Hokkaido Mathematical Journal Vol. 36 (2007) p. 535–562

Asymptotic behavior of positive solutions of $x'' = -t^{\alpha\lambda-2}x^{1+\alpha}$ with $\alpha < 0$ and $\lambda < -1$ or $\lambda > 0$

Ichiro TSUKAMOTO

(Received May 29, 2006)

Abstract. In this paper, we consider an initial value problem of the differential equation written in the title under an initial condition $x(T) = A$, $x'(T) = B$ ($0 < T < \infty$, $0 < A < \infty$, $-\infty < B < \infty$). In the case $\lambda > 0$, we conclude that if T and A are fixed arbitrarily, then there exists a number B_1 such that in every case of $B = B_1$, $B > B_1$, and $B < B_1$, we get analytical expressions of the solution of the initial value problem valid in the neighborhoods of the ends of the domain of the solution. Moreover we treat the case $\lambda < -1$. This case connects with the boundary layer theory of viscous fluids. The conclusions of this case are got directly from those of the case $\lambda > 0$. Finally we discuss the case $T = 0$ and $\lambda < -1$.

Key words: asymptotic behavior, an initial value problem, the analytical expressions, a first order rational differential equation, a two dimensional autonomous system.

1. Introduction

Let us consider a second order nonlinear differential equation

$$x'' = -t^{\alpha\lambda-2}x^{1+\alpha} \qquad \left(\ ' = \frac{d}{dt} \right) \tag{E}$$

in a domain

$$0 < t < \infty, \ 0 < x < \infty.$$

Here α and λ are parameters and

$$\alpha < 0, \ \lambda < -1 \quad \text{or} \quad \lambda > 0.$$

Notice that p^r ($p > 0$, $r \in \boldsymbol{R}$) takes its positive branch throughout this paper.

(E) is a useful equation to various fields. Indeed if $\alpha = -2$ and $\lambda = -3/2$ in particular, then (E) is easily deduced from a nonlinear singular

---

2000 Mathematics Subject Classification : Primary 34A12; Secondary 34A34.

boundary value problem

$$g(u)\frac{d^2g(u)}{du^2}+2u=0, \quad 0<u<1$$
$$\frac{dg(0)}{du}=0,\ g(1)=0$$

which expresses a boundary layer problem of viscous fluids called the Blasius problem (cf. [2] and [3]). Moreover in dynamics (E) is an equation of motion in a potential field and so Euler's equation of a variational problem. In the theory of partial differential equations, (E) is an equation which positive radial solutions of a nonlinear elliptic partial differential equation satisfy.

In many papers, (E) has been considered. Taking [7], [9], and [10] from references of [8] for example, solutions continuable to ∞ of a differential equation with more general form than (E) are treated there. However we have not solved an initial value problem of (E) and so shall consider this in the present paper. The initial condition is denoted as

$$x(T)=A, \quad x'(T)=B \tag{I}$$

where $0\le T<\infty$, $A>0$, and $B\in\boldsymbol{R}$. The case $0<T<\infty$ will be treated in Sections 1 through 5 and the case $T=0$, in Section 6 as a supplement. The method which we shall adopt was originally used in [12] and [13], and applied in [14] through [22]. Following this method, we shall first transform (E) into a first order rational differential equation and rewrite this as a two dimensional autonomous system. From considering these, we shall obtain analytical expressions of a solution of (E) valid in neighborhoods of ends of its domain. The analytical expressions just mentioned will show the asymptotic behavior of the solutions of the initial value problem (E) and (I).

2. Statement of our main conclusions

In Sections 1 through 5, we suppose that $T>0$ in the initial condition (I). Moreover, fix T and A arbitrarily and let $x(t)$ be a solution of an initial value problem (E) and (I).

First, suppose $\lambda>0$. Then we conclude the following:

Theorem 2.1 *There exists a number B_1 such that if $B = B_1$, then $x(t)$ is defined for (ω_-, ∞) $(0 < \omega_- < \infty)$ and has a representation*

$$x(t) = K\left(1 + \sum_{n=1}^{\infty} x_n t^{\alpha\lambda n}\right) \tag{2.1}$$

in the neighborhood of $t = \infty$. Here K and x_n are constants.

In the neighborhood of $t = \omega_-$, $x(t)$ has various representations depending on α as follows:

Theorem 2.2 *If $B = B_1$, then we get*

$$\begin{aligned} x(t) &= \Gamma(t-\omega_-) \\ &\times\left\{1 + \sum_{\ell+m+n>0} d_{\ell mn}(t-\omega_-)^{\ell}(t-\omega_-)^{-\alpha m/2}(t-\omega_-)^{(\alpha+2)n/2}\right\} \\ &\qquad\qquad\qquad\qquad \textit{if } -2 < \alpha < 0, \qquad (2.2) \\ x(t) &= \{\lambda(\lambda+1)\}^{-1/2} t^{-\lambda} U^{1-G(U,\,C)} e^{CG(U,\,C)} \quad \textit{if } \alpha = -2, \qquad (2.3) \end{aligned}$$

where

$$\begin{aligned} U &\sim \sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_-} \quad \textit{as } t \to \omega_-, \\ G(U,\,C) &= \frac{1}{2}(C-\log U)^{-1}\log(C-\log U) \\ &\quad + \sum_{\ell+m+n\ge 2} g_{\ell mn}\{U(C-\log U)^2\}^{\ell}(C-\log U)^{-m/2} \\ &\qquad \times \{(C-\log U)^{-1}\log(C-\log U)\}^n, \\ x(t) &= \left\{-\frac{2(\alpha+2)}{\alpha^2\omega_-^{\alpha\lambda-2}}\right\}^{1/\alpha}(t-\omega_-)^{-2/\alpha} \\ &\quad \times\left\{1 + \sum_{m+n>0} x_{mn}(t-\omega_-)^m(t-\omega_-)^{2(\alpha+2)n/\alpha}\right\} \\ &\qquad \textit{if } \alpha < -4 \quad \textit{or} \quad -4 < \alpha < -2, \qquad (2.4) \end{aligned}$$

and

$$\begin{aligned} x(t) &= \sqrt{\frac{2}{\omega_-^{2\lambda+1}}}(t-\omega_-)^{1/2} \\ &\quad \times\left\{1 + \sum_{m>0}(t-\omega_-)^m p_m(\log(t-\omega_-))\right\} \quad \textit{if } \alpha = -4 \qquad (2.5) \end{aligned}$$

in the neighborhood of $t = \omega_-$. *Here* Γ, C, $d_{\ell mn}$, $g_{\ell mn}$, *and* x_{mn} *are constants and* p_m *are polynomials whose degrees are not greater than* m.

In the case $B \neq B_1$, the following theorems are valid:

Theorem 2.3 *If* $B > B_1$, *then* $x(t)$ *is defined for* (ω_-, ∞) $(0 < \omega_- < \infty)$. *Moreover* $x(t)$ *has representations* (2.2) *through* (2.5) *in the neighborhood of* $t = \omega_-$ *and*

$$\begin{aligned} x(t) &= Kt\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha(\lambda+1)m-n}\right) \quad \text{if } -\frac{1}{\alpha(\lambda+1)} \notin \boldsymbol{N} \\ x(t) &= Kt\left\{1 + \sum_{k=1}^{\infty} t^{\alpha(\lambda+1)k} p_k(\log t)\right\} \quad \text{if } -\frac{1}{\alpha(\lambda+1)} \in \boldsymbol{N} \end{aligned} \tag{2.6}$$

in the neighborhood of $t = \infty$ *where* K *and* x_{mn} *are constants and* p_k *are polynomials whose degrees are not greater than* $[-\alpha(\lambda+1)k]$ ([ ] *denotes Gaussian symbol*).

Theorem 2.4 *If* $B < B_1$, *then* $x(t)$ *is defined for* (ω_-, ω_+) $(0 < \omega_- < \omega_+ < \infty)$. *Furthermore* $x(t)$ *has representations* (2.2) *through* (2.5) *in the neighborhood of* $t = \omega_-$ *and*

$$\begin{aligned} x(t) = \Gamma(\omega_+ - t)\Big\{1 + \sum_{\ell+m+n>0} & d_{\ell mn}(\omega_+ - t)^{\ell}(\omega_+ - t)^{-\alpha m/2} \\ & \times (\omega_+ - t)^{(\alpha+2)n/2}\Big\} \quad \text{if } -2 < \alpha < 0, \end{aligned} \tag{2.7}$$

$$x(t) = \{\lambda(\lambda+1)\}^{-1/2} t^{-\lambda} U^{1-G(U,C)} e^{CG(U,C)} \quad \text{if } \alpha = -2 \tag{2.8}$$

where

$$U \sim -\sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_+} \quad \text{as } t \to \omega_+$$

and $G(U, C)$ *has the same form as of* (2.3),

$$\begin{aligned} x(t) &= \left\{-\frac{2(\alpha+2)}{\alpha^2\omega_+^{\alpha\lambda-2}}\right\}^{1/\alpha}(\omega_+ - t)^{-2/\alpha} \\ &\quad \times \left\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m(\omega_+ - t)^{2(\alpha+2)n/\alpha}\right\} \\ &\quad \text{if } \alpha < -4 \quad \text{or} \quad -4 < \alpha < -2 \end{aligned} \tag{2.9}$$

and

$$x(t) = \sqrt{\frac{2}{\omega_+^{2\lambda+1}}}\,(\omega_+ - t)^{1/2} \times \left\{1 + \sum_{m>0} (\omega_+ - t)^m p_m(\log(\omega_+ - t))\right\} \quad \text{if } \alpha = -4 \quad (2.10)$$

in the neighborhood of $t = \omega_+$. *Here* Γ, C, $d_{\ell mn}$, x_{mn}, *and* p_m *are the same as in* Theorem 2.1.

Next, suppose $\lambda < -1$. Then we have the following:

Theorem 2.5 *There exists a number* B_2 *such that if* $B = B_2$, *then* $x(t)$ *is defined for* $(0, \omega_+)$ $(0 < \omega_+ < \infty)$ *and has representations*

$$x(t) = Kt\left\{1 + \sum_{n=1}^{\infty} x_n t^{\alpha(\lambda+1)n}\right\} \quad (2.11)$$

in the neighborhood of $t = 0$ *and* (2.7) *through* (2.10) *in the neighborhood of* $t = \omega_+$. *Here* K *is a constant.*

If $B \neq B_2$, then we conclude the following:

Theorem 2.6 *If* $B > B_2$, *then the conclusion of* Theorem 2.4 *follows. If* $B < B_2$, *then* $x(t)$ *is defined for* $(0, \omega_+)$ $(0 < \omega_+ < \infty)$ *and has representations*

$$\begin{aligned} x(t) &= K\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha\lambda m+n}\right) && \text{if } \frac{1}{\alpha\lambda} \notin \boldsymbol{N} \\ x(t) &= K\left\{1 + \sum_{k=1}^{\infty} t^{\alpha\lambda k} p_k(\log t)\right\} && \text{if } \frac{1}{\alpha\lambda} \in \boldsymbol{N} \end{aligned} \quad (2.12)$$

in the neighborhood of $t = 0$ *and* (2.7) *through* (2.10) *in the neighborhood of* $t = \omega_+$. *In* (2.12), K *and* x_{mn} *are constants and* p_k *are polynomials whose degrees are not greater than* $[\alpha\lambda k]$.

For proving Theorems 2.1 through 2.4, we adopt a transformation

$$x = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda} (-y)^{1/\alpha} \quad \left(\text{namely } y = -\frac{1}{\lambda(\lambda+1)} t^{\alpha\lambda} x^{\alpha}\right), \quad z = ty' \quad (2.13)$$

and transform (E) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{-\lambda(\lambda+1)\alpha^2 y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2 y^3}{\alpha yz}. \tag{2.14}$$

Moreover using a parameter s, we write this as a two dimensional autonomous system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz \\ \frac{dz}{ds} &= -\lambda(\lambda+1)\alpha^2 y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2 y^3. \end{aligned} \tag{2.15}$$

Here, notice the following:

(i) y got from (2.13) is negative, since we consider only positive solutions.

(ii) Only the origin is a critical point of (2.15) lying in $y \le 0$ in the yz plane.

(iii) An orbit of (2.15) is a solution of (2.14).

(iv) The z axis consists of the orbits and the origin.

Using a transformation written in [11], we shall show Theorems 2.5 and 2.6 from Theorems 2.1 through 2.4.

3. Orbits of (2.15) in the neighborhood of $y = 0$

First, suppose $\lambda > 0$. Then we consider asymptotic behavior of orbits of (2.15) as $y \to 0$.

Lemma 3.1 *If $z = z(y)$ is an orbit of* (2.15) *continuable to $y = 0$, then we get*

$$\lim_{y\to 0} \frac{z(y)}{y} = \alpha\lambda,\ \alpha(\lambda+1).$$

Proof. If $z(y)$ is unbounded as $y \to 0$, then putting $z = 1/\zeta$ in (2.14) we have

$$\frac{d\zeta}{dy} = \frac{\lambda(\lambda+1)\alpha^2 y^2\zeta^3 - (2\lambda+1)\alpha y\zeta^2 + (1-\alpha)\zeta - \lambda(\lambda+1)\alpha^2 y^3\zeta^3}{\alpha y} \tag{3.1}$$

and hence a contradiction

$$\zeta = \frac{1}{z(y)} \equiv 0.$$

Indeed from the usual discussion using Painlevé's theorem and the uniqueness of the solution, we conclude this (cf. [4]). Therefore $z(y)$ is bounded. If $z(y)$ tends to a nonzero number as $y \to 0$, then this contradicts the uniqueness of the solution of (2.15) since the z axis consists of the orbits and the critical point of (2.15). Thus we conclude

$$z(y) \to 0 \quad \text{as } y \to 0.$$

Hence we apply the proof of Lemma 1 of [19] and get

$$\lim_{y\to 0} \frac{z(y)}{y} = \alpha\lambda,\ \alpha(\lambda+1),\ \pm\infty.$$

Now if

$$\lim_{y\to 0} \frac{z(y)}{y} = \pm\infty, \tag{3.2}$$

then $w(y) = yz(y)^{-1}$ is a solution of a Briot-Bouquet differential equation

$$y\frac{dw}{dy} = \frac{1}{\alpha}w - (2\lambda+1)w^2 + \lambda(\lambda+1)\alpha w^3 - \lambda(\lambda+1)\alpha y w^3$$

with

$$w(y) \to 0 \quad \text{as } y \to 0.$$

However since $1/\alpha < 0$, it follows from Lemma 2.5 of [18] that

$$w(y) \equiv 0$$

which is a contradiction. Thus the case (3.2) is excluded and the proof is complete. □

Conversely from the conclusion of Lemma 3.1 we obtain the solutions of (2.14) and analytical expressions of these.

Lemma 3.2 *There exists a unique orbit $z = z_1(y)$ of* (2.15) *such that*

$$\lim_{y\to 0} \frac{z}{y} = \alpha\lambda. \tag{3.3}$$

Moreover in the neighborhood of $y = 0$, $z_1(y)$ is represented as

$$z_1(y) = \alpha\lambda y\left(1 + \sum_{n=1}^{\infty} z_n y^n\right) \tag{3.4}$$

where z_n are constants.

Proof. In (2.14) we put

$$v = y^{-1}z - \alpha\lambda \tag{3.5}$$

and get a Briot-Bouquet differential equation

$$y\frac{dv}{dy} = (\lambda + 1)y + \frac{1}{\alpha\lambda}v + \cdots \tag{3.6}$$

in the neighborhood of $(y, v) = (0, 0)$ where $\cdots$ denotes a convergent power series starting from a term whose degree is greater than that of the previous term. If z is a solution of (2.14) satisfying (3.3), then we get

$$v \to 0 \quad \text{as } y \to 0. \tag{3.7}$$

However since $1/\alpha\lambda < 0$, it follows from Lemma 2.5 of [18] that there exists the unique holomorphic solution

$$v = \sum_{n=1}^{\infty} v_n y^n \quad (v_n : \text{constants})$$

such that (3.7) holds. Therefore z is uniquely determined from using (3.5) to this. So if z is denoted as $z_1(y)$, then we have (3.4) and the proof is complete. □

Similarly we conclude the following:

Lemma 3.3 *There exists an orbit $z = z(y)$ of* (2.15) *such that*

$$\lim_{y\to 0} \frac{z(y)}{y} = \alpha(\lambda + 1). \tag{3.8}$$

Furthermore if $Y = -y$, then $z = z(y)$ is represented as

$$z = -\alpha(\lambda+1)Y \times \left[1 + \sum_{m+n>0} z_{mn} Y^m \left\{Y^{-1/\alpha(\lambda+1)}(h\log Y + C)\right\}^n\right] \tag{3.9}$$

in the neighborhood of $Y = 0$ *where* z_{mn}, h, *and* C *are constants and unless* $-1/\alpha(\lambda+1)$ *is an integer,* $h = 0$.

Proof. Put

$$v = y^{-1}z - \alpha(\lambda+1), \quad Y = -y.$$

Then from (2.14) we get a Briot-Bouquet differential equation

$$Y\frac{dv}{dY} = -\lambda Y - \frac{v}{\alpha(\lambda+1)} + \cdots . \tag{3.10}$$

Since $-1/\alpha(\lambda+1) > 0$, there exists a solution v of (3.10) such that

$$v \to 0 \quad \text{as } Y \to 0$$

and v is represented as

$$v = \sum_{m+n>0} v_{mn} Y^m \{Y^{-1/\alpha(\lambda+1)}(h\log Y + C)\}^n$$

in the neighborhood of $Y = 0$. Here $v_{01} = 1$ and v_{mn} are constants. Hence there exists $z = z(y)$ with (3.8) and $z = z(y)$ is represented as (3.9). □

4. Asymptotic behavior of orbits of (2.15) for decreasing y

First we put $y = -1/\eta$ in (2.14) and get

$$\frac{dz}{d\eta} = \frac{\lambda(\lambda+1)\alpha^2\eta + (2\lambda+1)\alpha\eta^2 z + (1-\alpha)\eta^3 z^2 + \lambda(\lambda+1)\alpha^2}{\alpha\eta^4 z}. \tag{4.1}$$

Here, recall that if we put $z = 1/\zeta$, then we have (3.1). In this we put $y = -1/\eta$ and obtain

$$\frac{d\zeta}{d\eta} = -\frac{\{\lambda(\lambda+1)\alpha^2\eta\zeta^2 + (2\lambda+1)\alpha\eta^2\zeta + (1-\alpha)\eta^3 + \lambda(\lambda+1)\alpha^2\zeta^2\}\zeta}{\alpha\eta^4}. \tag{4.2}$$

Moreover, put $w = \eta^{-3/2}\zeta$ and $\xi = \eta^{1/2}$. Then we get

$$\xi\frac{dw}{d\xi} = -\frac{\alpha+2}{\alpha}w - 2(2\lambda+1)\xi w^2 - 2\lambda(\lambda+1)\alpha w^3 - 2\lambda(\lambda+1)\alpha\xi^2 w^3. \tag{4.3}$$

Let $z = z(y)$ be an orbit of (2.15). Then we conclude the following:

Lemma 4.1 *$z(y)$ is continuable to $y = -\infty$ and*

$$\lim_{y \to -\infty} z(y) = \pm\infty. \tag{4.4}$$

Proof. If there exists a sequence $\{y_n\}$ such that y_n converges to c $(-\infty < c < 0)$ and $z(y_n)$ diverges to $\pm\infty$, then from (3.1) we have a contradiction

$$\zeta = \frac{1}{z(y)} \equiv 0.$$

Hence $z(y)$ is continuable to $y = -\infty$. Similarly if there exists a sequence $\{y_n\}$ such that y_n diverges to $-\infty$ and a sequence $\{z(y_n)\}$ is bounded, then from (4.1) we obtain a contradiction

$$\eta \equiv 0.$$

Therefore (4.4) holds and the proof is complete. □

Moreover the following lemma holds:

Lemma 4.2 *If $x = x(t)$ is a solution of* (E) *whose domain is denoted as (ω_-, ω_+) and (y, z) is defined as* (2.13), *then y tends to* 0 *or $-\infty$ as $t \to \omega_\pm$.*

Proof. This is almost the same as the proof of Lemma 2 of [19]. □

Owing to the previous section and this lemma, it is sufficient to consider (2.14) in the neighborhood of $y = -\infty$.

Lemma 4.3 *In the neighborhood of $y = -\infty$ an orbit $z = z(y)$ of* (2.15) *is represented as follows: If $-2 < \alpha < 0$, then we get*

$$\frac{1}{z} = C\xi^{(2\alpha-2)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\}, \tag{4.5}$$

if $\alpha = -2$, then

$$\begin{aligned} \frac{1}{z} &= \pm\xi^3\{-8\lambda(\lambda+1)(\log\xi + C)\}^{-1/2} \\ &\times\left[1 + \sum_{0<2j+k<2(N+1)} w_{jk}\xi^j\{-8\lambda(\lambda+1)(\log\xi + C)\}^{-k/2} + \Omega_N^{(1)}\right] \end{aligned} \tag{4.6}$$

where $\Omega_N^{(1)}$ *is a function of* ξ *and* $\log \xi$ *with*

$$|\Omega_N^{(1)}| \leq K_N^{(1)} |\log \xi|^{-N} \quad (K_N^{(1)} : \text{a constant}),$$

and if $\alpha < -2$, *then*

$$\frac{1}{z} = \xi^3 \left[\pm\rho + \sum_{m+n>0} u_{mn} \xi^m \{\xi^{2(\alpha+2)/\alpha} (h \log \xi + C)\}^n \right] \tag{4.7}$$

where

$$\rho = \frac{1}{\alpha} \sqrt{\frac{-\alpha - 2}{2\lambda(\lambda + 1)}}$$

and h *is a constant such that if* $\alpha \neq -4$, *then* $h = 0$. *Moreover in* (4.5), (4.6), *and* (4.7), C, w_{mn}, w_{jk}, *and* u_{mn} *are constants and* $u_{01} = 1$.

Proof. Let us consider (4.3). If its righthand side is equal to 0 in the case $\xi = 0$, then we get

$$w = 0$$

if $-2 \leq \alpha < 0$, and

$$w = 0, \pm\rho$$

if $\alpha < -2$. Here, let γ be a cluster point of a solution of (4.3) as $\xi \to 0$ (namely $y \to -\infty$). Then if $\gamma \neq 0, \pm\infty$ and besides $\gamma \neq \pm\rho$ in the case $\alpha < -2$, then from (4.3) we have

$$\frac{d\xi}{dw} = \frac{\xi}{\{-(\alpha+2)/\alpha\}w - 2(2\lambda+1)\xi w^2 - 2\lambda(\lambda+1)\alpha w^3 - 2\lambda(\lambda+1)\alpha\xi^2 w^3}$$

which implies a contradition

$$\xi \equiv 0.$$

Therefore we obtain

$$\gamma = 0, \pm\rho, \pm\infty$$

and γ is the limit. So, let $w = w_\gamma(\xi)$ be the solution of (4.3) tending to γ as $\xi \to 0$.

Now, suppose $\gamma = 0$. Then if $-2 < \alpha < 0$, we get

$$w_\gamma(\xi) = C\xi^{-(\alpha+2)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\},$$

since $-(\alpha+2)/\alpha > 0$ and w divides the righthand side of (4.3). Returning to the original variable z, we get (4.5). If $\alpha = -2$, then from (4.3) we have

$$\xi \frac{dw}{d\xi} = -2(2\lambda+1)\xi w^2 + 4\lambda(\lambda+1)w^3 + 4\lambda(\lambda+1)\xi^2 w^3.$$

Hence using a theory of [5] we obtain

$$\begin{aligned} w_\gamma(\xi) &= \pm\{-8\lambda(\lambda+1)(\log\xi + C)\}^{-1/2} \\ &\times\left[1 + \sum_{0<2j+k<2(N+1)} w_{jk}\xi^j\{-8\lambda(\lambda+1)(\log\xi + C)\}^{-k/2} + \Omega_N^{(1)}\right] \end{aligned}$$

and returning to the original variable z, (4.6). Finally if $\alpha < -2$, then since $-(\alpha+2)/\alpha < 0$ we get a contradiction

$$w_\gamma(\xi) \equiv 0$$

from Lemma 2.5 of [18].

Next, suppose $\gamma = \pm\rho$. Then $\alpha < -2$. Moreover, put

$$u = w_\gamma(\xi) - \gamma.$$

Then u is a solution of

$$\xi\frac{du}{d\xi} = \frac{(2\lambda+1)(\alpha+2)}{\lambda(\lambda+1)\alpha^2}\xi + \frac{2(\alpha+2)}{\alpha}u + \cdots$$

and tends to 0 as $\xi \to 0$. Since $2(\alpha+2)/\alpha > 0$, we obtain

$$u = \sum_{m+n>0} u_{mn}\xi^m\left\{\xi^{2(\alpha+2)/\alpha}(h\log\xi + C)\right\}^n$$

and returning to the original variable z, (4.7).

Finally, suppose $\gamma = \pm\infty$. Then if we put $\theta = 1/w_\gamma(\xi)$, θ is a solution of

$$\frac{d\xi}{d\theta} = \frac{\alpha\xi\theta}{(\alpha+2)\theta^2 + 2(2\lambda+1)\alpha\xi\theta + 2\lambda(\lambda+1)\alpha^2 + 2\lambda(\lambda+1)\alpha^2\xi^2}$$

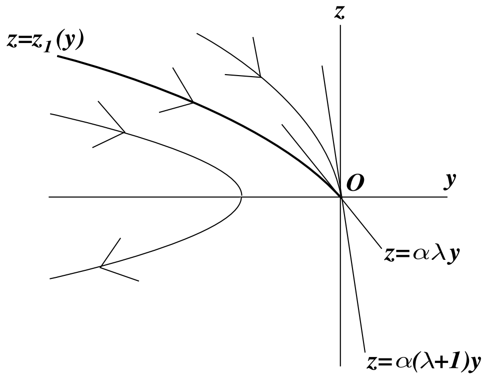

Fig. 1.

and tends to 0 as $\xi \to 0$. Therefore we conclude a contradiction

$$\xi \equiv 0.$$

Namely the case $\gamma = \pm\infty$ never occurs.

Since we have just examined all cases of γ, the proof is complete. □

5. Proof of theorems

Now, suppose $\lambda > 0$ again. Then it follows from Lemma 3.1 that orbits of (2.15) continuable to $y = 0$ satisfy (3.3) or (3.8). Moreover due to Lemma 3.2, the orbit satisfying (3.3) is only $z = z_1(y)$. It follows from Lemma 4.1 that every orbit $z = z(y)$ of (2.15) is continuable to $y = -\infty$ and $z(y)$ diverges to $\pm\infty$ as $y \to -\infty$. As s increases, y increases in a region $y < 0$ and $z > 0$, and decreases in a region $y < 0$ and $z < 0$, since $dy/ds = \alpha yz$ from (2.15). Furthermore on the y axis we have

$$\frac{dy}{ds} = 0, \ \frac{dz}{ds} = \lambda(\lambda+1)\alpha^2 y^2(y-1) < 0$$

from (2.15), namely the orbit of (2.15) passes the y axis vertically and decreasingly. Noticing the above, we draw a phase portrait of (2.15) as in Fig. 1.

Now we take (I) into account. If we define (y, z) as (2.13) from a solution $x = x(t)$ of (E) and (I), then (y, z) is a solution of (2.15) and satisfies

$$z = \alpha y\left(\lambda + \frac{tx'(t)}{x(t)}\right).$$

Therefore if $(y,\, z)$ passes a point $(y_0,\, z_0)$ at $t = T$, then we get

$$y_0 = -\frac{T^{\alpha\lambda}A^{\alpha}}{\lambda(\lambda+1)},\ z_0 = \alpha y_0\left(\lambda + \frac{TB}{A}\right). \tag{5.1}$$

Moreover an orbit $z = z(y)$ of $(y,\, z)$ is a solution of (2.14) with an initial condition

$$z(y_0) = z_0. \tag{5.2}$$

Conversely from an orbit $z = z(y)$ passing $(y_0,\, z_0)$ we have a solution $x = x(t)$ of (E) and (I).

Here for the proof of our theorems, fix $T(>0)$ and A arbitrarily. Then from (5.1), y_0 is fixed and $(y_0,\, z_0)$ draws a line parallel to the z axis as B varies. Let L be this line.

Proof of Theorem 2.1. Take $(y_0,\, z_0)$ to be the intersection of L and $z = z_1(y)$, and suppose $B = B_1$ in this case. Then from (2.13) and (3.4) we get a differential equation

$$ty' = \alpha\lambda y\left(1 + \sum_{n=1}^{\infty} z_n y^n\right).$$

Solving this, we determine y and have (2.1) from substituting this into (2.13) namely

$$x = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda}(-y)^{1/\alpha}.$$

This completes the proof. □

In order to prove Theorem 2.2, we prepare the following:

Lemma 5.1 *If*

$$I_{\mu\nu} = \int x^{\mu-1}(C - \log x)^{\nu} dx \quad (\mu \in \boldsymbol{N},\, \nu \in \boldsymbol{Q} - \boldsymbol{N})$$

and $\beta = -\beta_1/\beta_2$ *where* β_1 *and* β_2 *are relatively prime positive integers with* $1 \le \beta_1 \le \beta_2 - 1$ *or* $\beta_1 = \beta_2 = 1$, *then in the neighborhood of* $x = 0$ *we get*

$$I_{\mu\beta} = \frac{1}{\mu}x^{\mu}(C - \log x)$$

$$\times \left\{ (C - \log x)^{-(\beta_1+\beta_2)/\beta_2} + \sum_{j,k} p_{jk} x^j (C - \log x)^{-k/\beta_2} \right\} + R_{\mu\beta}.$$

Here $\sum_{j,k}$ *is the sum for all integers* j *and* k *satisfying*

$$1 \le \beta_2 j + k < \beta_2(N + 1 - \mu)$$

and

$$k \ge \beta_1 + \beta_2 + 1 \quad \text{or} \quad j \ge 1.$$

Moreover p_{jk} *are constants and* $R_{\mu\beta}$ *is a function such that*

$$|R_{\mu\beta}| \le K_N |x|^{\mu} \big|\log |x|\big|^{1-N} \quad (K_N : \textit{a constant})$$

if

$$\sup_x |\arg x| < \infty.$$

This is obtained from replacing C with $-C$ in Lemma 2.7 of [21].

Proof of Theorem 2.2. If $B = B_1$, then from applying (2.13) to a solution $x = x(t)$ of (E) and (I) we get the orbit $z = z_1(y)$ since (y_0, z_0) lies on $z = z_1(y)$. Furthermore if (ω_-, ω_+) denotes the domain of $x(t)$, then from Lemma 4.2 and (2.13), namely

$$\frac{dy}{dt} = \frac{z_1(y)}{t} > 0$$

we have

$$y \to -\infty \quad \text{as } t \to \omega_-.$$

Therefore from Lemma 4.3, $z = z_1(y)$ is represented as (4.5), (4.6), and (4.7) respectively if $-2 < \alpha < 0$, $\alpha = -2$, and $\alpha < -2$.

If $-2 < \alpha < 0$, then from (4.5) and (2.13) we get

$$\left(C\eta^{-(\alpha+1)/\alpha} + \sum_{m+n>0} w_{mn} \eta^{m/2 - ((\alpha+2)/2\alpha)n - (\alpha+1)/\alpha} \right) \eta' = \frac{1}{t}$$

$$(w_{mn} : \text{constants})$$

since $y = -1/\eta$, $\xi = \eta^{1/2}$, and $z = ty' = t\eta'/\eta^2$. Integrating both sides, we have

$$-\alpha C\eta^{-1/\alpha}\Bigl(1+\sum_{m+n>0}a_{mn}\eta^{m/2-((\alpha+2)/2\alpha)n}\Bigr)=\log t+D$$

where a_{mn} are constants and D is an integral constant. Since $\eta\to 0$ as $y\to-\infty$, the lefthand side is bounded. Hence $\log t$ is also bounded and $\omega_->0$. Therefore we obtain

$$\eta^{-1/\alpha}\Bigl(1+\sum_{m+n>0}a_{mn}\eta^{m/2-((\alpha+2)/2\alpha)n}\Bigr)=\frac{\log t/\omega_-}{-\alpha C} \tag{5.3}$$

and from this

$$\eta^{1/2}\Bigl(1+\sum_{m+n>0}b_{mn}\eta^{m/2-((\alpha+2)/2\alpha)n}\Bigr)=\Bigl(\frac{\log t/\omega_-}{-\alpha C}\Bigr)^{-\alpha/2} \tag{5.4}$$

$$\eta^{-(\alpha+2)/2\alpha}\Bigl(1+\sum_{m+n>0}c_{mn}\eta^{m/2-((\alpha+2)/2\alpha)n}\Bigr)=\Bigl(\frac{\log t/\omega_-}{-\alpha C}\Bigr)^{(\alpha+2)/2} \tag{5.5}$$

where b_{mn} and c_{mn} are constants. Now we apply the inverse function theorem to (5.3), (5.4), and (5.5), and determine $\eta^{-1/\alpha}$, $\eta^{1/2}$, and $\eta^{-(\alpha+2)/2\alpha}$. In particular we have

$$\eta^{-1/\alpha}=\frac{\log t/\omega_-}{-\alpha C}\Bigl\{1+\sum_{\ell+m+n>0}\tilde{d}_{\ell mn}\Bigl(\frac{\log t/\omega_-}{-\alpha C}\Bigr)^{\ell}\times\Bigl(\frac{\log t/\omega_-}{-\alpha C}\Bigr)^{-(\alpha/2)m}\Bigl(\frac{\log t/\omega_-}{-\alpha C}\Bigr)^{((\alpha+2)/2)n}\Bigr\}$$

where $\tilde{d}_{\ell mn}$ are constants. Therefore from (2.13) we obtain (2.2), since

$$\log\frac{t}{\omega_-}=\frac{t-\omega_-}{\omega_-}-\frac{1}{2}\Bigl(\frac{t-\omega_-}{\omega_-}\Bigr)^2+\cdots,\quad t^{-\lambda}=\omega_-^{-\lambda}(1+\cdots).$$

Next, let us consider the case $\alpha=-2$. Then we merely follow the line of the discussion for obtaining Corollary 2.6 of [21]. Since $z=ty'=2t\xi'/\xi^3$, we get from (4.6)

$$\pm\{-8\lambda(\lambda+1)(\log\xi+C)\}^{-1/2}$$

$$\times\left[1+\sum_{0<2j+k<2(N+1)} w_{jk}\xi^j\{-8\lambda(\lambda+1)(\log\xi+C)\}^{-k/2}+\Omega_N^{(1)}\right]\xi'=\frac{1}{2t}$$

where w_{jk} are constants. Replacing C with $-C$ here, we have

$$\pm(C-\log\xi)^{-1/2}\left[1+\sum_{0<2j+k<2(N+1)} \tilde{w}_{jk}\xi^j(C-\log\xi)^{-k/2}+\Omega_N^{(1)}\right]\xi'$$
$$=\frac{\sqrt{2\lambda(\lambda+1)}}{t}.$$

Integrating both sides from 0 to ξ, we obtain

$$I_{1\ -1/2}+\sum_{0<2j+k<2(N+1)} \tilde{w}_{jk} I_{j+1\ -(k+1)/2}+\Omega_N^{(2)}$$
$$=\sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_-} \quad (5.6)$$

where $\tilde{w}_{jk}$ are constants and $\Omega_N^{(2)}$ is a function with

$$|\Omega_N^{(2)}|=\left|\int_0^\xi (C-\log\xi)^{-1/2}\Omega_N^{(1)}d\xi\right|\le K_N^{(2)}|\xi||\log\xi|^{-N-1/2}$$
$$(K_N^{(2)}\ \text{: a constant}).$$

Parting the sum into a sum with even k and a sum with odd k we write (5.6) as

$$I_{1\ -1/2}+\sum_{0<2j+2m-1<2(N+1)} \tilde{w}_{j\ 2m-1} I_{j+1\ -m}$$
$$+\sum_{0<2j+2m<2(N+1)} \tilde{w}_{j\ 2m} I_{j+1\ -m-1/2}+\Omega_N^{(2)}=\sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_-}. \quad (5.7)$$

On the other hand, from Lemma 5.1 we get

$$I_{1\ -1/2}=\xi(C-\log\xi)$$
$$\times\left\{(C-\log\xi)^{-3/2}+\sum_{1\le 2j+k<2N} p_{jk}\xi^j(C-\log\xi)^{-k/2}\right\}+R_{1\ -1/2} \quad (5.8)$$

where in the sum $k \geq 4$ or $j \geq 1$. Moreover from Lemma 5.1 and

$$I_{j+1\ -m} = \sum_{\ell=1}^{m-1} b_{m\ell}\xi^{j+1}(C-\log\xi)^{-m+\ell} + b_m I_{j+1\ -1}$$

where

$$b_{1\ell} = 0, \quad b_1 = 1, \quad b_{m\ell} = -\frac{(j+1)^{\ell-1}}{(-m+1)(-m+2)\cdots(-m+\ell)},$$
$$b_m = \frac{(-1)^{m-1}(j+1)^{m-1}}{(m-1)!} \quad (m \geq 2),$$

we obtain

$$I_{j+1\ -m} = \sum_{1\leq J+K\leq N+1} q^{(1)}_{jJK}\xi^{j+1+J}(C-\log\xi)^{1-K} + b_m R_{j+1\ -1} \quad (5.9)$$

where $q^{(1)}_{jJK}$ are constants and in the sum $K \geq 2$ or $J \geq 1$. Since

$$I_{j+1\ -m-1/2} = \sum_{\ell=1}^{m} b_{m\ell}\xi^{j+1}(C-\log\xi)^{-m-1/2+\ell} + b_m I_{j+1\ -1/2}$$

where

$$b_{m\ell} = -\frac{(j+1)^{\ell-1}}{(-m+1/2)(-m+3/2)\cdots(-m-1/2+\ell)},$$
$$b_m = \frac{(j+1)^m}{(-m+1/2)(-m+3/2)\cdots(-1/2)},$$

we get from Lemma 5.1

$$I_{j+1\ -m-1/2}$$
$$= \sum_{1\leq 2J+K\leq 2(N+1)} q^{(2)}_{jJK}\xi^{j+1+J}(C-\log\xi)^{1-K/2} + b_m R_{j+1\ -1/2} \quad (5.10)$$

where $q^{(2)}_{jJK}$ are constants and in the sum $K \geq 4$ or $J \geq 1$. Indeed in $I_{j+1\ -m-1/2}$ the coefficient $b_{mm}+b_m/(j+1)$ of $\xi^{j+1}(C-\log\xi)^{-1/2}$ vanishes. From (5.7) through (5.10) we have

$$\xi(C-\log\xi)^{-1/2}\left\{1 + \sum_{1\leq j+k<3(N+1)} q_{jk}\xi^j(C-\log\xi)^{(3-k)/2}\right\}$$

$$+\,\Omega_N^{(3)} = \sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_-} \quad (5.11)$$

where q_{jk} are constants, in the sum $k \geq 4$ or $j \geq 1$,

$$\Omega_N^{(3)} = R_{1\ -1/2} + \sum_{0<2j+2m-1<2(N+1)} w_{j\ 2m-1} b_m R_{j+1\ -1} + \sum_{0<2j+2m<2(N+1)} w_{j\ 2m} b_m R_{j+1\ -1/2} + \Omega_N^{(2)},$$

and

$$|\Omega_N^{(3)}| \leq K_N^{(3)} |\xi| |\log\xi|^{1-N} \quad (K_N^{(3)} : \text{a constant}).$$

Now, put

$$\Omega_N^{(4)} = \xi^{-1}(C-\log\xi)^{1/2} \times \left\{1 + \sum_{1\leq j+k<3(N+1)} q_{jk}\xi^j (C-\log\xi)^{(3-k)/2}\right\}^{-1} \Omega_N^{(3)},$$

$$U = \frac{\sqrt{2\lambda(\lambda+1)}\log t/\omega_-}{1+\Omega_N^{(4)}}.$$

Then from (5.11) we obtain

$$\xi(C-\log\xi)^{-1/2} \times \left\{1 + \sum_{1\leq j+k<3(N+1)} q_{jk}\xi^j (C-\log\xi)^{(3-k)/2}\right\} = U \quad (5.12)$$

and

$$|\Omega_N^{(4)}| \leq K_N^{(4)} |\log\xi|^{3/2-N} \quad (K_N^{(4)} : \text{a constant}),$$

$$U \sim \sqrt{2\lambda(\lambda+1)}\log\frac{t}{\omega_-}.$$

Since (5.12) is similar to (2.11) of [21], we follow the discussion of [21] after this and get (2.3).

Finally if $\alpha < -2$, then we have (4.7) which is similar to (16) and (16') of [13]. Therefore for getting (2.4) and (2.5) it suffices to follow the discussion of [13] after those. Thus the proof is complete. □

Proof of Theorem 2.3. Define (y_0, z_0) as (5.1) in the case $B > B_1$. Then z_0 is increasing in B for fixed T and A, and (y_0, z_0) is a point lying above $z = z_1(y)$ in the yz plane. Therefore if we define an orbit $z = z(y)$ of (2.15) passing (y_0, z_0) from (2.13), then this lies above $z = z_1(y)$. On the other hand, it follows from the discussion of Sections 3 and 4 that $z = z(y)$ is represented as (3.9) in the neighborhood of $y = 0$ and (4.5), (4.6), and (4.7) in the neighborhood of $y = -\infty$.

Now applying (2.13) to (3.9), we have

$$\left[1 + \sum_{m+n>0} a_{mn} Y^m \{Y^{-1/\alpha(\lambda+1)}(h \log Y + C)\}^n\right] \frac{Y'}{Y} = \frac{\alpha(\lambda+1)}{t}$$

and from the integration of both sides

$$Y\left[1 + \sum_{m+n>0} b_{mn} Y^m \{Y^{-1/\alpha(\lambda+1)}(h \log Y + C)\}^n\right] = \Gamma t^{\alpha(\lambda+1)}$$

where a_{mn}, b_{mn}, and Γ are constants. Therefore we obtain

$$t \to \infty \quad \text{as } y \to 0 \quad (\text{namely } Y \to 0)$$

and from Smith's lemma (Lemma 1 of [13])

$$Y = \Gamma t^{\alpha(\lambda+1)}\left[1 + \sum_{m+n>0} c_{mn} t^{\alpha(\lambda+1)m} \{t^{-1}(h \log t + C)\}^n\right]$$

where c_{mn} are constants and $\alpha(\lambda+1)h$, $h \log \Gamma + C$ are replaced with h, C respectively. Hence from (3.9) we get

$$x(t) = Kt\left[1 + \sum_{m+n>0} \tilde{x}_{mn} t^{\alpha(\lambda+1)m} \{t^{-1}(h \log t + C)\}^n\right]$$

where $\tilde{x}_{mn}$ are constants. Furthermore putting

$$k = m - \frac{n}{\alpha(\lambda+1)},$$

we have

$$n = -\alpha(\lambda+1)(k-m) \leq [-\alpha(\lambda+1)k]$$

and (2.6) in the neighborhood of $t = \infty$.

Moreover we have (2.2) through (2.5) in the neighborhood of $y = -\infty$ as in the proof of Theorem 2.2. □

Proof of Theorem 2.4. In the case $B < B_1$, (y_0, z_0) defined as (5.1) and the orbit $z = z(y)$ of (2.15) passing (y_0, z_0) lie below $z = z_1(y)$. Moreover from Fig. 1, we get

$$z(y) \to \pm\infty \quad \text{as } y \to -\infty$$

and from Lemma 4.3, $z(y)$ is represented as (4.5), (4.6), and (4.7) in the neighborhood of $y = -\infty$. Here, notice that $z(y)$ is not a single valued function of y. On the other hand, it follows from (2.13) that if $z(y) > 0$, then y is an increasing function of t and if $z(y) < 0$, a decreasing function of t. Therefore as t decreases, $z(y) \to \infty$ and as t increases, $z(y) \to -\infty$. Noticing this, we follow the line of the calculations done in the proof of Theorem 2.2 and have the desired representations of the solution $x = x(t)$ of (E). Thus the proof is complete. □

Here, let us obtain the theorems of the case $\lambda < -1$ from those of the case $\lambda > 0$.

Proof of Theorems 2.5 *and* 2.6. Suppose $\lambda > 0$ in (E) and adopt a transformation

$$x = \frac{w(\tau)}{\tau}, \quad t = \frac{1}{\tau} \tag{5.13}$$

for (E) (cf. [11]). Then we get

$$w'' = -\tau^{\alpha\tilde{\lambda}-2} w^{1+\alpha} \quad \left({}' = \frac{d}{d\tau},\ \tilde{\lambda} = -\lambda - 1\right). \tag{5.14}$$

Since $\lambda > 0$, we have

$$\tilde{\lambda} < -1.$$

Moreover applying (5.13) to (I), we obtain the initial condition

$$w(\tilde{T}) = \tilde{A}, \quad w'(\tilde{T}) = \tilde{B}$$

where

$$\tilde{T} = \frac{1}{T}, \quad \tilde{A} = \frac{A}{T}, \quad \tilde{B} = A - BT.$$

Indeed we get

$$x'(t) = -w'(\tau)\tau + w(\tau)$$

from (5.13).

Now, put

$$\tilde{B}_1 = A - B_1 T.$$

Then $B = B_1$, $B > B_1$, and $B < B_1$ are equivalent to $\tilde{B} = \tilde{B}_1$, $\tilde{B} < \tilde{B}_1$, and $\tilde{B} > \tilde{B}_1$ respectively. Moreover, put

$$\tilde{\omega}_{\pm} = \frac{1}{\omega_{\mp}}.$$

Then $\omega_- < t < \infty$ and $\omega_- < t < \omega_+$ are equivalent to $\tilde{\omega}_+ > \tau > 0$ and $\tilde{\omega}_+ > \tau > \tilde{\omega}_-$ respectively. Furthermore we have

$$t - \omega_- = \frac{\tilde{\omega}_+ - \tau}{\tilde{\omega}_+^2}\left(1 + \frac{\tilde{\omega}_+ - \tau}{\tilde{\omega}_+} + \cdots\right),$$

$$\omega_+ - t = \frac{\tau - \tilde{\omega}_-}{\tilde{\omega}_-^2}\left(1 - \frac{\tau - \tilde{\omega}_-}{\tilde{\omega}_-} + \cdots\right),$$

$$w(\tau) = \{\tilde{\omega}_+ - (\tilde{\omega}_+ - \tau)\}x\left(\frac{1}{\tau}\right) = \{\tilde{\omega}_- + (\tau - \tilde{\omega}_-)\}x\left(\frac{1}{\tau}\right).$$

Noticing these, we obtain analytical expressions of solutions of (5.14) from (2.1) through (2.10). For completing the proof, it suffices to denote $\tilde{B}_1$, τ, w, $\tilde{\omega}_\pm$, $\tilde{\lambda}$ as B_2, t, x, $\omega_\pm$, λ respectively. □

6. On the case $T = 0$

In this section, suppose $T = 0$ in the initial condition (I). Then recalling our theorems, we conclude that a solution of (E) satisfying (I) is represented only as (2.12). Therefore if $\lambda > 0$, then there exists no solution of the initial value problem (E) and (I). So we suppose $\lambda < -1$ and state the answer of (E) and (I) as follows:

Corollary 6.1 *If $1/\alpha\lambda \notin \boldsymbol{N}$ and $\alpha\lambda > 1$, then there exists uniquely a solution $x = x(t)$ of* (E) *and* (I) *such that*

$$x(t) = A + Bt - \frac{A^{1+\alpha}}{\alpha\lambda(\alpha\lambda - 1)}t^{\alpha\lambda} + \sum_{m+n>1} x_{mn} t^{\alpha\lambda m + n} \tag{6.1}$$

in the neighborhood of $t = 0$. Here x_{mn} are constants with $x_{0n} = 0$ $(n = 2, 3, \ldots)$.

Proof. Substitute (2.12) of the case $1/\alpha\lambda \notin \boldsymbol{N}$ into (E). Then we get

$$\sum_{m+n>0} (\alpha\lambda m + n)(\alpha\lambda m + n - 1)x_{mn}t^{\alpha\lambda m+n} = -K^\alpha t^{\alpha\lambda}\left(1 + \sum_{m+n>0} x_{mn}t^{\alpha\lambda m+n}\right)^{1+\alpha}.$$

Since $1/\alpha\lambda \notin \boldsymbol{N}$, the series appearing here are regarded as double power series of $t^{\alpha\lambda}$ and t. Therefore we have

$$\sum_{m+n>0} (\alpha\lambda m + n)(\alpha\lambda m + n - 1)x_{mn}t^{\alpha\lambda m+n} = -K^\alpha t^{\alpha\lambda} - \sum_{m+n>1} K^\alpha P_{m-1\ n}(x_{MN}\colon M \le m-1,\ N \le n)t^{\alpha\lambda m+n} \tag{6.2}$$

where $P_{m-1\ n}(x_{MN}\colon M \le m-1,\ N \le n)$ are polynomials of x_{MN} with $M \le m-1$, $N \le n$ and is equal to 0 if $m = 0$. Putting $(m, n) = (1, 0)$, we get

$$x_{10} = -\frac{K^\alpha}{\alpha\lambda(\alpha\lambda - 1)}.$$

In the case $(m, n) = (0, 1)$, (6.2) becomes a trivial equation and none of x_{mn} is determined. If $m + n > 1$, then we have

$$x_{0n} = 0 \ (n \ge 2),$$
$$x_{mn} = -\frac{K^\alpha P_{m-1\ n}(x_{MN}\colon M \le m-1,\ N \le n)}{(\alpha\lambda m + n)(\alpha\lambda m + n - 1)} \quad \text{if } m > 0.$$

Therefore x_{01} is arbitrary, but the other x_{mn} are determined uniquely from x_{MN} with $M \le m-1$, $N \le n$. Hence we obtain

$$x(t) = K\left\{1 - \frac{K^\alpha}{\alpha\lambda(\alpha\lambda - 1)}t^{\alpha\lambda} + x_{01}t + \sum_{m+n>1} x_{mn}t^{\alpha\lambda m+n}\right\} \tag{6.3}$$

$$x'(t) = K\left\{-\frac{K^\alpha}{\alpha\lambda - 1}t^{\alpha\lambda-1} + x_{01} + \sum_{m+n>0} (\alpha\lambda m + n)x_{mn}t^{\alpha\lambda m+n-1}\right\}. \tag{6.4}$$

Since $\alpha\lambda > 1$, we get

$$x'(0) = Kx_{01}$$

and from (I)

$$K = A, \quad x_{01} = \frac{B}{A}.$$

Thus we have (6.1). Since x_{mn} are uniquely determined, the existence of (6.1) is unique. This completes the proof. □

Corollary 6.2 *Suppose that* $1/\alpha\lambda \notin \boldsymbol{N}$ *and* $0 < \alpha\lambda < 1$. *Then there exists a solution* $x = x(t)$ *of* (E) *and* (I) *if and only if* $B = \infty$. *Moreover* $x(t)$ *is represented as*

$$x(t) = A - \frac{A^{1+\alpha}}{\alpha\lambda(\alpha\lambda - 1)} t^{\alpha\lambda} + Ct + \sum_{m+n>1} x_{mn} t^{\alpha\lambda m+n} \tag{6.5}$$

in the neighborhood of $t = 0$ *where* x_{mn} *are the same as of* Corollary 6.1 *and* C *is an arbitrary constant.*

Proof. In the similar way, we get (6.3), (6.4) and $K = A$. From $\alpha\lambda < 1$ and (6.4) we have

$$x'(0) = \infty.$$

Therefore if we put $C = Ax_{01}$, then we obtain (6.5) and the proof is complete. □

Suppose $1/\alpha\lambda \in \boldsymbol{N}$ in the following corollary:

Corollary 6.3 *There exists a solution* $x = x(t)$ *of* (E) *and* (I) *if and only if* $B = \infty$. *Furthermore if* $0 < \alpha\lambda < 1$, *then the existence of* $x(t)$ *is unique and* $x(t)$ *has a representation*

$$x(t) = A - \frac{A^{1+\alpha}}{\alpha\lambda(\alpha\lambda - 1)} t^{\alpha\lambda} + \sum_{k=2}^{\infty} t^{\alpha\lambda k} q_k(\log t) \tag{6.6}$$

in the neighborhood of $t = 0$ *where* $q_k(\log t) = Ap_k(\log t)$. *If* $\alpha\lambda = 1$, *then* $x(t)$ *is represented as*

$$x(t) = A + t(C - A^{1+\alpha} \log t) + \sum_{k=2}^{\infty} t^k q_k(\log t) \tag{6.7}$$

in the neighborhood of $t = 0$ where C is an arbitrary constant.

Proof. Substitute (2.12) of the case $1/\alpha\lambda \in \boldsymbol{N}$ into (E). Then we obtain

$$\sum_{k=1}^{\infty} t^{\alpha\lambda k}\{\ddot{p}_k(s) + (2\alpha\lambda k - 1)\dot{p}_k(s) + \alpha\lambda k(\alpha\lambda k - 1)p_k(s)\}$$
$$= -t^{\alpha\lambda}K^{\alpha} - \sum_{k=2}^{\infty} t^{\alpha\lambda k}K^{\alpha}P_{k-1}(p_l\colon l \le k-1) \tag{6.8}$$

where $s = \log t$, $\dot{} = d/ds$, and $P_{k-1}(p_l\colon l \le k-1)$ are polynomials of $p_l(s)$ with $l \le k-1$. Hence if $k = 1$, then we get

$$\ddot{p}_1(s) + (2\alpha\lambda - 1)\dot{p}_1(s) + \alpha\lambda(\alpha\lambda - 1)p_1(s) = -K^{\alpha}. \tag{6.9}$$

On the other hand, we have $\deg p_1(s) \le [\alpha\lambda]$ and $\alpha\lambda \le 1$ from $1/\alpha\lambda \in \boldsymbol{N}$. Hence if $\alpha\lambda < 1$, then we obtain

$$\deg p_1(s) = 0.$$

Namely $p_1(s)$ is a constant and from (6.9) we get

$$p_1(s) = -\frac{K^{\alpha}}{\alpha\lambda(\alpha\lambda - 1)}.$$

If $\alpha\lambda = 1$, then we have

$$\deg p_1(s) \le 1.$$

So we put

$$p_1(s) = as + b \quad (a,\ b \text{ are constants})$$

and obtain

$$a = -K^{\alpha}.$$

Hence we get

$$p_1(s) = -K^{\alpha}s + b.$$

Moreover if $k \ge 2$, then from (6.8) we have a second order inhomogeneous linear differential equation

$$\ddot{p}_k(s) + (2\alpha\lambda k - 1)\dot{p}_k(s) + \alpha\lambda k(\alpha\lambda k - 1)p_k(s)$$

$$= -K^\alpha P_{k-1}(p_l\colon l \le k-1).$$

Solving this and recalling that $p_k(s)$ are polynomials of s, we obtain

$$p_k(s) = K^\alpha \left\{ e^{(\alpha\lambda k-1)t} \int P_{k-1}(q_l\colon l \le k-1) e^{-(\alpha\lambda k-1)s} ds - e^{\alpha\lambda kt} \int P_{k-1}(q_l\colon l \le k-1) e^{-\alpha\lambda ks} ds \right\}.$$

Therefore if we determine b, then $p_k(s)$ are uniquely determined.

Owing to the above discussion, we get

$$x(t) = K\left\{1 - \frac{K^\alpha}{\alpha\lambda(\alpha\lambda-1)} t^{\alpha\lambda} + \sum_{k=2}^{\infty} t^{\alpha\lambda k} p_k(s)\right\} \tag{6.10}$$

$$x'(t) = -\frac{K^{1+\alpha}}{\alpha\lambda-1} t^{\alpha\lambda-1} + K\sum_{k=2}^{\infty} t^{\alpha\lambda k-1}\{\alpha\lambda k p_k(s) + \dot p_k(s)\}$$

if $\alpha\lambda < 1$, and

$$x(t) = K\left\{1 + t(b - K^\alpha s) + \sum_{k=2}^{\infty} t^k p_k(s)\right\} \tag{6.11}$$

$$x'(t) = -K^{1+\alpha} + K(b - K^\alpha s) + K\sum_{k=2}^{\infty} t^{k-1}\{k p_k(s) + \dot p_k(s)\}$$

if $\alpha\lambda = 1$. Thus we have

$$x'(0) = \infty$$

in both cases. Finally applying $x(0) = A$ to (6.10), (6.11), and putting $C = Kb$ in (6.11), we obtain (6.6), (6.7). Now the proof is complete. □

Since C is arbitrary, we conclude from (6.5) and (6.7) that the initial value problem (E) and (I) has infinitely many solutions in the case $1/\alpha\lambda \notin \boldsymbol{N}$, $0 < \alpha\lambda < 1$, and the case $\alpha\lambda = 1$. Moreover it follows from Theorem 2.6 that the solutions got in the above corollaries are continuable to $t = \omega_+$ $(0 < \omega_+ < \infty)$ and represented as (2.7) through (2.10), since these solutions are obtained from (2.12).

Finally let us consider the Blasius problem introduced in Section 1. In [2] it has been already shown that the solution $g(u)$ of this problem exists.

Putting

$$\alpha = -2, \quad \lambda = -\frac{3}{2}, \quad g(u) = \sqrt{2}x, \quad u = t$$

in (E), we have this problem. Therefore we get the representation of $g(u)$ in the neighborhood of $u = 0$ from Corollary 6.1, while this was obtained also in [2]. In addition we have the representation of $g(u)$ in the neighborhood of $u = 1$ from (2.8) (where $\omega_+ = 1$). From this we obtain an asymptotic expression

$$g(u) \sim 2(1-u) \quad \text{as } u \to 1-0.$$

References

[1] Bellman R., *Stability Theory of Differential Equations.* McGraw-Hill, New York, 1953.

[2] Callegari A.J. and Friedman M.B., *An analytical solution of a nonlinear, singular boundary value problem in the theory of viscous fluids.* J. Math. Anal. Appl. **21** (1968), 510–529.

[3] Callegari A. and Nachman A., *Some singular, nonlinear differential equations arising in boundary layer theory.* J. Math. Anal. Appl. **64** (1978), 96–105.

[4] Hille E., *Ordinary differential equations in the complex domain.* Wiley, 1976.

[5] Hukuhara M., *Sur les points singuliers d'une équation différentielle ordinaire du premier ordre,* I. Mem. Fac. Eng. Kyushu Imp. Univ. **8** (1937), 203–247.

[6] Hukuhara M., Kimura T. and Matuda T., *Équations Différentielles Ordinaires du Premier Ordre dans le Champ Complexe.* Publ. Math. Soc. Japan (1961).

[7] Naito M., *Asymptotic behavior of solutions of second order differential equations with integrable coefficients.* Trans. Amer. Math. Soc. **282** (1984), 577–588.

[8] Naito M., *Emden-Fowler gata jobibun hoteishiki ni taisuru shindo riron* (in Japanese). Sugaku **37** (1985).

[9] Naito M., *Nonoscillatory solutions of second order differential equations with integrable coefficients.* Proc. Amer. Math. Soc. **109** (1990), 769–774.

[10] Nehari Z., *On a class of nonlinear second-order differential equations.* Trans. Amer. Math. Soc. **95** (1960), 101–123.

[11] Panayotounakos D.E. and Sotiropoulos N., *Exact analytic solutions of unsolvable classes of first-and second-order nonlinear ODEs* (*Part* II: *Emden-Fowler and relative equations*). Appl. Math. Lett. **18** (2005), 367–374.

[12] Saito T., *On bounded solutions of* $x'' = t^{\beta}x^{1+\alpha}$. Tokyo J. Math. **1** (1978), 57–75.

[13] Saito T., *Solutions of* $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ *with movable singularity.* Tokyo J. Math. **2** (1979), 262–283.

[14] Tsukamoto I., Mishina T. and Ono M., *On solutions of* $x'' = e^{\alpha\lambda t}x^{1+\alpha}$. Keio Sci. Tech. Rep. **35** (1982), 1–36.

[15] Tsukamoto I., *On solutions of* $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$. Tokyo J. Math. **12** (1989), 181–203.
[16] Tsukamoto I., *Remarks on the papers "On solutions of* $x'' = \pm e^{\alpha\lambda t}x^{1+\alpha}$*"*. Funkcial. Ekvac. **37** (1994), 363–371.
[17] Tsukamoto I., *On the generalized Thomas-Fermi differential equations and applicability of Saito's transformation.* Tokyo J. Math. **20** (1997), 107–121.
[18] Tsukamoto I., *Asymptotic behavior of solutions of* $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ *where* $-1 < \alpha < 0$. Osaka J. Math. **40** (2003), 595–620.
[19] Tsukamoto I., *On solutions of* $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ *starting at some positive* t. Hokkaido Math. J. **32** (2003), 523–538.
[20] Tsukamoto I., *On asymptotic behavior of positive solutions of* $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ *with* $\alpha < -1$. to appear in Hiroshima Math. J.
[21] Tsukamoto I., *On solutions of* $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$ *where* $\alpha < 0$. Japan. J. Math. **31** (2005), 1–24.
[22] Tsukamoto I., *On solutions of* $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ *where* $\alpha > 0$ *and* $\lambda = 0, -1$. Hokkaido Math. J. **35** (2006), 41–60.

3-10-38, Higashi-kamagaya
Kamagaya-shi, Chiba 273-0104, Japan
E-mail: itkmt@k9.dion.ne.jp