \rangle = 0$ on each U_α and vice versa. Note that in this correspondence we have $S(\mathcal{F}) = S(\mathcal{E})$ and the integral curves of the vector field v_α coincide with the solutions of the differential equation $\omega_\alpha = 0$ on each U_α.

Let K_X be the canonical bundle of the ambient space X. Calculating the systems of transition functions of E, F and K_X, we get $E = K_X \otimes F$. (See [Br1], [HS].) Then

$$C^2 - F \cdot C = C^2 + K_X \cdot C - E \cdot C.$$

From the adjunction formula ([K]), we have

$$C^2 - F \cdot C = \sum_{p \in T} \mu_p(C) - \chi(C) - E \cdot C,$$

where $\mu_p(C)$ is the Milnor number of C at p and $\chi(C)$ is the Euler-Poincaré characteristic of C.

Corollary 1.7 *In the above situation,*

$$\sum_{p \in T} \big(I_p(\mathcal{F}, C) - \mu_p(C)\big) = -\chi(C) - E \cdot C.$$

2. Values of tangential indices in some situations

We can calculate the tangential index $I_p(v, C)$ explicitly in some simple situations, namely, when the vector field v is non-singular at p or v is *simple* at p.

Proposition 2.1 *Let v be a vector field near $o \in \mathbf{C}^2$ and C a curve near 0. If 0 is a tangent point and v is non-singular at 0, then*

$$I_0(v, C) = (L, C)_0 + \mu_0(C) - 1,$$

where L is the integral curve of v through 0, $\mu_0(C)$ is the Milnor number of C at 0 and $(L, C)_0$ is the intersection number of L and C at 0.

Proof. At first, assume that C is irreducible at 0. Since the vector field v is non-singular at 0, the Frobenius Theorem allows us to take the coordinate system (x, y) near 0 such that $v = \partial/\partial x$. Let f be a defining function of C near 0. Note that L is defined by y in this coordinate system, and we can assume that f and $\partial f/\partial x$ are relatively prime at 0, i.e. there exist no non-trivial common factors, since f is reduced and regular in y; $f(x,\,0)\not\equiv 0$. Thus

$$I_0(v,\,C)=\big(f,\,v(f)\big)_0=\left(f,\,\frac{\partial f}{\partial x}\right)_0.$$

For a sufficiently small $\varepsilon>0$, put $\Delta=\{t\in\mathbf{C}\mid |t|<\varepsilon\}$. Let $\pi\colon\Delta\to\mathbf{C}^2$ be the Puiseux parametrization of C such that $\pi(0)=\big(x(0),\,y(0)\big)=0$. Then

$$\begin{aligned}I_0(v,\,C)=&\mathrm{ord}_0\left(\frac{\partial f}{\partial x}\big(\pi(t)\big)\right)+\mathrm{ord}_0\,\frac{dx}{dt}-\mathrm{ord}_0\,\frac{dx}{dt}\\=&\mathrm{ord}_0\left(\frac{\partial f}{\partial x}\big(\pi(t)\big)\frac{dx}{dt}\right)-(x,\,f)_0+1.\end{aligned}$$

Now we have $f(\pi(t))=0$ for any $t\in\Delta$ since $\pi(t)=(x(t),\,y(t))$ is the parametrization of C. So

$$0=\frac{df}{dt}\big(\pi(t)\big)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Therefore we have

$$\begin{aligned}I_0(v,\,C)=&\mathrm{ord}_0\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right)-(x,\,f)_0+1\\=&\mathrm{ord}_0\,\frac{\partial f}{\partial y}(x(t),\,y(t))_0+\mathrm{ord}_0\,\frac{dy}{dt}-(x,\,f)_0+1\\=&\left(f,\,\frac{\partial f}{\partial y}\right)_0-(x,\,f)_0+(L,\,C)_0.\end{aligned}$$

On the other hand, from [Ln], we have

$$\mu_0(C)=\left(f,\,\frac{\partial f}{\partial y}\right)_0-(x,\,f)_0+1$$

Consequently we have

$$I_0(v,\,C)=(L,\,C)_0+\mu_0(C)-1,$$

when C is irreducible at 0.

Next we probe the general case. Let $f=f_1f_2\cdots f_k$ be the irreducible decomposition of f at 0 and C_j the curve defined by each f_j.

$$I_0(v,\,C)=\big(f_1f_2\cdots f_k,v(f_1f_2\cdots f_k)\big)_0$$

$$
\begin{aligned}
&=\sum_{i=1}^{k}\Big(f_i,\ \sum_{j=1}^{k} f_1\cdots\hat{f_j}\cdots f_k v(f_j)\Big)_0\\
&=\sum_{i=1}^{k}\big(f_i,\ f_1\cdots\hat{f_i}\cdots f_k v(f_i)\big)_0\\
&=\sum_{i=1}^{k}\Big(\sum_{j\neq i}(C_i,\ C_j)+\big(f_i,\ v(f_j)\big)\Big)_0\\
&=2\sum_{1\le i<j\le k}(C_i\,C_j)_0+\sum_{j=1}^{k}\big((L,\ C_j)+\mu_0(C_j)-1\big)\\
&=(L,\ C)_0+\sum_{j=1}^{k}\mu_0(C_j)+2\sum_{1\le i<j\le k}(C_i\,C_j)_0-k.
\end{aligned}
$$

On the other hand, from [Su1], we have

$$\mu_0(C)=\sum_{i=1}^{k}\mu_0(C_i)+2\sum_{1\le i<j\le k}(C_i,\ C_j)_0-k+1.$$

Therefore in general

$$I_0(v,\ C)=(L,\ C)_p+\mu_0(C)-1.$$

□

Proposition 2.2 *Let $v=\lambda x\partial/\partial x+\mu y\partial/\partial y$ be a vector field near the origin $0\in\mathbf{C}^2$ such that λ/μ is not a positive rational number, S_1 and S_2 the y-axis and x-axis, respectively, C a curve irreducible at 0 such that the intersection numbers $(C,\ S_1)_0$ and $(C,\ S_2)_0$ are relatively prime. Then we have*

$$I_0(v,\ C)=(C,\ S_1)_0\cdot(C,\ S_2)_0.$$

Proof. Put $k=(C,\ S_1)_0$ and $l=(C,\ S_2)_0$. From the Weierstrass preparation theorem there exist a holomorphic function $u\in\mathcal{O}_0$ and a Weierstrass polynomial $w(x,\ y)\in\mathbf{C}\{x\}[y]$ such that

$$
\begin{aligned}
f(x,\ y)&=u(x,\ y)w(x,\ y),\quad u(0,\ 0)\neq 0\\
w(x,\ y)&=y^k+a_1(x)y^{k-1}+\cdots+a_{k-1}(x)y+a_k(x)\\
&\qquad \operatorname{ord}_0 a_i(x)>i\quad\text{for each } i.
\end{aligned}
$$

We may assume that $f=w$ without loss of generality. Then

$$v(f)=\lambda x\sum_{i=1}^{k-1}a_i'(x)y^{k-1}+\lambda x a_k'(x)+\mu k y^k+\mu\sum_{i=1}^{k-1}(k-i)a_i(x)y^{k-i}.$$

Setting $\mathrm{ord}_0\, a_i(x)=l_i$ and $a_i(x)=a_{il_i}x^{l_i}+$ higher order terms, and substituting this, we have

$$\begin{aligned}v(f)=&\lambda\alpha_{kl}lx^l+\text{higher order terms in } x\\ &+\mu k y^k\\ &+\sum_{i=1}^{k-1}\big(\lambda\alpha_{il_i}l_i x^{l_i}y^{k-i}+(\text{higher order terms in } x)y^{k-i}\big)\\ &+\sum_{i=1}^{k-1}\big(\mu(k-i)\alpha_{il_i}x^{l_i}y^{k-i}+(\text{higher order terms in } x)y^{k-i}\big).\end{aligned}$$

Note that $l_k=l$.

On the other hand, let $\pi(t)=\big(x(t),\, y(t)\big)$ be the Puiseux parametrization of C at 0 such that $x(t)=t^k$ and $y(t)=t^l\tilde{y}(t)$, $\tilde{y}(0)\neq 0$. Put $\beta=\tilde{y}(0)$. Then we have

$$\begin{aligned}f(\pi(t))=&\beta^k t^{kl}+\text{higher order terms}\\ &+\sum_{i=1}^{k-1}\big(\alpha_{il_i}\beta^{k-i}t^{kl_i+l(k-i)}+\text{higher order terms}\big)\\ &+\alpha_{kl}t^{kl}+\text{higher order terms.}\end{aligned}$$

Now we consider the Newton polygon of f. The edge of this polygon is a line $x/l+y/k-1=0$. A point $(l_i,\, k-i)$ is in the interior of the polygon. Hence $l_i/l+(k-i)/k-1>0$ i.e. $kl_i+l(k-i)>kl$. Thus we get

$$f(\pi(t))=(\beta^k+\alpha_{kl})t^{kl}+\text{higher order terms.}$$

Since $f(\pi(t))\equiv 0$, we have $\beta^k=-\alpha_{kl}$. Therefore

$$\begin{aligned}v(f)(\pi(t))=&(\lambda\alpha_{kl}l+\mu k\beta^k)t^{kl}+\text{higher order terms}\\ =&\alpha_{kl}(\lambda l-\mu k)t^{kl}+\text{higher order terms.}\end{aligned}$$

Recall that $\alpha_{kl}\neq 0$ and μ/λ is not rational. Thus we get

$$I_0(v,\, C)=\mathrm{ord}_0\, v(f)(\pi(t))$$

$$
\begin{aligned}
&= kl \\
&= (C,\, S_1)_0 \cdot (C,\, S_2)_0.
\end{aligned}
$$

□

Remark 2.3 In Proposition 2.2, the assumption that $(C, S_1)_0$ and $(C, S_2)_0$ are relatively prime is necessary. In fact, let C be the curve defined by

$$w(x,\, y) = y^6 - 2(1+x)x^5y^3 + x^{10}.$$

Then it can be shown that it is irreducible at 0. However, we have $(C,\, S_1)_0 = 6$ and $(C,\, S_2)_0 = 10$. Let v be a vector field as in the proposition. Then, using the parametrization

$$
\begin{cases}
x = t^6, \\
y = t^{10}\left(1 + \dfrac{\sqrt{2}}{3}t^3 + \dfrac{1}{9}t^6 + \cdots\right)
\end{cases}
$$

of C, we compute $I_0(v,\, C) = 63 \neq (C,\, S_1) \cdot (C,\, S_2)_0$.

The author would like to thank the referee for pointing out an error in the previous version of this paper and for providing this example.

In the rest of this section, we consider singular foliations defined by meromorphic functions, following [HS].

Let φ be a meromorphic function on X, $\{U_\alpha\}$ a coordinate covering of X on which the differential $d\varphi$ of φ is written as $d\varphi = \varphi_\alpha \omega_\alpha$, where φ_α is a meromorphic function on U_α and ω_α is a holomorphic 1-form on U_α with isolated zeros. Then the system $(\{U_\alpha\},\, \omega_\alpha,\, \varphi_\alpha/\varphi_\beta)$ defines a codimension one reduced foliation $\mathcal{E}$ on X and its conormal bundle E is defined by the cocycle $\{\varphi_\alpha/\varphi_\beta\}$. We denote the zero and pole divisor of φ by $D^{(0)}$ and $D^{(\infty)}$, respectively. Let $D^{(\infty)} = \sum_{i=1}^{r} m_i D_i^{(\infty)}$ be the irreducible decomposition of $D^{(\infty)}$ with positive integers m_i. For a divisor D on X, we denote the support and associated line bundle of D by $|D|$ and $[D]$, respectively.

We assume that the critical points of φ in $X \setminus |D^{(\infty)}|$ are all isolated through this section. Under this assumption we have

$$E = \left[-\sum_{i=1}^{r}(m_i + 1)D_i^{(\infty)}\right],$$

and the foliation $\mathcal{E}$ is generated by

$$\omega = g_1 \cdots g_r dh - h\left(\sum_{i=1}^{r} m_i g_1 \cdots \hat{g}_i \cdots g_r dg_i\right)$$

near $p \in |D^{(\infty)}|$, where $\varphi = h/g$ and $g = g_1^{m_1} \cdots g_r^{m_r}$ is the irreducible decomposition of g. (See [HS] Lemma (2.1)).

Definition 2.4 Let C be a reduced curve in the ambient space X and ψ a holomorphic function near $p \in C$. Assume that ψ is not constant on C. Then we define an integer $\mu_p(\psi, C)$ by $(L, C)_p - 1$, where L is the level set of ψ through p;

$$\mu_p(\psi, C) = (L, C)_p - 1.$$

Lemma 2.5 *Let $C = C_1 \cup \cdots \cup C_k$ is the irreducible decomposition of C at $p \in C$. Then we have*

$$\mu_p(\psi, C) = \sum_{i=1}^{k} \mu_p(\psi, C_i) + k - 1.$$

Proof.

$$\begin{aligned}\mu_p(\psi, C) &= (L, C)_p - 1 \\ &= \sum_{i=1}^{k} (L, C_i)_p - 1 \\ &= \sum_{i=1}^{k} \mu_p(\psi, C_i) + k - 1\end{aligned}$$

□

Proposition 2.6 *Let C be a reduced curve in X. Assume that φ is not constant on C. If $p \in C \cap (X \backslash |D^{(\infty)}|)$, we have*

$$I_p(\mathcal{F}, C) = \mu_p(\varphi, C) + \mu_p(C),$$

where $\mathcal{F}$ is the annihilator of $\mathcal{E}$.

Proof. The foliation $\mathcal{E}$ is defined by $d\varphi$ near p. Hence, if (x, y) is a coordinate system near p such that $p = (0, 0)$, $\mathcal{F}$ is defined by $v = (\partial\varphi/\partial y)\partial/\partial x - (\partial\varphi/\partial x)\partial/\partial y$.

At first we assume that C is irreducible at p. Let $\pi(t) = (x(t), y(t))$ be the Puiseux parametrization of C at p and f a defining function of C.

Similarly to the proof of Proposition (2.1), we have

$$\begin{aligned}
I_p(\mathcal{F}, C) =& \big(f, v(f)\big)_p = \left(f, \frac{\partial\varphi}{\partial y}\frac{\partial f}{\partial x} - \frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial y}\right)_p \\
=& \operatorname{ord}_0\left(\frac{\partial\varphi}{\partial y}\frac{\partial f}{\partial x} - \frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial y}\right)\big(\pi(t)\big) \\
=& \operatorname{ord}_0\left(\frac{\partial\varphi}{\partial y}\frac{\partial f}{\partial x}\frac{dx}{dt} - \frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial y}\frac{dx}{dt}\right) - \operatorname{ord}_0\frac{dx}{dt} \\
=& \operatorname{ord}_0\left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt}\right) + \operatorname{ord}_0\frac{\partial f}{\partial y}\big(\pi(t)\big) - \operatorname{ord}_0\frac{dx}{dt} \\
=& \operatorname{ord}_0\frac{d}{dt}\varphi\big(\pi(t)\big) + \left(f, \frac{\partial f}{\partial y}\right)_p - (x, f)_p + 1 \\
=& \mu_p(\varphi, C) + \mu_p(C).
\end{aligned}$$

Now we consider the case where C is reducible at p. Let $C = C_1 \cup \cdots C_k$ be the irreducible decomposition of C at p. From the proof of Proposition (2.1) and Lemma (2.5), we have

$$\begin{aligned}
I_p(\mathcal{F}, C) =& \sum_{j\neq i}(C_i, C_j)_p + \sum_{i=1}^{k} I_p(\mathcal{F}, C_i) \\
=& \sum_{j\neq i}(C_i, C_j)_p + \sum_{i=1}^{k}\mu_p(C_i) + \sum_{i=1}^{k}\mu_p(\varphi, C_i) \\
=& \mu_p(C) + k - 1 + \sum_{i=1}^{k}\mu_p(\varphi, C_i) \\
=& \mu_p(C) + \mu_p(\varphi, C).
\end{aligned}$$

□

Proposition 2.7 *Let C be a reduced curve on which φ is not constant. If $p \in C \cap |D^{(\infty)}|$, then*

$$I_p(\mathcal{F}, C) = (D^{(0)}, C)_p + \sum_{i=1}^{r}(D_i^{(\infty)}, C)_p + \mu_p(C) - 1$$

Proof. The foliation $\mathcal{F}$ is defined by $v = a\partial/\partial x + b\partial/\partial y$ near p, where

$$a = g_1 \cdots g_r \frac{\partial h}{\partial y} - \sum_{i=1}^{r} m_i h g_1 \cdots \hat{g}_i \cdots g_r \frac{\partial g_i}{\partial y}$$
$$b = -\left(g_1 \cdots g_r \frac{\partial h}{\partial y} - \sum_{i=1}^{r} m_i h g_1 \cdots \hat{g}_i \cdots g_r \frac{\partial g_i}{\partial y} \right).$$

Similarly to the previous proposition, at first, we assume that C is irreducible at p and let f be a defining function of C, (x, y) a coordinate system near p such that $p = (0, 0)$ and $\pi(t) = (x(t), y(t))$ the Puiseux parametrization of C.

$$\begin{aligned}
I_p(\mathcal{F}, C) &= \operatorname{ord}_0 \left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} \right) (\pi(t)) \\
&= \operatorname{ord}_0 \left(a\frac{\partial f}{\partial x}\frac{dx}{dt} + b\frac{\partial f}{\partial y}\frac{dx}{dt} \right) (\pi(t)) - \operatorname{ord}_0 \frac{dx}{dt} \\
&= \operatorname{ord}_0 \left(a\frac{dy}{dt} - b\frac{dx}{dt} \right) (\pi(t)) + \left(f, \frac{\partial f}{\partial y} \right)_p + (x, f)_p - 1 \\
&= \operatorname{ord}_0 \left(\frac{d}{dt}\varphi(\pi(t)) \cdot g_1^{m_1+1}(\pi(t)) \cdots g_r^{m_r+1}(\pi(t)) \right) + \mu_p(C) \\
&= (D^{(0)}, C)_p + \sum_{i=1}^{r} (D_i^{(\infty)}, C)_p + \mu_p(C) - 1.
\end{aligned}$$

We consider the case where C is not irreducible at p. Let $C = C_1 \cup \cdots \cup C_k$ be the irreducible decomposition of C at p. Then we have

$$\begin{aligned}
I_p(\mathcal{F}, C) &= \sum_{j \neq i} (C_i, C_j)_p + \sum_{i=1}^{k} I_p(\mathcal{F}, C_i) \\
&= (D^{(0)}, C)_p + \sum_{i=1}^{r} (D_i^{(\infty)}, C)_p + \sum_{i=1}^{k} \mu_p(C_i) \\
&\qquad + \sum_{j \neq i} (C_i, C_j)_p - k \\
&= (D^{(0)}, C)_p + \sum_{i=1}^{r} (D_i^{(\infty)}, C)_p + \mu_p(C) - 1.
\end{aligned}$$

$\square$

We denote

$$(D^{(0)},\, C)_p + \sum_{i=1}^{r} (D_i^{(\infty)},\, C)_p - 1$$

by $\mu_p(\varphi,\, C)$ for $p \in C \cap |D^{(\infty)}|$. From Corollary (1.7) we get the following.

Corollary 2.8 *If C is compact,*

$$\sum_{p \in T} \mu_p(\varphi,\, C) = -\chi(C) + \sum_{i=1}^{r} (m_i + 1) C \cdot D_i^{(\infty)}.$$

3. Invariant curves and the tangential index

Let v be a holomorphic vector field which defines a foliation $\mathcal{F}$ near $0 \in \mathbf{C}^2$ with an isolated singularity at 0 and C a smooth curve defined by $f = 0$ near $0 \in \mathbf{C}^2$;

$$\begin{aligned} v =& a(x,\, y)\frac{\partial}{\partial x} + b(x,\, y)\frac{\partial}{\partial y}, \\ a(x,\, y) =& \sum_{i+j \geq 1} a_{ij} x^i y^j, \quad b(x,\, y) = \sum_{i+j \geq 1} b_i j x^i y^j, \\ f(x,\, y) =& \alpha x + \beta y + \text{higher order terms.} \end{aligned}$$

We consider the tangential index $I_0(\mathcal{F},\, C)$.

We define the order $\mathrm{ord}_0(v)$ of a vector field v at 0 by

$$\mathrm{ord}_0(v) = \min\{\mathrm{ord}_0\, a,\, \mathrm{ord}_0\, b\}$$

and let $\varphi(t) = (\beta t,\, -\alpha t + \text{higher order terms})$ be the Puiseux parametrization of C. Put $n = \mathrm{ord}_0(v)$ and

$$\begin{aligned} I_0(v,\, C) =& \big(v(f), f\big)_0 \\ =& \mathrm{ord}_0\, v(f)\big(\varphi(t)\big) \\ =& \mathrm{ord}_0\Bigg(\Big\{\alpha \sum_{i+j=n} a_{ij}\beta^i(-\alpha)^j + \beta \sum_{i+j=n} b_{ij}\beta^i(-\alpha)^j\Big\} t^n \\ & \qquad + \text{higher order terms.}\Bigg) \end{aligned}$$

We take a homogeneous polynomial $P(x,\, y) = x b_n(x,\, y) - y a_n(x,\, y)$, where $a_n = \sum_{i+j=n} a_{ij} x^i y^j$ and $b_n \sum_{i+j=n} b_{ij} x^i y^j$. If the polynomial $P(x,\, y) \equiv$

0, then the singularity 0 of v is said to be dicritical and otherwise to be non-dicritical. (See also [CLS] or [CS].) If 0 is a non-dicritical singularity, the tangential index of v with a smooth curve C at the singularity 0 of v is equal to the order of v at 0 in general. However it is larger than the order if C is tangent to a line whose direction is given by the equation $P(x, y) = 0$.

Now consider the blowing up of a singular foliation $\mathcal{F}$ at the singularity 0. ([CS] or [CLS]) Let v be a holomorphic vector field which generates the foliation $\mathcal{F}$ and $\pi\colon \tilde{U} \to U$ the proper map of the blowing up at 0, where U is a neighborhood of 0. Let D be an exceptional divisor, ω an annihilator of v; $\omega = b(x, y)dx - a(x, y)dy$ and $\pi_1 = \pi|_{\tilde{U}_1}$, $\pi_2 = \pi|_{\tilde{U}_2}$ the restrictions of π to $\tilde{U}_1$ and $\tilde{U}_2$, respectively, where $\tilde{U}_1$ and $\tilde{U}_2$ are coordinate neighborhoods of $\tilde{U}$.

Since we can write $\pi_1\colon (x, \eta) \mapsto (x, x\eta)$ and $\pi_2\colon (\xi, y) \mapsto (\xi y, y)$, the pull backs of ω can be written as follows;

$$\begin{aligned}\pi_1^*\omega &= \{x^n P(1, \eta) + x^{n+1}R_1(x, \eta)\}dx - x^{n+1}Q_1(x, \eta)d\eta,\\ \pi_2^*\omega &= y^{n+1}Q_2(\xi, y)d\xi + \{y^n P(\xi, 1) + y^{n+1}R_2(\xi, y)\}dy.\end{aligned}$$

Then we define the blowing up $\tilde{\mathcal{F}}$ of $\mathcal{F}$ at 0 by the foliation generated by the following vector fields on $\tilde{U}$ if 0 is a non-dicritical singularity of v;

$$\begin{aligned}\tilde{v_1} &= xQ_1(x, \eta)\frac{\partial}{\partial x} + \big(P(1, \eta) + xR_1(x, \eta)\big)\frac{\partial}{\partial \eta},\\ \tilde{v_2} &= \big(P(\xi, 1) + yR_2(\xi, y)\big)\frac{\partial}{\partial \xi} - yQ_2(\xi, y)\frac{\partial}{\partial y}.\end{aligned}$$

If 0 is a dicritical singularity, then the generators of the blowing up $\tilde{\mathcal{F}}$ of $\mathcal{F}$ are

$$\begin{aligned}\tilde{v_1} &= Q_1(x, \eta)\frac{\partial}{\partial x} + R_1(x, \eta)\frac{\partial}{\partial \eta},\\ \tilde{v_2} &= R_2(\xi, y)\frac{\partial}{\partial \xi} - Q_2(\xi, y)\frac{\partial}{\partial y}.\end{aligned}$$

Note that $\tilde{\mathcal{F}}$ is reduced in both cases.

Assume that 0 is a non-dicritical singularity. The singularities of the blowing up $\tilde{\mathcal{F}}$ of $\mathcal{F}$ are given by the solution of the homogeneous polynomial $P(x, y) = 0$. These singularities are on the exceptional divisor and indicate the directions of the tangent cones of the invariant curves of $\mathcal{F}$ at 0. On the other hand the tangential index of $\mathcal{F}$ with smooth curve which has a tangent line with the direction given by $P(x, y) = 0$ are not general value. So we can say this index can catch the directions of invariant curve of $\mathcal{F}$.

Here is an explicit formula which describes how the tangential indices behave under blowing up.

Proposition 3.1 *Let C be an analytic curve of order m at 0 and $\mathcal{F}$ a foliation with a singularity 0. Then*

$$I_0(\mathcal{F}, C) = \sum_{\tilde{p} \in \tilde{T} \cap D} I_{\tilde{p}}(\tilde{\mathcal{F}}, \tilde{C}) + (m + \nu - 1)\tilde{C} \cdot D,$$

where $\tilde{C}$ is the proper transform of C, $\tilde{T}$ the set of tangent points of $\tilde{\mathcal{F}}$ to $\tilde{C}$ and $\nu = \operatorname{ord}_p v$ if p is dicritical, $\nu = (\operatorname{ord}_p v) + 1$ if p is non-dicritical.

Proof. Let E be the conormal bundle of the annihilator $\mathcal{E}$ of $\mathcal{F}$, $\tilde{\mathcal{E}}$ the blowing up of $\mathcal{E}$ at 0, $\tilde{E}$ the conormal bundle of $\tilde{\mathcal{E}}$ and w_i, which is an annihilator of v_i, a generator of $\tilde{\mathcal{E}}$ on $\tilde{U}_i$. Since $\pi_1^*\omega = \pi_2^*\omega$ on $\tilde{U}_1 \cap \tilde{U}_2$, $x^\nu \omega_1 = \pi_1^*\omega$ and $y^\nu \omega_2 = \pi_2^*\omega$, we have

$$\omega_1 = \left(\frac{y}{x}\right)^\nu \omega_2.$$

This means that $\pi^*E = [-\nu D] \otimes \tilde{E}$. Similarly we have $\pi^*N = [mD] \otimes \tilde{N}$, where $\tilde{N}$ is the normal bundle of $\tilde{C}$ in the sense of Section 1. Moreover $K_{\tilde{X}} = \pi^*K_X \otimes [\cdot \cdot D]$ (See [GH]). Note that $E = F \otimes K_X$. Therefore

$$\pi^*c_1(N \otimes F^*) = c_1(\tilde{N} \otimes \tilde{F}^*) + (m + \nu - 1)c_1(D).$$

From the proof of the index formula, we have

$$I_p(\mathcal{F}, C) = \int_{U \cap C} c_1(N \otimes F^*),$$

where $p \in C$, U is an open neighborhood of p, i.e. the tangential index is a localization of the Chern class $c_1(N \otimes F^*)$. Hence we have

$$\begin{aligned} I_0(\mathcal{F}, C) &= \int_{U \cap C} c_1(N \otimes F^*) \\ &= \int_{\tilde{U} \cap \tilde{C}} \pi^*c_1(N \otimes F^*) \\ &= \int_{\tilde{U} \cap \tilde{C}} c_1(\tilde{N} \otimes \tilde{F}^*) + (m + \nu - 1) \int_{\tilde{U} \cap \tilde{C}} c_1(D) \end{aligned}$$

$$= \sum_{\tilde{p} \in \tilde{T} \cap D} I_{\tilde{p}}(\tilde{\mathcal{F}}, \tilde{C}) + (m + \nu - 1)\tilde{C} \cdot D.$$

□

After the preparation of the manuscript, the author was informed of a preprint [Me]. In this paper, a fact similar to Proposition (3.1) is described under the assumption that the ambient space is compact.

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