

## Well-posedness for a modified Zakharov system

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**Abstract.** The Cauchy problem for a modified Zakharov system is proven to be locally well-posed for rough data in two and three space dimensions. In the three dimensional case the problem is globally well-posed for data with small energy. Under this assumption there also exists a global classical solution for sufficiently smooth data.

*Key words:* Zakharov system, well-posedness, Fourier restriction norm method.

### 0. Introduction

The following system describes in plasma physics the nonlinear coupling of lower-hybrid waves, characterized by the complex amplitude  $\varphi$  of the wave potential, with the much lower-frequency quasineutral density perturbations  $\chi$  of the ion-acoustic type. It was introduced in [14] as a variant of the standard Zakharov system which describes the phenomenon of Langmuir turbulence in a plasma. For details of the physical background and its derivation we refer to [14]. The  $(2+1)$ -dimensional version reads as follows:

$$i \frac{\partial}{\partial t} \Delta \varphi + \Delta^2 \varphi + \frac{1}{i} \nabla \varphi \cdot \bar{\nabla} \chi = 0 \quad (1)$$

$$\frac{\partial^2}{\partial t^2} \chi - \Delta \chi - \frac{1}{i} \Delta (\nabla \bar{\varphi} \cdot \bar{\nabla} \varphi) = 0. \quad (2)$$

Here  $\nabla$  denotes the usual gradient and  $\bar{\nabla} = (\partial/\partial x_2, -\partial/\partial x_1)$ , and  $\varphi$  and  $\chi$  are respectively a complex-valued and a real-valued function defined for  $(x, t) \in \mathbf{R}^2 \times \mathbf{R}^+$ .

The initial conditions are

$$\varphi(x, 0) = \varphi_0(x), \quad \chi(x, 0) = \chi_0(x), \quad \frac{\partial \chi}{\partial t}(x, 0) = \chi_1(x). \quad (3)$$

The functions  $\varphi_0$ ,  $\chi_0$ ,  $\chi_1$  are given in suitable Sobolev spaces.

A similar  $(3+1)$ -dimensional version of the Cauchy problem will also be considered, which reads as follows:

$$i \frac{\partial}{\partial t} \Delta \varphi + \Delta^2 \varphi + \frac{1}{i} (\nabla \varphi \times \nabla \chi) \cdot e = 0 \quad (4)$$

$$\frac{\partial^2}{\partial t^2} \chi - \Delta \chi - \frac{1}{i} \Delta (\nabla \bar{\varphi} \times \nabla \varphi) \cdot e = 0. \quad (5)$$

Here  $e$  is a constant vector in  $\mathbf{R}^3$  and  $\times$  denotes the vector product.

The most important question concerning the Cauchy problem is whether global smooth solutions exist for a class of smooth data. One way to attack this problem is to give a local well-posedness result for data with low regularity and then to use the conservation laws, especially the energy conservation, to extend this solution globally. It then remains to show that regular data lead to regular solutions. This program can in fact successfully be carried out, at least in  $3 + 1$  dimensions.

We are going to use the Fourier restriction norm method introduced by Bourgain [2], [3] to prove local existence and uniqueness of the problems also for rough data. It turns out that in  $3 + 1$  dimensions such a result is true for the problem (4), (5), (3) provided  $B\varphi_0 \in H^k(\mathbf{R}^3)$ ,  $B\chi_0 \in H^l(\mathbf{R}^3)$ ,  $B\chi_1 \in H^{l-1}(\mathbf{R}^3)$ , where  $B := (-\Delta)^{1/2}$ ,  $l \geq -1$ ,  $l + 1 \leq k \leq l + 2$  and  $k \geq (l + 2)/2$ . So the lowest admissible pair is  $(k, l) = (1/2, -1)$  (cf. Theorem 2.1). It is also possible to treat the case  $B\varphi_0 \in H^1(\mathbf{R}^3)$ ,  $\chi_0 \in L^2(\mathbf{R}^3)$ ,  $B^{-1}\chi_1 \in L^2(\mathbf{R}^3)$ . This is of particular interest, because in this case the conservation laws belonging to our problem (cf. (11), (12) below) can be used to give an a-priori bound for  $\|B\varphi\|_{H^1} + \|\chi\|_{L^2} + \|B^{-1}\chi_t\|_{L^2}$ , provided  $\|B\varphi_0\|_{H^1} + \|\chi_0\|_{L^2} + \|B^{-1}\chi_1\|_{L^2}$  is sufficiently small. This allows to extend the solution globally in time, thus showing global well-posedness of the problem in energy space (Theorem 2.2).

It is also possible to refine these results in such a way (cf. Theorem 2.3) that one can show global well-posedness of the Cauchy problem for smoother data, especially proving the existence of global classical solutions under the above mentioned (weak) smallness assumption on the data (Theorem 2.4).

In  $2+1$  dimensions local well-posedness is proven for  $B^{1+\epsilon}\varphi_0 \in H^{k-\epsilon}(\mathbf{R}^2)$ ,  $B^{1-\delta}\chi_0 \in H^{l+\delta}(\mathbf{R}^2)$ ,  $B^{-\delta}\chi_1 \in H^{l+\delta}(\mathbf{R}^2)$ , if  $l \geq -1$ ,  $l + 1 \leq k \leq l + 2$ ,  $k \geq (l + 2)/2$  for  $0 < \epsilon, \delta < 1$  (Theorem 3.1). It is also possible to treat the case  $B^{1+\epsilon}\varphi_0 \in H^{1-\epsilon}(\mathbf{R}^2)$ ,  $\chi_0 \in L^2(\mathbf{R}^2)$ ,  $B^{-1}\chi_1 \in L^2(\mathbf{R}^2)$  for  $0 < \epsilon < 1$ , but for global well-posedness one would need  $\epsilon = 0$ , which is excluded here. The latter has to do with low frequency problems and the lack of a Sobolev embedding  $\dot{H}^1 \subset L^\infty$  in two space dimensions.

This paper leaves open the question whether the results are optimal.

In order to show the sharpness of the bilinear estimates one would need a number of counterexamples showing the necessity of the various conditions on the parameters involved. But even if this could be done this would not directly imply ill-posedness. A remarkable progress has been made in a recent paper by Holmer ([10]) for the original Zakharov system in dimension  $1 + 1$ , who made precise in which sense ill-posedness holds, if certain conditions on the parameters are violated. An idea could be to adapt these methods to the present more complicated higher dimensional situation, but I am not going to make such an attempt in this paper.

The technique of the proof relies on the pioneering works of Bourgain [2] and Kenig, Ponce and Vega [11], and especially on the paper of Ginibre-Tsutsumi-Velo [5] for the corresponding problem for the original Zakharov system, which reads as follows:

$$\begin{aligned} i \frac{\partial}{\partial t} u + \Delta u &= nu \\ \frac{\partial^2}{\partial t^2} n - \Delta n &= \Delta(|u|^2) \\ u(0) &= u_0, \quad n(0) = n_0, \quad \frac{\partial n}{\partial t}(0) = n_1. \end{aligned}$$

In  $2 + 1$  and  $3 + 1$  dimensions they showed local well-posedness for data  $u_0 \in H^{k'}$ ,  $n_0 \in H^{l'}$ ,  $n_1 \in H^{l'-1}$  under the assumptions  $l' \geq 0$ ,  $l' \leq k' \leq l' + 1$ ,  $k' \geq (l' + 2)/2$ . These conditions are in principle the same as ours (with  $l' = l + 1$  and  $k' = k$ ), if one remarks that somehow  $u$  can be identified with  $(-\Delta)^{1/2}\varphi$  and  $n$  with  $\chi$ . Namely, after this identification and applying  $(-\Delta)^{1/2}$  to the first equation of the Zakharov system we arrive at

$$\begin{aligned} -i \frac{\partial}{\partial t} \Delta \varphi - \Delta^2 \varphi &= (-\Delta)^{1/2} (\chi (-\Delta)^{1/2} \varphi) \\ \frac{\partial^2}{\partial t^2} \chi - \Delta \chi &= \Delta (|(-\Delta)^{1/2} \varphi|^2), \end{aligned}$$

which has a similar form as (4), (5) (just counting the number of derivatives), although the nonlinearities are of a different type.

Global well-posedness for the Zakharov system also holds for small data in two and three space dimensions [4]. A problem which is somehow related to the problem considered in the paper at hand has been treated in [9]. They however consider the 2-dimensional version with a weaker nonlinearity in the wave equation and prove global well-posedness for smooth data.

We will often use the notation  $a+ = a + \epsilon$  for a small  $\epsilon > 0$ . Similarly,  $a- = a - \epsilon$  and  $a++ = a + 2\epsilon$ .

The solution spaces are defined as follows: For  $k, l, b \in \mathbf{R}$  we denote by  $X^{k,b}$  and  $X_{\pm}^{l,b}$  the space such that  $f \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R})$  and

$$\|f\|_{X^{k,b}}^2 := \int \langle \tau + |\xi|^2 \rangle^{2b} \langle \xi \rangle^{2k} |\widehat{f}(\xi, \tau)|^2 d\xi d\tau < \infty$$

and

$$\|f\|_{X_{\pm}^{l,b}}^2 := \int \langle \tau \pm |\xi| \rangle^{2b} \langle \xi \rangle^{2l} |\widehat{f}(\xi, \tau)|^2 d\xi d\tau < \infty,$$

respectively.  $\dot{X}^{k,b}$  and  $\dot{X}_{\pm}^{l,b}$  are defined by replacing  $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$  by  $|\xi|$ .  $Y^k$  is defined with respect to

$$\|f\|_{Y^k} := \|\langle \tau + |\xi|^2 \rangle^{-1} \langle \xi \rangle^k \widehat{f}(\xi, \tau)\|_{L_{\xi}^2(L_{\tau}^1)}$$

and  $Y_{\pm}^l$  similarly by replacing  $\langle \tau + |\xi|^2 \rangle^{-1}$  by  $\langle \tau \pm |\xi| \rangle^{-1}$ .  $\dot{Y}^k$  and  $\dot{Y}_{\pm}^l$  are defined by replacing  $\langle \xi \rangle$  by  $|\xi|$ . We also use the corresponding restriction norm spaces  $X^{k,b}[0, T]$  by its norm  $\|f\|_{X^{k,b}[0, T]} := \inf_{\tilde{f}|_{[0, T]} = f} \|\tilde{f}\|_{X^{k,b}}$  and similarly the other cases.

We use the following standard facts about these spaces. Let  $\psi$  denote a cut-off function in  $C_0^\infty(\mathbf{R})$  with  $\text{supp } \psi \subset (-2, 2)$ ,  $\psi = 1$  on  $[-1, 1]$ ,  $\psi(t) = \psi(-t)$ ,  $\psi(t) \geq 0$ ,  $\psi_\delta(t) := \psi(t/\delta)$ ,  $0 < \delta \leq 1$ . Then the following estimates hold:

$$\|\psi_\delta e^{it\Delta} f\|_{X^{k,b}} \leq c\delta^{1/2-b} \|f\|_{H_x^k}, \quad b \geq 0$$

and similarly

$$\|\psi_\delta e^{\pm itB} f\|_{X_{\pm}^{l,b}} \leq c\delta^{1/2-b} \|f\|_{H_x^l}, \quad b \geq 0.$$

Moreover

$$\left\| \psi_\delta \int_0^t e^{-i(t-s)\Delta} f(s) ds \right\|_{X^{k,b}} \leq c\delta^{1-b+b'} \|f\|_{X^{k,b'}} \tag{6}$$

for  $b' \leq 0 \leq b \leq b' + 1$ ,  $b' > -1/2$ ,  $\delta \leq 1$ , and

$$\left\| \psi_\delta \int_0^t e^{-i(t-s)\Delta} f(s) ds \right\|_{X^{k,1/2}} \leq c(\|f\|_{X^{k,-1/2}} + \|f\|_{Y^k}) \tag{7}$$

as well as

$$\|\psi_\delta f\|_{X^{k,b}} \leq c\delta^{-\epsilon} \|f\|_{X^{k,b}} \tag{8}$$

for  $b \geq 0, \epsilon > 0$ .

Similar estimates hold for  $X_{\pm}^{k,b}$ , where  $-\Delta$  is replaced by  $B := (-\Delta)^{1/2}$ . Proofs can be found in [5].

The Strichartz estimates for the Schrödinger equation in  $\mathbf{R}^n$  are given by

$$\|e^{it\Delta} u_0\|_{L_t^q(L_x^r)} \leq c \|u_0\|_{L_x^2},$$

if  $0 \leq 2/q = n(1/2 - 1/r) < 1$ . A direct consequence is (cf. [5], Lemma 2.4):

$$\|f\|_{L_t^q(L_x^r)} \leq c \|f\|_{X^{0,b}}, \tag{9}$$

if  $b_0 > 1/2, 0 \leq b \leq b_0, 0 \leq \eta \leq 1, 2/q = 1 - \eta(b/b_0), n(1/2 - 1/r) = (1 - \eta)(b/b_0)$ .

For the wave equation we only use

$$\|e^{\pm itB} u_0\|_{L_t^\infty(L_x^2)} \leq c \|u_0\|_{L_x^2}$$

and its consequence

$$\|f\|_{L_t^q(L_x^2)} \leq c \|f\|_{X_{\pm}^{0,b}}, \tag{10}$$

if  $b_0 > 1/2, 2/q = 1 - b/b_0$ .

An important consequence for functions with a suitable support property is given by [5], Lemma 3.1, which we state as follows (for the Schrödinger equation):

**Lemma 0.1** *Let  $\sigma = \tau + |\xi|^2, b_0 > 1/2, a \geq 0, 0 \leq \gamma \leq 1, (1 - \gamma)a \leq b_0, a' \geq \gamma a$ . Define  $2/q = 1 - \eta(1 - \gamma)(a/b_0), n(1/2 - 1/r) := (1 - \eta)(1 - \gamma)(a/b_0)$ . Let  $v \in L^2$  be given such that  $\mathcal{F}^{-1}(\langle \sigma \rangle^{-a'} \widehat{v})$  has support in  $\{|t| \leq cT\}$ . Then the following estimate holds:*

$$\|\mathcal{F}^{-1}(\langle \sigma \rangle^{-a} |\widehat{v}|)\|_{L_t^q(L_x^r)} \leq cT^\Theta \|v\|_{L_x^2},$$

where  $\Theta = \gamma a(1 - [a' - 1/2]_+/a'), [a' - 1/2]_+ := a' - 1/2$ , if  $a' > 1/2, := \epsilon$ , if  $a' = 1/2, := 0$ , if  $a' < 1/2$ .

The proof is a combination of (9), the support property and Hölder's inequality.

**Remark 1.** The same estimate is true for the wave equation with  $\sigma := \tau \pm |\xi|$  in the special case  $\eta = 1, r = 2$  (by use of (10)).

2. The statement of the Lemma without the factor  $T^\Theta$  remains true, if no support property is assumed (with even a simpler proof).

For details we refer to [5].

### 1. Conservation laws

We now show that the system (4), (5) has two conserved quantities, namely

$$I_1 := \int_{\mathbf{R}^3} |\nabla\varphi|^2 dx \tag{11}$$

$$I_2 := \int_{\mathbf{R}^3} |\Delta\varphi|^2 dx + \frac{1}{2} \int_{\mathbf{R}^3} (|(-\Delta)^{-1/2}\chi_t|^2 + |\chi|^2) dx \tag{12}$$

$$+ \frac{1}{i} \int_{\mathbf{R}^3} \chi(\nabla\bar{\varphi} \times \nabla\varphi) \cdot e dx$$

In order to show that  $I_1$  is conserved we take the imaginary part of the scalar product of (4) with  $\varphi$ . We use

$$\Im \frac{1}{i} \langle (\nabla\varphi \times \nabla\chi) \cdot e, \varphi \rangle$$

$$= -\frac{1}{2} \int [(\varphi_{x_1}\chi_{x_2} - \varphi_{x_2}\chi_{x_1})\bar{\varphi} + \varphi(\bar{\varphi}_{x_1}\chi_{x_2} - \bar{\varphi}_{x_2}\chi_{x_1})] e_3 dx$$

2+ similar terms by permutation of the indices.

The first term is treated as follows

$$\dots = -\frac{e_3}{2} \int [(\varphi_{x_1}\chi)_{x_2}\bar{\varphi} - \varphi_{x_1x_2}\chi\bar{\varphi} - (\varphi_{x_2}\chi)_{x_1}\bar{\varphi} + \varphi_{x_2x_1}\chi\bar{\varphi}$$

$$+ \varphi(\bar{\varphi}_{x_1}\chi)_{x_2} - \varphi(\bar{\varphi}_{x_1x_2}\chi) - \varphi(\bar{\varphi}_{x_2}\chi)_{x_1} + \varphi\bar{\varphi}_{x_2x_1}\chi] dx$$

$$= 0.$$

This implies that  $I_1$  is conserved.

Next we show that  $I_2$  is conserved. We take the real part of the scalar product of (4) with  $\varphi_t$ . We remark that

$$\Re \langle i\Delta\varphi_t, \varphi_t \rangle = 0, \quad \Re \langle \Delta^2\varphi, \varphi_t \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta\varphi\|^2$$

and

$$\begin{aligned} & \Re \frac{1}{i} \langle (\nabla \varphi \times \nabla \chi) \cdot e, \varphi_t \rangle \\ &= \frac{1}{2i} (\langle (\nabla \varphi \times \nabla \chi) \cdot e, \varphi_t \rangle - \langle (\nabla \bar{\varphi} \times \nabla \chi) \cdot e, \bar{\varphi}_t \rangle). \end{aligned}$$

Calculating  $(\nabla \varphi \times \nabla \chi) \cdot e$  and taking its third term (the others are similar) we get

$$\begin{aligned} & \frac{e_3}{2i} \int ((\varphi_{x_1} \chi_{x_2} - \varphi_{x_2} \chi_{x_1}) \bar{\varphi}_t - (\bar{\varphi}_{x_1} \chi_{x_2} - \bar{\varphi}_{x_2} \chi_{x_1}) \varphi_t) dx \\ &= \frac{e_3}{2i} \int [(\varphi_{x_1} \chi)_{x_2} \bar{\varphi}_t - \varphi_{x_1 x_2} \chi \bar{\varphi}_t - (\varphi_{x_2} \chi)_{x_1} \bar{\varphi}_t + \varphi_{x_2 x_1} \chi \bar{\varphi}_t \\ &\quad - (\bar{\varphi}_{x_1} \chi)_{x_2} \varphi_t + \bar{\varphi}_{x_1 x_2} \chi \varphi_t + (\bar{\varphi}_{x_2} \chi)_{x_1} \varphi_t - \bar{\varphi}_{x_2 x_1} \chi \varphi_t] dx \\ &= \frac{e_3}{2i} \int (-\varphi_{x_1} \chi \bar{\varphi}_{tx_2} + \varphi_{x_2} \chi \bar{\varphi}_{tx_1} + \bar{\varphi}_{x_1} \chi \varphi_{tx_2} - \bar{\varphi}_{x_2} \chi \varphi_{tx_1}) dx \\ &= \frac{e_3}{2i} \int \chi (-\varphi_{x_1} \bar{\varphi}_{tx_2} + (\bar{\varphi}_{x_1} \varphi_{x_2})_t - \bar{\varphi}_{x_1} \varphi_{tx_2} \\ &\quad + \bar{\varphi}_{x_1} \varphi_{tx_2} - (\bar{\varphi}_{x_2} \varphi_{x_1})_t + \bar{\varphi}_{tx_2} \varphi_{x_1}) dx \\ &= \frac{e_3}{2i} \int \chi (\bar{\varphi}_{x_1} \varphi_{x_2} - \bar{\varphi}_{x_2} \varphi_{x_1})_t dx. \end{aligned}$$

Thus we arrive at

$$\begin{aligned} \Re \frac{1}{i} \langle (\nabla \varphi \times \nabla \chi) \cdot e, \varphi_t \rangle &= \frac{1}{2i} \int \chi ((\nabla \bar{\varphi} \times \nabla \varphi) \cdot e)_t dx \\ &= \frac{1}{2i} \frac{d}{dt} \int \chi (\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx - \frac{1}{2i} \int \chi_t (\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx \\ &= \frac{1}{2i} \frac{d}{dt} \int \chi (\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx - \frac{1}{2} \int \chi_t (\Delta^{-1} \chi_{tt} - \chi) dx \end{aligned}$$

by using (5). Now we have

$$\begin{aligned} -\frac{1}{2} \int \chi_t (\Delta^{-1} \chi_{tt} - \chi) dx &= \frac{1}{2} (\langle (-\Delta)^{-1/2} \chi_t, (-\Delta)^{-1/2} \chi_{tt} \rangle + \langle \chi_t, \chi \rangle) \\ &= \frac{1}{4} \frac{d}{dt} (\|(-\Delta)^{-1/2} \chi_t\|^2 + \|\chi\|^2). \end{aligned}$$

Summarizing we get

$$\frac{d}{dt} \left( \|\Delta \varphi\|^2 + \frac{1}{2} (\|(-\Delta)^{-1/2} \chi_t\|^2 + \|\chi\|^2) \right)$$

$$+ \frac{1}{i} \int \chi(\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx \Big) = 0.$$

These two conservation laws imply an a-priori bound for the solution of our system (4), (5), (3), provided suitable norms of the data are sufficiently small.

**Proposition 1.1** *Let  $(\varphi, \chi)$  be a solution of (4), (5), (3) with  $B\varphi \in C^0([0, T], H^1(\mathbf{R}^3))$ ,  $\chi \in C^0([0, T], L^2(\mathbf{R}^3))$ ,  $B^{-1}\chi_t \in C^0([0, T], L^2(\mathbf{R}^3))$ . Assume that the data fulfill*

$$\|B\varphi_0\|_{H^1} + \|\chi_0\|_{L^2} + \|B^{-1}\chi_t\|_{L^2} < \epsilon_0$$

for a sufficiently small  $\epsilon_0$  dependent only on the vector  $e$  and some Sobolev embedding constants. Then for  $t \in [0, T]$ :

$$\|B\varphi(t)\|_{H^1} + \|\chi(t)\|_{L^2} + \|B^{-1}\chi_t(t)\|_{L^2} \leq C_0,$$

where  $C_0$  is independent of  $T$ .

*Proof.* Consider the conserved quantity

$$\begin{aligned} E(\varphi, \chi, \chi_t) &:= \|\Delta\varphi\|^2 + \frac{1}{2}\|\chi\|^2 + \frac{1}{2}\|B^{-1}\chi_t\|^2 \\ &\quad + \frac{1}{i} \int \chi(\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx + \|\nabla\varphi\|^2. \end{aligned}$$

Now by the Sobolev embedding  $H^1(\mathbf{R}^3) \subset L^4(\mathbf{R}^3)$ :

$$\begin{aligned} \frac{1}{2} \left| \int \chi(\nabla \bar{\varphi} \times \nabla \varphi) \cdot e dx \right| &\leq c \int |\chi| |\nabla\varphi|^2 dx \\ &\leq \frac{1}{4} \int |\chi|^2 dx + c' \int |\nabla\varphi|^4 dx \\ &\leq \frac{1}{4} \int |\chi|^2 dx + c_0 (\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2)^2 \end{aligned} \tag{13}$$

Defining

$$\begin{aligned} \tilde{E}(\varphi_0, \chi_0, \chi_1) &:= \|\Delta\varphi_0\|^2 + \frac{1}{2}\|\chi_0\|^2 + \frac{1}{2}\|B^{-1}\chi_1\|^2 \\ &\quad + \left| \int \chi_0(\nabla \bar{\varphi}_0 \times \nabla \varphi_0) \cdot e dx \right| + \|\nabla\varphi_0\|^2, \end{aligned}$$



we get

$$\begin{aligned} m(t) &:= \|\Delta\varphi\|^2 + \frac{1}{4}\|\chi\|^2 + \frac{1}{2}\|B^{-1}\chi_t\|^2 + \|\nabla\varphi\|^2 \\ &\leq \tilde{E}(\varphi_0, \chi_0, \chi_1) + c_0(\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2)^2, \end{aligned}$$

thus

$$m(t) \leq \tilde{E}(\varphi_0, \chi_0, \chi_1) + c_0m(t)^2 \quad \forall t \in [0, T].$$

Defining

$$f(m) := \tilde{E}(\varphi_0, \chi_0, \chi_1) - m + c_0m^2$$

we get  $f(m(t)) \geq 0 \forall t \in [0, T]$ .  $f$  has its only minimum in  $m_0 = 1/2c_0$ . For a suitably chosen  $C_0$  our smallness assumption implies  $\tilde{E}(\varphi_0, \chi_0, \chi_1) < 1/4c_0$  using (13) above. This implies

$$f(m_0) < \frac{1}{4c_0} - m_0 + c_0m_0^2 = \frac{1}{4c_0} - \frac{1}{2c_0} + c_0\frac{1}{4c_0^2} = 0.$$

Because  $m(0) \leq \tilde{E}(\varphi_0, \chi_0, \chi_1) < 1/4c_0 < m_0$  and  $f(m(0)) \geq 0$ , this implies  $m(0) \leq m_1$ , where  $m_1$  is the smaller zero of  $f(m)$ . Because  $m(t)$  is continuous and  $f(m(t)) \geq 0$  we conclude  $m(t) \leq m_1 \forall t \in [0, T]$  and especially  $m(t) \leq m_0 \forall t \in [0, T]$ . Thus we have an a-priori bound for  $m(t)$ , and the claim follows.  $\square$

Concerning the (2+1)-dimensional problem the system (1), (2), (3) has also two conserved quantities, namely

$$\begin{aligned} I_1 &:= \int_{\mathbf{R}^2} |\nabla\varphi|^2 dx \\ I_2 &:= \int_{\mathbf{R}^2} |\Delta\varphi|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} (|(-\Delta)^{-1/2}\chi_t|^2 + |\chi|^2) dx \\ &\quad + \frac{1}{i} \int_{\mathbf{R}^2} \chi(\nabla\bar{\varphi} \cdot \bar{\nabla}\varphi) dx. \end{aligned}$$

This is shown in the same manner as in 3 dimensions. Moreover it is easy to see that these conservation laws imply an a-priori bound for  $\|B\varphi\|_{H^1} + \|\chi\|_{L^2} + \|B^{-1}\chi_t\|_{L^2}$ , provided  $\|B\varphi_0\|_{L^2}$  is sufficiently small. This follows immediately from a Gagliardo-Nirenberg type inequality for the cubic term

in  $I_2$ , namely

$$\begin{aligned} \left| \int_{\mathbf{R}^2} \chi(\nabla\varphi \cdot \bar{\nabla}\varphi) dx \right| &\leq \frac{1}{4} \|\chi\|_{L^2}^2 + c \|\nabla\varphi\|_{L^2}^2 \|\Delta\varphi\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\chi\|_{L^2}^2 + \frac{1}{2} \|\Delta\varphi\|^2, \end{aligned}$$

provided  $c\|\nabla\varphi_0\|_{L^2}^2 \leq 1/2$ .

The systems in 2 + 1 as well as in 3 + 1 dimensions can be transformed into a first order system in  $t$  by defining

$$\begin{aligned} \chi_{\pm} &:= \chi \pm i(-\Delta)^{-1/2} \frac{\partial\chi}{\partial t}, \quad \chi = \frac{1}{2}(\chi_+ + \chi_-), \\ \chi_{\pm 0} &:= \chi_0 \pm i(-\Delta)^{-1/2} \chi_1. \end{aligned}$$

In 3 + 1 dimensions this leads to the system

$$\begin{aligned} i \frac{\partial}{\partial t} \Delta\varphi + \Delta^2\varphi + \frac{1}{2i} (\nabla\varphi \times \nabla(\chi_+ + \chi_-)) \cdot e &= 0 \\ i \frac{\partial}{\partial t} \chi_{\pm} \mp (\Delta)^{-1/2} \chi_{\pm} \pm \frac{1}{i} (\Delta)^{-1/2} (\nabla\bar{\varphi} \times \nabla\varphi) \cdot e &= 0 \end{aligned}$$

and

$$\varphi(0) = \varphi_0, \quad \chi_{\pm}(0) = \chi_{\pm 0}.$$

The corresponding system of integral equations reads as follows:

$$\begin{aligned} (-\Delta)^{1/2} \varphi(t) &= (-\Delta)^{1/2} e^{it\Delta} \varphi_0 \\ &\quad - \frac{1}{2i} \int_0^t e^{i(t-s)\Delta} (-\Delta)^{-1/2} ((\nabla\varphi \times \nabla(\chi_+ + \chi_-)) \cdot e) ds \\ (-\Delta)^{1/2} \chi_{\pm}(t) &= (-\Delta)^{1/2} e^{\mp it(-\Delta)^{1/2}} \chi_{\pm 0} \\ &\quad \mp \frac{1}{i} \int_0^t e^{\mp i(t-s)(-\Delta)^{1/2}} (-\Delta) ((\nabla\bar{\varphi} \times \nabla\varphi) \cdot e) ds. \end{aligned}$$

## 2. Local and global existence in 3 + 1 dimensions

Concerning the system (4), (5), (3), in order to prove local existence and uniqueness for solutions  $B\varphi \in X^{k,b}[0, T]$  and  $B\chi \in X_+^{l,b_1}[0, T] + X_-^{l,b_1}[0, T]$  we have to give estimates for the nonlinearities in spaces of the type  $X^{k,b'}$  and  $X_{\pm}^{l,b'_1}$  for some  $b', b'_1 \leq 0$ , and in some limiting cases also in the spaces  $Y^k$  and  $Y_{\pm}^l$ , respectively, because in these cases we are forced to choose

$b' = -1/2$  or  $b'_1 = -1/2$  (cf. (6) and (7)).

In the sequel we use the notation

$$\xi := \xi_1 - \xi_2, \quad \tau := \tau_1 - \tau_2, \quad \sigma_i := \tau_i + |\xi_i|^2 \quad (i = 1, 2), \quad \sigma := \tau \pm |\xi|.$$

Then we have

$$|\xi_1|^2 - |\xi_2|^2 \mp |\xi| = \sigma_1 - \sigma_2 - \sigma. \quad (14)$$

Later we need the following elementary algebraic inequalities, which were essentially proven in ([5]), Lemma 3.3. Here  $\phi_E$  denotes the characteristic function of the set  $E$ .

**Lemma 2.1** 1. *Let  $y_1, y_2 \in \mathbf{R}$  and  $z = y_1 - y_2$ . Then for any  $\lambda > 1$*

$$|z| \leq \lambda|y_2| + \frac{\lambda}{\lambda-1}|y_1|\phi_{\{\lambda/(\lambda+1) \leq |z|/|y_1| \leq \lambda/(\lambda-1)\}}. \quad (15)$$

2. *Let  $|\xi_1| \geq 2|\xi_2|$ . Then*

$$\langle \xi_1 \rangle^2 \leq c(\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle) \quad (16)$$

$$\langle \xi_1 \rangle^2 \leq c(\langle \sigma \rangle + \langle \sigma_2 \rangle + \langle \sigma_1 \rangle \phi_{\{c_1|\sigma_1| \leq |\xi_1|^2 \leq c_2|\sigma_1|\}}) \quad (17)$$

$$\langle \xi_1 \rangle^2 \leq c(\langle \sigma_1 \rangle + \langle \sigma_2 \rangle + \langle \sigma \rangle \phi_{\{c_1|\sigma| \leq |\xi|^2 \leq c_2|\sigma|\}}), \quad (18)$$

where  $c, c_1, c_2 > 0$ .

*Proof.* (15) follows from the fact that  $\{(\lambda-1)/\lambda\}|z| \leq |y_1| \leq \{(\lambda+1)/\lambda\}|z|$ , if  $|z| \geq \lambda|y_2|$ .

(16) is implied by (14) and the fact that  $|\xi_1|^2 - |\xi_2|^2 \mp |\xi| \sim |\xi_1|$  for large  $|\xi_1|$ , and that  $|\xi_1|^2 - |\xi_2|^2 \mp |\xi|$  is bounded for small  $|\xi_1|$ .

In order to prove (17) we use (15) with  $z = |\xi_1|^2 - |\xi_2|^2 \mp |\xi|$ , and get for large  $|\xi_1|$ :

$$\begin{aligned} |\xi_1|^2 &\sim ||\xi_1|^2 - |\xi_2|^2 \mp |\xi|| \\ &\leq \lambda(|\sigma| + |\sigma_2|) + \frac{\lambda}{\lambda-1}|\sigma_1|\phi_{\{\lambda/(\lambda+1) \leq ||\xi_1|^2 - |\xi_2|^2 \mp |\xi||/|\sigma_1| \leq \lambda/(\lambda-1)\}} \\ &\leq c(\langle \sigma \rangle + \langle \sigma_2 \rangle + \langle \sigma_1 \rangle \phi_{\{c_1|\sigma_1| \leq |\xi_1|^2 \leq c_2|\sigma_2|\}}). \end{aligned}$$

But (17) is trivially also true for small  $|\xi_1|$ .

Finally, (18) follows from (17) by interchanging  $\sigma$  and  $\sigma_1$  and using  $|\xi| \sim |\xi_1|$ .  $\square$

**Lemma 2.2** *In space dimensions  $n = 2$  or  $n = 3$  let  $m > 0$ ,  $1/2 \geq a, a_1, a_2 \geq 0$  satisfy  $2(a + a_1 + a_2) + m > n/2 + 1$  and  $a + a_1 + a_2 > 1/2$ . Let  $v, v_1, v_2 \in L^2_{xt}$  be given such that  $\mathcal{F}^{-1}(\langle \sigma \rangle^{-b} \widehat{v})$  and  $\mathcal{F}^{-1}(\langle \sigma_i \rangle^{-b_i} \widehat{v}_i)$  are supported in  $\{|t| \leq cT\}$  for some  $B \geq b \geq a, B \geq b_i \geq a_i$  ( $i = 1, 2$ ). Then the following estimates hold with  $\Theta = \Theta(a, a_1, a_2, m, B) > 0$ :*

$$\int \frac{|\widehat{v} \widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \xi \rangle^m} \leq cT^\Theta \|v\|_{L^2_{xt}} \|v_1\|_{L^2_{xt}} \|v_2\|_{L^2_{xt}},$$

$$\int \frac{|\widehat{v} \widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \xi_2 \rangle^m} \leq cT^\Theta \|v\|_{L^2_{xt}} \|v_1\|_{L^2_{xt}} \|v_2\|_{L^2_{xt}}.$$

**Remark** Here and in the following integrals are always taken over  $d\xi_1 d\xi_2 d\tau_1 d\tau_2$  and  $\widehat{v} = \widehat{v}(\xi, \tau)$ ,  $\widehat{v}_1 = \widehat{v}_1(\xi_1, \tau_1)$ ,  $\widehat{v}_2 = \widehat{v}_2(\xi_2, \tau_2)$ .

*Proof.* For the proof of the second inequality we refer to Lemma 2.3 below. Just remark that we can assume  $m < n/2$  w.l.o.g. under our assumptions  $2(a + a_1 + a_2) + m > n/2 + 1$  and  $a + a_1 + a_2 > 1/2$ .

Next we prove the first inequality along the lines of [5], Lemma 3.2. We estimate using Hölder’s inequality by

$$c \|\mathcal{F}^{-1}(\langle \xi \rangle^{-m} \langle \sigma \rangle^{-a} |\widehat{v}|)\|_{L^q_t(L^r_x)} \cdot \|\mathcal{F}^{-1}(\langle \sigma_1 \rangle^{-a_1} |\widehat{v}_1|)\|_{L^{q_1}_t(L^{r_1}_x)} \cdot \|\mathcal{F}^{-1}(\langle \sigma_2 \rangle^{-a_2} |\widehat{v}_2|)\|_{L^{q_2}_t(L^{r_2}_x)} \tag{19}$$

with

$$\frac{1}{q} + \frac{1}{q_1} + \frac{1}{q_2} = 1, \tag{20}$$

$$\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} = 1. \tag{21}$$

Choose  $b_0 = 1/2 + \epsilon$ ,  $\epsilon$  sufficiently small, and  $0 < \gamma, \eta < 1$  such that

$$\frac{2}{q_i} = 1 - \eta(1 - \gamma) \frac{a_i}{b_0} \quad (i = 1, 2), \quad \frac{2}{q} = 1 - (1 - \gamma) \frac{a}{b_0}$$

(remark that  $(1 - \gamma) \max(a, a_1, a_2) < b_0$ , because  $a, a_1, a_2 \leq 1/2$ , so that  $q, q_1, q_2 \geq 2$ ). Now (20) is equivalent to

$$(1 - \gamma)(a + \eta(a_1 + a_2)) = b_0. \tag{22}$$

Concerning the  $x$ -integration we use the Sobolev embedding  $H^{m,2}_x \subset L^r_x$  for

$$m > n \left( \frac{1}{2} - \frac{1}{r} \right) \geq 0 \tag{23}$$

and choose

$$n\left(\frac{1}{2} - \frac{1}{r_i}\right) = (1 - \gamma)(1 - \eta)\frac{a_i}{b_0}. \tag{24}$$

With these choices an application of Lemma 0.1 (+Remark 1) gives the desired bound. Now (21) by use of (24) reduces to

$$\begin{aligned} n\left(\frac{1}{2} - \frac{1}{r}\right) &= n\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{2}\right) = -n\left(\frac{1}{2} - \frac{1}{r_1}\right) - n\left(\frac{1}{2} - \frac{1}{r_2}\right) + \frac{n}{2} \\ &= \frac{n}{2} - (1 - \gamma)(1 - \eta)\frac{a_1 + a_2}{b_0}. \end{aligned}$$

From (22) we get  $(1 - \gamma)\{\eta(a_1 + a_2)/b_0\} = 1 - (1 - \gamma)a/b_0$  and thus  $n(1/2 - 1/r) = 1 + n/2 - (1 - \gamma)\{(a + a_1 + a_2)/b_0\}$  so that (23) reduces to the condition

$$m > 1 + \frac{n}{2} - (1 - \gamma)\frac{a + a_1 + a_2}{b_0}. \tag{25}$$

It remains to check (22) and (25). (25) can be fulfilled for a suitable  $0 < \gamma < 1$  close to 0, if  $b_0$  is close enough to  $1/2$  under our assumption  $2(a + a_1 + a_2) + m > n/2 + 1$ . Concerning (22) we only remark that  $(1 - \gamma)a < 1/2 < b_0$ , whereas  $(1 - \gamma)(a + a_1 + a_2) > b_0$  for small  $\gamma > 0$  and  $b_0$  close to  $1/2$  by the assumption  $a + a_1 + a_2 > 1/2$ . So (22) can be fulfilled for a suitable  $0 < \eta < 1$ . □

**Remark** Lemma 2.2 remains true, if one of the three factors does not fulfill the support property and at least one of the exponents  $a, a_1, a_2$  belonging to the other two factors is strictly positive. This follows by using Remark 2 to Lemma 0.1.

We also need the following variant of the previous Lemma.

**Lemma 2.3** *In space dimensions  $n = 2$  or  $n = 3$  let  $n/2 > m \geq 0, 1/2 \geq a, a_1, a_2 \geq 0, a_1 > 0$  satisfy  $2(a + a_1 + a_2) + m > n/2 + 1$ . Let  $v, v_1, v_2 \in L^2_{xt}$  be given such that  $\mathcal{F}^{-1}(\langle \sigma \rangle^{-b} \widehat{v})$  and  $\mathcal{F}^{-1}(\langle \sigma_i \rangle^{-b_i} \widehat{v}_i)$  are supported in  $\{|t| \leq cT\}$  for some  $B \geq b \geq a, B \geq b_i \geq a_i (i = 1, 2)$ . Then the following estimate holds with  $\Theta = \Theta(a, a_1, a_2, m, B) > 0$ :*

$$\int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} |\xi_2|^m} \leq cT^\Theta \|v\|_{L^2_{xt}} \|v_1\|_{L^2_{xt}} \|v_2\|_{L^2_{xt}}.$$

*Proof.* Again using a variant of the proof of [5], Lemma 3.2 we estimate the l.h.s. by Hölder’s inequality as follows:

$$c\|\mathcal{F}^{-1}(\langle\sigma\rangle^{-a}|\widehat{v}|)\|_{L_t^q(L_x^2)}\cdot\|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-a_1}|\widehat{v}_1|)\|_{L_t^{q_1}(L_x^{r_1})} \\ \cdot\|\mathcal{F}^{-1}(|\xi_2|^{-m}\langle\sigma_2\rangle^{-a_2}|\widehat{v}_2|)\|_{L_t^{q_2}(L_x^{r_2})} \quad (26)$$

with

$$\frac{1}{q}+\frac{1}{q_1}+\frac{1}{q_2}=1, \quad (27)$$

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{2}. \quad (28)$$

Choose  $b_0 = 1/2 + \epsilon$ ,  $\epsilon$  sufficiently small, and  $0 < \gamma, \eta < 1$  such that

$$\frac{2}{q_i}=1-\eta(1-\gamma)\frac{a_i}{b_0} \quad (i=1,2), \quad \frac{2}{q}=1-(1-\gamma)\frac{a}{b_0}$$

(remark that  $(1-\gamma)\max(a, a_1, a_2) < b_0$ , because  $a, a_1, a_2 \leq 1/2$ , so that  $q, q_1, q_2 \geq 2$ ). Now (27) is equivalent to

$$(1-\gamma)(a+\eta(a_1+a_2))=b_0. \quad (29)$$

Concerning the x-integration we use the Sobolev embedding  $\dot{H}_x^{m,r'_2} \subset L_x^{r_2}$  provided

$$m=n\left(\frac{1}{r'_2}-\frac{1}{r_2}\right)\geq 0 \quad (30)$$

and  $r_2 \neq \infty$ . This last condition is by (28) equivalent to  $r_1 \neq 2$ . We now choose  $r_1$  such that

$$n\left(\frac{1}{2}-\frac{1}{r_1}\right):=(1-\gamma)(1-\eta)\frac{a_1}{b_0}. \quad (31)$$

This is strictly positive, because  $a_1 > 0$ . Thus  $r_1 \neq 2$  and  $r_2 \neq \infty$  is fulfilled. Now we choose  $r'_2$  such that

$$n\left(\frac{1}{2}-\frac{1}{r'_2}\right):=(1-\gamma)(1-\eta)\frac{a_2}{b_0}. \quad (32)$$

With these choices we can estimate (26) by  $cT^\Theta\|v\|_{L_{xt}^2}\|v_1\|_{L_{xt}^2}\|v_2\|_{L_{xt}^2}$  using Lemma 0.1 (+Remark 1). Now we compute using (28), (31), (32):

$$n\left(\frac{1}{r'_2}-\frac{1}{r_2}\right)=n\left(\frac{1}{r'_2}+\frac{1}{r_1}-\frac{1}{2}\right)=\frac{n}{2}-(1-\gamma)(1-\eta)\frac{a_1+a_2}{b_0}$$

$$= \frac{n}{2} - (1 - \gamma) \frac{a_1 + a_2}{b_0} + \eta(1 - \gamma) \frac{a_1 + a_2}{b_0}.$$

From (29) we get  $(1 - \gamma)\{\eta(a_1 + a_2)/b_0\} = 1 - (1 - \gamma)(a/b_0)$  and thus

$$n\left(\frac{1}{r_2'} - \frac{1}{r_2}\right) = 1 + \frac{n}{2} - (1 - \gamma) \frac{a + a_1 + a_2}{b_0}.$$

Thus (30) reduces to

$$m = 1 + \frac{n}{2} - (1 - \gamma) \frac{a + a_1 + a_2}{b_0} \\ \iff (1 - \gamma)(a + a_1 + a_2) = b_0\left(\frac{n}{2} + 1 - m\right). \quad (33)$$

It remains to fulfill (29) and (33). (33) can be fulfilled with a suitable  $0 < \gamma < 1$ , if  $b_0$  is close enough to  $1/2$  under our assumption  $2(a + a_1 + a_2) + m > n/2 + 1$ . It remains to fulfill (29). By (33) and  $m < n/2$  we have  $(1 - \gamma)(a + a_1 + a_2) > b_0$ , whereas  $(1 - \gamma)a < 1/2 < b_0$ , so that (29) can be fulfilled by a suitable choice of  $\eta \in (0, 1)$ .  $\square$

**Remark** Similarly as for Lemma 2.2 it is sufficient here to have the support property for only two of the three factors, provided at least one of the exponents  $a, a_1, a_2$  belonging to the other two factors is strictly positive.

In the following  $D$  denotes any first order spatial derivative.

**Lemma 2.4** *In space dimension  $n = 3$  assume  $l \geq -1, k \geq l + 1, k < l + 2$  with the exception of  $(k, l) = (0, -1)$ .  $\varphi$  and  $\chi$  are given with support in  $\{|t| \leq cT\}$ . Then the following estimate holds:*

$$\|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{k,(-1/2)+}} \leq cT^\Theta \|D\varphi\|_{X^{k,1/2}} \|D\chi\|_{X_\pm^{l,1/2}}$$

with  $\Theta = \Theta(k, l) > 0$ .

**Remark** Trivially we can replace  $\|D\chi\|_{X_\pm^{l,1/2}}$  by  $\|\chi\|_{\dot{H}_\pm^{l+1,1/2}}$ , if  $l \leq 0$ .

*Proof.* Defining  $\widehat{v} := \langle \xi \rangle^l \langle \sigma \rangle^{1/2} \widehat{D\chi}$ ,  $\widehat{v}_2 := \langle \xi_2 \rangle^k \langle \sigma_2 \rangle^{1/2} \widehat{D\varphi}$  and  $\widehat{\psi} := \langle \xi_1 \rangle^k \langle \sigma_1 \rangle^{-1/2+} \widehat{v}_1$ , where  $v_1 \in L_{xt}^2$ , we have  $\|v\|_{L_{xt}^2} = \|D\chi\|_{X_\pm^{l,1/2}}$ ,  $\|v_2\|_{L_{xt}^2} = \|D\varphi\|_{X^{k,1/2}}$  and  $\|v_1\|_{L_{xt}^2} = \|\psi\|_{X^{-k,1/2-}}$ . This generic function  $\psi$  in  $X^{-k,1/2-}$  can be assumed to have support in  $\{|t| \leq cT\}$ , too. Thus we have: the support of  $\mathcal{F}^{-1}(\langle \sigma \rangle^{-1/2} \widehat{v})$ ,  $\mathcal{F}^{-1}(\langle \sigma_2 \rangle^{-1/2} \widehat{v}_2)$  and  $\mathcal{F}^{-1}(\langle \sigma_1 \rangle^{-1/2+} \widehat{v}_1)$  is contained in  $\{|t| \leq cT\}$ . We thus have to show:

$$S := \left| \int \frac{\widehat{v}_1 \widehat{v}_2 |\xi_1|^{-1} \langle \xi_1 \rangle^k}{\langle \xi \rangle^l \langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}} \right| \leq c T^\Theta \|v\|_{L_{xt}^2} \|v_1\|_{L_{xt}^2} \|v_2\|_{L_{xt}^2}. \quad (34)$$

**Region A:**  $|\xi_1| \leq (1/2)|\xi_2|$ .

In this case we have  $|\xi| \sim |\xi_2|$ , thus

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1} \langle \xi_1 \rangle^k}{\langle \xi_2 \rangle^{k+l} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

Case 1:  $k < 1, k + l \leq 0$ .

We use the estimate (cf. (16))  $\langle \xi_2 \rangle \leq (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{1/2}$  and get

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(-k-l)/2}}{|\xi_1| \langle \xi_1 \rangle^{-k} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

Because under our assumptions  $-k-l < 1$ , we get three terms with positive powers of the  $\sigma$ -modules in the denominator.

a. We consider first the case  $|\xi_1| \geq 1$ , where we have

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(-k-l)/2}}{\langle \xi_1 \rangle^{1-k} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

We use Lemma 2.2 with e.g.  $a = 1/2 + (k+l)/2, a_1 = 1/2-, a_2 = 1/2, m = 1-k$  (and similar choices in the other cases) and get  $2(a+a_1+a_2) + m = l+4- > 5/2$  for  $l > -3/2, a+a_1+a_2 = 3/2 + (k+l)/2- > 1/2$  and  $a, a_1, a_2 \leq 1/2$ , because  $k+l \leq 0$ .

b. In the case  $|\xi_1| \leq 1$  we get

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(-k-l)/2}}{|\xi_1| \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

Similarly as before we use Lemma 2.3 with  $m = 1$  and get  $2(a+a_1+a_2) + m = k+l+4 \geq -1+4 = 3$ , thus the desired estimate.

Case 2:  $k \leq 1, k + l > 0$ .

We get

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1} \langle \xi_1 \rangle^k}{\langle \xi_1 \rangle^{k+l} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}} \\ &= c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_1| \langle \xi_1 \rangle^l \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$



**a.**  $|\xi_1| \geq 1$ .

By  $l \geq -1$  we get

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

This can be handled by Lemma 2.3 with  $a = a_2 = 1/2$ ,  $a_1 = 1/2-$ ,  $m = 0$ .

**b.**  $|\xi_1| \leq 1$ .

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_1| \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

We use Lemma 2.3 with  $a = a_2 = 1/2$ ,  $a_1 = 1/2-$ ,  $m = 1$ .

*Case 3:*  $k \geq 1$ .

**a.**  $|\xi_1| \geq 1$ .

Using  $|\xi_1| \leq (1/2)|\xi_2|$  and  $l \geq -1$  we get

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{k-1}}{\langle \xi_2 \rangle^{k+l} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \xi_2 \rangle^{l+1} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

This can be handled by Lemma 2.3 with  $a = a_2 = 1/2$ ,  $a_1 = 1/2-$ ,  $m = 0$ .

**b.**  $|\xi_1| \leq 1$ .

Using  $k + l \geq 1 + l \geq 0$  we get

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_1| \langle \xi_2 \rangle^{k+l} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_1| \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

Now we use Lemma 2.3 with  $a = a_2 = 1/2$ ,  $a_1 = 1/2-$ ,  $m = 1$ .

**Region B:**  $(1/2)|\xi_2| \leq |\xi_1| \leq 2|\xi_2|$  ( $\Rightarrow |\xi| \leq 3|\xi_1|, 3|\xi_2|$ ).

We have

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi \rangle^{-l}}{|\xi_1| \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

If  $l \geq 0$  we arrive at the same integral as in Region A, Case 3b.

If  $-1 \leq l < 0$  we estimate as follows:

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle \xi_1 \rangle^{-l}}{|\xi_1|\langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}} \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle \xi_1 \rangle}{|\xi_1|\langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}}.$$

In the case  $|\xi_1| \leq 1$  and  $|\xi_1| \geq 1$  we arrive at the same integral as in Region A, Case 3b and Case 3a, respectively.

**Region C:**  $|\xi_1| \geq 2|\xi_2|$  ( $\Rightarrow |\xi| \sim |\xi_1|$ ).

We get

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2||\xi_1|^{-1}\langle \xi_1 \rangle^{k-l}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}}.$$

**a.**  $|\xi_1| \leq 1$ .

This implies  $|\xi_2| \leq 1/2$ , so that we again arrive at the same term as in Region A, Case 3b.

**b.**  $|\xi_1| \geq 1$ .

Because  $k \geq l + 1$  by assumption, we get by (16):

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle \xi_1 \rangle^{k-l-1}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|(\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(k-l-1)/2}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

We remark that our assumption  $k < l + 2$  implies that the exponents of the  $\sigma$ -modules in the denominator are positive. Using Lemma 2.2 with e.g.  $a = 1/2 = a_2$ ,  $a_1 = 1/2 - (k - l - 1)/2-$ ,  $m = k > 0$ , thus  $2(a + a_1 + a_2) + m = 4 + l - > 5/2$  for  $l > -3/2$ , we get the desired bound.  $\square$

**Corollary 2.1** *Under the assumptions of Lemma 2.4 we have for  $k \geq 1$ :*

$$\begin{aligned} &\|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{k,-1/2+}} \\ &\leq cT^\Theta(\|D\varphi\|_{X^{1,1/2}}\|\chi\|_{X_\pm^{l+1,1/2}} + \|D\varphi\|_{X^{k,1/2}}\|\chi\|_{X_\pm^{0,1/2}}). \end{aligned}$$

*Proof.* We use Lemma 2.4 with  $k = 1-$ ,  $l = -1$ :

$$\begin{aligned} &\|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{1-,( -1/2)+}} \leq cT^\Theta\|D\varphi\|_{X^{1-,1/2}}\|D\chi\|_{X_\pm^{-1,1/2}} \\ &\leq cT^\Theta\|D\varphi\|_{X^{1-,1/2}}\|\chi\|_{X_\pm^{0,1/2}}. \end{aligned}$$

Applying the elementary inequality  $\langle \xi_1 \rangle^{k-1+} \leq c(\langle \xi \rangle^{k-1+} + \langle \xi_2 \rangle^{k-1+})$  in the Fourier variables we arrive at

$$\begin{aligned} & \|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{k,(-1/2)+}} \\ & \leq cT^\Theta (\|D\varphi\|_{X^{1-,1/2}} \|\chi\|_{X_\pm^{k-1+,1/2}} + \|D\varphi\|_{X^{k,1/2}} \|\chi\|_{X_\pm^{0,1/2}}) \\ & \leq cT^\Theta (\|D\varphi\|_{X^{1,1/2}} \|\chi\|_{X_\pm^{l+1,1/2}} + \|D\varphi\|_{X^{k,1/2}} \|\chi\|_{X_\pm^{0,1/2}}). \end{aligned}$$

□

**Lemma 2.5** *In space dimension  $n = 3$  assume  $l \geq -1$ ,  $k \geq (l + 2)/2$ ,  $k > l + 1$ , and let  $\varphi_1, \varphi_2$  be supported in  $\{|t| \leq cT\}$ . Then the following estimate holds:*

$$\|D\bar{\varphi}_1 D\varphi_2\|_{X_\pm^{l+2,-1/2+}} \leq cT^\Theta \|D\varphi_1\|_{X^{k,1/2}} \|D\varphi_2\|_{X^{k,1/2}}$$

with  $\Theta = \Theta(k, l) > 0$ .

**Remark** Trivially we can replace  $X_\pm^{l+2,-1/2+}$  by  $\dot{X}_\pm^{l+2,-1/2+}$ .

*Proof.* Defining  $\widehat{v}_1 := \langle \xi_1 \rangle^k \langle \sigma_1 \rangle^{1/2} \widehat{D\varphi_1}$ ,  $\widehat{v}_2 := \langle \xi_2 \rangle^k \langle \sigma_2 \rangle^{1/2} \widehat{D\varphi_2}$  and  $\widehat{\psi} := \langle \xi \rangle^{l+2} \langle \sigma \rangle^{-1/2+} \widehat{v}$ , where  $v \in L^2$ , we have to show

$$\begin{aligned} W &= \left| \int \frac{\widehat{v}\widehat{v}_1\widehat{v}_2\langle \xi \rangle^{l+2}}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2-} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \right| \\ &\leq cT^\Theta \|v\|_{L^2_{xt}} \|v_1\|_{L^2_{xt}} \|v_2\|_{L^2_{xt}}. \end{aligned}$$

**Region A:**  $|\xi_2|/2 \leq |\xi_1| \leq 2|\xi_2|$  ( $\Rightarrow |\xi| \leq 3|\xi_1|, 3|\xi_2|$ ).

This gives

$$W \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle \xi_1 \rangle^{l+2-2k}}{\langle \sigma \rangle^{1/2-} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle \sigma \rangle^{1/2-} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}$$

by our assumption  $k \geq (l + 2)/2$ . This integral is treated by Lemma 2.3 as before.

**Region B:**  $|\xi_1| \geq 2|\xi_2|$  ( $\Rightarrow |\xi| \sim |\xi_1|$ ) (and similarly  $|\xi_2| \geq 2|\xi_1|$ ).

Using  $k \leq l + 2$  w.l.o.g. and (16) we get

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle \xi_1 \rangle^{l+2-k}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2-} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|(\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(l+2-k)/2}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2-} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

The condition  $k > l + 1$  is required to produce positive exponents of the  $\sigma$ -modules in the denominator. Moreover we have  $k > 0$  so that we can apply Lemma 2.2 with e.g.  $a = 1/2 - (l + 2 - k)/2-$ ,  $a_1 = a_2 = 1/2$  and  $m = k$ , so that  $2(a + a_1 + a_2) + m = k + (k - l) + 1- > 5/2$ , because  $k - l > 1$  and  $k \geq (l + 2)/2 \geq 1/2$ . This completes the proof of Lemma 2.5.  $\square$

**Corollary 2.2** *Under the assumptions of Lemma 2.5 we get for  $k \geq 1$ :*

$$\begin{aligned} & \|(-\Delta)^{1/2}(D\bar{\varphi}_1 D\varphi_2)\|_{X_{\pm}^{l+1,(-1/2)+}} \\ & \leq cT^{\Theta}(\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}}\|D\varphi_2\|_{X^{1,1/2}}). \end{aligned}$$

*Proof.* Using Lemma 2.5 with  $k = 1, l = 0-$  we get

$$\|D\bar{\varphi}_1 D\varphi_2\|_{X_{\pm}^{2-,(-1/2)+}} \leq cT^{\Theta}\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{1,1/2}}, \tag{35}$$

which gives as in the proof of Corollary 2.1 for  $l \geq 0-$ :

$$\begin{aligned} & \|(-\Delta)^{1/2}(D\bar{\varphi}_1 D\varphi_2)\|_{X_{\pm}^{l+1,(-1/2)+}} \leq \|D\bar{\varphi}_1 D\varphi_2\|_{X_{\pm}^{l+2,(-1/2)+}} \\ & \leq cT^{\Theta}(\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{l+1+,1/2}} + \|D\varphi_1\|_{X^{l+1+,1/2}}\|D\varphi_2\|_{X^{1,1/2}}) \\ & \leq cT^{\Theta}(\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}}\|D\varphi_2\|_{X^{1,1/2}}), \end{aligned}$$

whereas for  $l \leq 0-$  we get obviously by (35):

$$\begin{aligned} & \|(-\Delta)^{1/2}D\bar{\varphi}_1 D\varphi_2\|_{X_{\pm}^{l+1,(-1/2)+}} \leq \|D\bar{\varphi}_1 D\varphi_2\|_{X_{\pm}^{2-, -1/2+}} \\ & \leq cT^{\Theta}\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{1,1/2}} \\ & \leq cT^{\Theta}(\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}}\|D\varphi_2\|_{X^{1,1/2}}). \end{aligned}$$

$\square$

**Lemma 2.6** *Let  $n = 3, l \geq -1, l + 1 \leq k \leq l + 2$ , and let  $\varphi, \chi$  be given with support in  $\{|t| \leq cT\}$ . Then the following estimate holds:*

$$\|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{k,-1/2}} \leq cT^{\Theta}\|D\varphi\|_{X^{k,1/2}}\|D\chi\|_{X_{\pm}^{l,1/2}}$$

with  $\Theta = \Theta(k, l) > 0$ .

**Remark** For  $l \leq 0$  we can obviously replace  $\|D\chi\|_{X_{\pm}^{l,1/2}}$  by  $\|\chi\|_{\dot{X}_{\pm}^{l+1,1/2}}$ .

*Proof.* We repeat the proof of Lemma 2.4 replacing everywhere  $\langle \sigma_1 \rangle^{1/2-}$  by  $\langle \sigma_1 \rangle^{1/2}$ . Then we can allow  $(k, l) = (0, -1)$  in Region A, Case 1. The strong inequality  $k < l + 2$  was only used in Region Cb. Here the case

$k = l + 2$  is also possible, if  $\langle \sigma_1 \rangle^{1/2}$  appears instead of  $\langle \sigma_1 \rangle^{1/2-}$ . Just remark that in the limiting case  $k = l + 2$  we have  $k > 0$  so that Lemma 2.2 can be applied.  $\square$

**Corollary 2.3** *Under the assumptions of Lemma 2.6 we have*

$$\begin{aligned} & \|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{X^{k,-1/2}} \\ & \leq cT^\Theta (\|D\varphi\|_{X^{1,1/2}}\|\chi\|_{X_\pm^{l+1,1/2}} + \|D\varphi\|_{X^{k,1/2}}\|\chi\|_{X_\pm^{0,1/2}}). \end{aligned}$$

**Lemma 2.7** *Let  $n = 3$ ,  $l \geq -1$ ,  $k \geq (l + 2)/2$ ,  $k = l + 1$  and suppose  $\varphi_1$  and  $\varphi_2$  are supported in  $\{|t| \leq cT\}$ . Then*

$$\|D\bar{\varphi}_1 D\varphi_2\|_{X_\pm^{l+2,-1/2}} \leq cT^\Theta \|D\varphi_1\|_{X^{k,1/2+}} \|D\varphi_2\|_{X^{k,1/2+}}$$

with  $\Theta = \Theta(k, l) > 0$ .

**Remark** We can replace  $X_\pm^{l+2,-1/2}$  by  $\dot{X}_\pm^{l+2,-1/2}$ .

*Proof.* Replacing  $\langle \sigma \rangle^{1/2-}$  by  $\langle \sigma \rangle^{1/2}$  and  $\langle \sigma_i \rangle^{1/2}$  by  $\langle \sigma_i \rangle^{1/2+}$  everywhere we repeat the proof of Lemma 2.5. The strong condition  $k > l + 1$  was only required in Region B to produce positive exponents of the  $\sigma$ -modules in the denominator. In the limiting case  $k = l + 1$  (remark that  $k > 0$  here) we use Lemma 2.2 with e.g.  $a = 0$ ,  $a_1 = 1/2+$ ,  $a_2 = 1/2+$  and  $m = k$  and get the inequality

$$2(a + a_1 + a_2) + m = 2 + k + > \frac{5}{2}, \tag{36}$$

if  $k \geq 1/2$ . This completes the proof.  $\square$

**Remark** For  $k > 1/2$  we can replace  $X^{k,1/2+}$  by  $X^{k,1/2}$  in the statement of Lemma 2.7.

This follows immediately, because in this case condition (36) with  $a = 0$ ,  $a_1 = a_2 = 1/2$  is also satisfied.

**Corollary 2.4** *Under the assumptions of Lemma 2.7 and  $k \geq 1$  we get*

$$\begin{aligned} & \|(-\Delta)^{1/2}(D\bar{\varphi}_1 D\varphi_2)\|_{X_\pm^{l+1,-1/2}} \\ & \leq cT^\Theta (\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}}\|D\varphi_2\|_{X^{1,1/2}}). \end{aligned}$$

Because we were forced to replace  $X^{k,-1/2+}$  by  $X^{k,-1/2}$  in the limiting case  $k = l + 2$  in Lemma 2.4 we have to give an additional estimates where

$X^{k,-1/2}$  is replaced by  $Y^k$  (in order to apply (7) later). Similarly, because  $X_{\pm}^{l+2,(-1/2)^+}$  had to be replaced by  $X_{\pm}^{l+2,-1/2}$  in the limiting case  $k = l + 1$  in Lemma 2.5 we need an estimate where  $X_{\pm}^{l+2,-1/2}$  is replaced by  $Y_{\pm}^{l+2}$ .

**Lemma 2.8** *Let  $n = 3$ ,  $l \geq -1$ ,  $l + 1 \leq k \leq l + 2$  be given and let  $\varphi$  and  $\chi$  be supported in  $\{|t| \leq cT\}$ . Then*

$$\|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{Y^k} \leq cT^\Theta \|D\varphi\|_{X^{k,1/2}} \|D\chi\|_{X_{\pm}^{l,1/2}}$$

with  $\Theta = \Theta(k, l) > 0$ .

**Remark** For  $l \leq 0$  we can replace  $\|D\chi\|_{X_{\pm}^{l,1/2}}$  by  $\|\chi\|_{\dot{X}_{\pm}^{l+1,1/2}}$ .

**Corollary 2.5** *Under the assumptions of Lemma 2.8 we have*

$$\begin{aligned} \|(-\Delta)^{-1/2}(D\varphi D\chi)\|_{Y^k} \\ \leq cT^\Theta (\|D\varphi\|_{X^{1,1/2}} \|\chi\|_{X_{\pm}^{l+1,1/2}} + \|D\varphi\|_{X^{k,1/2}} \|\chi\|_{X_{\pm}^{0,1/2}}). \end{aligned}$$

*Proof of Lemma 2.8.* Defining  $v$  and  $v_2$  as in the proof of Lemma 2.4 and  $\widehat{\psi}(\xi_1) := \langle \xi_1 \rangle^k \widehat{w}_1(\xi_1)$  with  $w_1 \in L_x^2$ , so that  $\psi$  denotes a generic function in  $H_x^{-k}$ , we have to show

$$\tilde{S} := \int \frac{|\widehat{v} \widehat{w}_1 \widehat{v}_2| |\xi_1|^{-1} \langle \xi_1 \rangle^k}{\langle \xi \rangle^l \langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}} \leq cT^\Theta \|v\|_{L_{xt}^2} \|w_1\|_{L_x^2} \|v_2\|_{L_{xt}^2}.$$

The only case where the strict inequality  $k < l + 2$  was used in the proof of Lemma 2.4 was the region  $|\xi_1| \geq 2|\xi_2|$  and  $|\xi_1| \geq 1$ . In all other regions we define  $\widehat{v}_1 := \langle \sigma_1 \rangle^{-1/2} \widehat{w}_1$ . Then one easily checks  $\|v_1\|_{L_{xt}^2} \leq c\|w_1\|_{L_x^2}$  and  $\tilde{S}$  can be replaced by

$$\int \frac{|\widehat{v} \widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1} \langle \xi_1 \rangle^k}{\langle \xi \rangle^l \langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}.$$

This is exactly the integral treated in the proof of Lemma 2.4, so that the desired result in these regions follows using the remarks to Lemma 2.2 and Lemma 2.3 taking into account that  $w_1$  fulfills no support property.

It remains to consider the region where  $|\xi_1| \geq 2|\xi_2|$  and  $|\xi_1| \geq 1$  and  $l + 1 \leq k \leq l + 2$ . In this case we get as in Lemma 2.4

$$\tilde{S} \leq c \int \frac{|\widehat{v} \widehat{w}_1 \widehat{v}_2| \langle \xi_1 \rangle^{k-l-1}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}}$$

$$\leq c \int \frac{|\widehat{v}\widehat{w}_1\widehat{v}_2|(\langle\sigma\rangle + \langle\sigma_2\rangle + \langle\sigma_1\rangle\phi_{\{c_1|\sigma_1|\leq|\xi_1|^2\leq c_2|\sigma_1|\}})^{(k-l-1)/2}}{\langle\xi_2\rangle^k\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle\langle\sigma_2\rangle^{1/2}}.$$

Here we used (17). The two terms coming from  $\langle\sigma\rangle$  and  $\langle\sigma_2\rangle$  in the numerator are treated by defining  $\widehat{v}_1$  as before by Lemma 2.2 with e.g.  $a = 1/2 - (k - l - 1)/2 \geq 0$ ,  $a_1 = 1/2-$ ,  $a_2 = 1/2$ ,  $m = k$ , which implies  $2(a + a_1 + a_2) + m = 4 + l - > 5/2$ , whereas the term coming from  $\langle\sigma_1\rangle$  is treated by defining  $\widehat{v}_1 := \langle\sigma_1\rangle^{-1/2}\widehat{w}_1\phi_{\{c_1|\sigma_1|\leq|\xi_1|^2\leq c_2|\sigma_1|\}}$ . One can easily show  $\|v_1\|_{L_{xt}^2} \leq c\|w_1\|_{L_x^2}$ , so that we only have to give the estimate

$$\int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle\xi_2\rangle^k\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-(k-l-1)/2}\langle\sigma_2\rangle^{1/2}} \leq cT^\Theta \|v\|_{L_{xt}^2} \|v_1\|_{L_{xt}^2} \|v_2\|_{L_{xt}^2}.$$

This can be done by Lemma 2.2 (+remark) with  $a = a_2 = 1/2$ ,  $a_1 = 1/2 - (k - l - 1)/2 \geq 0$  and  $m = k$  which implies  $2(a + a_1 + a_2) + m = 4 - l > 5/2$ . □

We also get

**Lemma 2.9** *Let  $n = 3$ ,  $l \geq -1$ ,  $k \geq (l + 2)/2$ ,  $k = l + 1$  and suppose  $\varphi_1$  and  $\varphi_2$  are supported in  $\{|t| \leq cT\}$ . Then*

$$\|D\bar{\varphi}_1 D\varphi_2\|_{Y_{\pm}^{l+2}} \leq cT^\Theta \|D\varphi_1\|_{X^{k,1/2+}} \|D\varphi_2\|_{X^{k,1/2+}}$$

with  $\Theta = \Theta(k, l) > 0$ . If  $k > 1/2$ , we can replace  $X^{k,1/2+}$  by  $X^{k,1/2}$ .

**Remark** We can obviously replace  $Y_{\pm}^{l+2}$  by  $\dot{Y}_{\pm}^{l+2}$  and  $k = l + 1$  by  $k \geq l + 1$ .

*Proof.* Defining  $v_1$  and  $v_2$  similarly as in the proof of Lemma 2.5 and  $\widehat{\psi}(\xi) := \langle\xi\rangle^{l+2}\widehat{w}(\xi)$  with  $w \in L_x^2$  (so that  $\psi$  is a generic function in  $H_x^{-l-2}$ ), we have to show for any  $\epsilon > 0$ :

$$\begin{aligned} \tilde{W} &:= \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|\langle\xi\rangle^{l+2}}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k\langle\sigma\rangle\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}} \\ &\leq cT^\Theta \|w\|_{L_x^2} \|v_1\|_{L_{xt}^2} \|v_2\|_{L_{xt}^2}. \end{aligned}$$

In region A of the proof of Lemma 2.5 we define  $\widehat{v} := \langle\sigma\rangle^{-1/2-\epsilon/2}\widehat{w}$  such that  $\|v\|_{L_{xt}^2} \leq c\|w\|_{L_x^2}$  and  $\tilde{W}$  is estimated by

$$c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle\xi_1\rangle^{l+2-2k}}{\langle\sigma\rangle^{1/2-\epsilon/2}\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}}$$

$$\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle\sigma\rangle^{1/2-\epsilon/2}\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}},$$

which can be estimated by  $cT^\Theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}$  by Lemma 2.3 (+remark) as before. In region B of the proof of Lemma 2.5 we get using  $k = l + 1$  and (18):

$$\begin{aligned} \widetilde{W} &\leq c \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|\langle\xi_1\rangle}{\langle\xi_2\rangle^k\langle\sigma\rangle\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}} \\ &\leq c \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|(\langle\sigma_1\rangle + \langle\sigma_2\rangle + \langle\sigma\rangle\phi_{\{c_1|\sigma|\leq|\xi|^2\leq c_2|\sigma|\}})^{1/2}}{\langle\xi_2\rangle^k\langle\sigma\rangle\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}}. \end{aligned}$$

The two terms coming from  $\langle\sigma_1\rangle$  and  $\langle\sigma_2\rangle$  in the numerator are treated by defining  $\widehat{v}$  as before by Lemma 2.2 with e.g.  $a_1 = \epsilon$ ,  $a_2 = 1/2 + \epsilon$ ,  $a = 1/2 - \epsilon/2$ ,  $m = k \geq 1/2$ , so that

$$2(a + a_1 + a_2) + m > \frac{5}{2}. \tag{37}$$

The term coming from  $\langle\sigma\rangle$  is treated by defining  $\widehat{v} := \langle\sigma\rangle^{-1/2}\widehat{w}\phi_{\{c_1|\sigma|\leq|\xi|^2\leq c_2|\sigma|\}}$ , so that  $\|v\|_{L^2_{xt}} \leq c\|w\|_{L^2_x}$ . Thus it remains to show

$$\int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle\xi_2\rangle^k\langle\sigma_1\rangle^{1/2+\epsilon}\langle\sigma_2\rangle^{1/2+\epsilon}} \leq cT^\Theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}.$$

This is true by Lemma 2.2 with  $a_1 = a_2 = 1/2 + \epsilon$ ,  $a = 0$ ,  $m = k \geq 1/2$ , thus  $2(a + a_1 + a_2) + m > 5/2$ .

If  $k > 1/2$ , we can easily modify the proof by replacing  $\langle\sigma_j\rangle^{1/2+\epsilon}$  by  $\langle\sigma_j\rangle^{1/2}$  ( $j = 1, 2$ ), because the decisive condition (37) in this case also holds.  $\square$

**Corollary 2.6** *Under the assumptions of Lemma 2.9 and  $k \geq 1$  we get*

$$\begin{aligned} &\|(-\Delta)^{1/2}(D\bar{\varphi}_1D\varphi_2)\|_{Y^\pm_{l+1}} \\ &\leq cT^\Theta(\|D\varphi_1\|_{X^{1,1/2}}\|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}}\|D\varphi_2\|_{X^{1,1/2}}). \end{aligned}$$

*Proof.* Follows from Lemma 2.9 and the remark to that Lemma.  $\square$

**Theorem 2.1** *In space dimension  $n = 3$  assume  $l \geq -1$ ,  $l + 1 \leq k \leq l + 2$ ,  $k \geq (l + 2)/2$ , and*

$$B\varphi_0 \in H^k(\mathbf{R}^3), \quad B\chi_0 \in H^l(\mathbf{R}^3), \quad \chi_1 \in H^l(\mathbf{R}^3).$$



Then there exists  $1 \geq T > 0$ ,  $T = T(\|B\varphi_0\|_{H^k}, \|B\chi_0\|_{H^l}, \|\chi_1\|_{H^l})$ , such that the problem (4), (5), (3) has a unique solution  $(\varphi, \chi)$  with

$$B\varphi \in X^{k,b}[0, T], \quad B\chi, \chi_t \in X_+^{l,b_1}[0, T] + X_-^{l,b_1}[0, T].$$

Here  $b = 1/2+$ ,  $b_1 = 1/2+$ , if  $l + 1 < k < l + 2$ ,  $b = 1/2$ ,  $b_1 = 1/2+$ , if  $k = l + 2$ , and  $b = 1/2+$ ,  $b_1 = 1/2$ , if  $k = l + 1$ . This solution satisfies

$$B\varphi \in C^0([0, T], H^k(\mathbf{R}^3)), \quad B\chi, \chi_t \in C^0([0, T], H^l(\mathbf{R}^3)).$$

If  $l \leq 0$  we can replace  $B\chi_0, \chi_1 \in H^l$  by  $\chi_0 \in \dot{H}^{l+1}$ ,  $\chi_1 \in \dot{H}^l$ , and  $B\chi, \chi_t \in X_+^{l,b_1}[0, T] + X_-^{l,b_1}[0, T]$  by  $\chi \in \dot{X}_+^{l+1,b_1}[0, T] + \dot{X}_-^{l+1,b_1}[0, T]$ ,  $\chi_t \in \dot{X}_+^{l,b_1}[0, T] + \dot{X}_-^{l,b_1}[0, T]$ , and we have  $\chi \in C^0([0, T], \dot{H}^{l+1}(\mathbf{R}^3))$ ,  $\chi_t \in C^0([0, T], \dot{H}^l(\mathbf{R}^3))$ .

*Proof.* We replace our system of integral equations by the cut-off system

$$\begin{aligned} B\varphi(t) &= \psi_1(t)Be^{it\Delta}\varphi_0 - \frac{1}{2i}\psi_T(t) \int_0^t e^{i(t-s)\Delta} B^{-1} \left( (\psi_{2T}(s)\nabla\varphi(s)) \right. \\ &\quad \left. \times (\psi_{2T}(s)\nabla(\chi_+(s) + \chi_-(s))) \cdot e \right) ds \\ B\chi_{\pm}(t) &= \psi_1(t)Be^{\pm itB}\chi_{\pm 0} \mp \frac{1}{i}\psi_T(t) \int_0^t e^{\mp i(t-s)B} B^2 \left( (\psi_{2T}(s)\nabla\bar{\varphi}(s)) \right. \\ &\quad \left. \times (\psi_{2T}(s)\nabla\varphi(s)) \cdot e \right) ds, \end{aligned}$$

which we want to solve globally in  $t$ . This gives a solution of the original system in  $[0, T]$ . The factors  $\psi_{2T}$  here allow to assume that the factors in the nonlinearities are supported in  $\{|t| \leq 2T\}$ . We want to use the contraction mapping principle and consider the case  $l + 1 < k < l + 2$  first.

The linear parts are treated as follows:

$$\|\psi_1(t)Be^{it\Delta}\varphi_0\|_{X^{k,b}} \leq c\|B\varphi_0\|_{H^k}$$

and

$$\|\psi_1(t)Be^{\pm itB}\chi_{\pm 0}\|_{X_{\pm}^{l,b_1}} \leq c\|B\chi_{\pm 0}\|_{H^l}.$$

Using (6) the integral term in the first equation can be estimated in the  $X^{k,1/2+}$ -norm by

$$cT^{0+} \|B^{-1}((\psi_{2T}\nabla\varphi) \times (\psi_{2T}\nabla(\chi_+ + \chi_-))) \cdot e\|_{X^{k,(-1/2)+}},$$

which by Lemma 2.4 and (8) is majorized by

$$\begin{aligned} & cT^{\Theta^+} \|B(\psi_{2T}\varphi)\|_{X^{k,1/2}} (\|B(\psi_{2T}\chi_+)\|_{X_+^{l,1/2}} + \|B(\psi_{2T}\chi_-)\|_{X_-^{l,1/2}}) \\ & \leq cT^{\Theta^-} \|B\varphi\|_{X^{k,1/2}} (\|B\chi_+\|_{X_+^{l,1/2}} + \|B\chi_-\|_{X_-^{l,1/2}}), \end{aligned}$$

where  $\Theta > 0$ .

The integral term in the second equation can be estimated in the  $X_{\pm}^{l,1/2+}$ -norm similarly by use of Lemma 2.5 instead of Lemma 2.4 and leads to the bound  $cT^{\Theta^-} \|B\varphi\|_{X^{k,1/2}}^2$ .

The standard contraction argument then gives a unique solution  $B\varphi \in X^{k,b}$ ,  $B\chi_{\pm} \in X_{\pm}^{l,b_1}$  of the cut-off system for small enough  $T$ .

If  $k = l + 1$  the estimates for the first equation remain unchanged whereas Lemma 2.5 is no longer true and forces us to choose  $b_1 = 1/2$ , so that the integral term in the  $X_{\pm}^{l,b_1}$ -norm is estimated by (7) by

$$\begin{aligned} & \|B^2(((\psi_{2T}\nabla\bar{\varphi}) \times (\psi_{2T}\nabla\varphi)) \cdot e)\|_{X_{\pm}^{l,-1/2}} \\ & \quad + \|B^2(((\psi_{2T}\nabla\bar{\varphi}) \times (\psi_{2T}\nabla\varphi)) \cdot e)\|_{Y^l}. \end{aligned}$$

The first term can be treated by Lemma 2.7 and (8) and gives the bound  $cT^{\Theta} \|B(\psi_{2T}\varphi)\|_{X^{k,1/2+}}^2 \leq cT^{\Theta^-} \|B\varphi\|_{X^{k,1/2+}}^2$ , whereas the second term gives the same bound by Lemma 2.9. So we get a unique solution  $B\varphi \in X^{k,1/2+}$ ,  $B\chi_{\pm} \in X_{\pm}^{l,1/2}$ .

If  $k = l + 2$  the estimates for the second equation remain unchanged, whereas Lemma 2.4 is no longer true and thus requires  $b = 1/2$  so that the integral term in the  $X^{k,b}$ -norm is bounded by

$$\begin{aligned} & \|B(((\psi_{2T}\nabla\varphi) \times (\psi_{2T}\nabla(\chi_+ + \chi_-))) \cdot e)\|_{X^{k,-1/2}} \\ & \quad + \|B(((\psi_{2T}\nabla\varphi) \times (\psi_{2T}\nabla(\chi_+ + \chi_-))) \cdot e)\|_{Y^k}. \end{aligned}$$

These terms are treated by Lemma 2.6 and Lemma 2.8, which gives the bound

$$\begin{aligned} & cT^{\Theta} \|B\psi_{2T}\varphi\|_{X^{k,1/2}} (\|B\psi_{2T}\chi_+\|_{X_+^{l,\frac{1}{2}}} + \|B\psi_{2T}\chi_-\|_{X_-^{l,\frac{1}{2}}}) \\ & \leq cT^{\Theta^-} \|B\varphi\|_{X^{k,1/2}} (\|B\chi_+\|_{X_+^{l,1/2}} + \|B\chi_-\|_{X_-^{l,1/2}}), \end{aligned}$$

which leads to a unique solution  $B\varphi \in X^{k,1/2}$ ,  $B\chi_{\pm} \in X_{\pm}^{l,1/2+}$  of the cut-off system.

To prove uniqueness for the original system of integral equations in  $[0, T]$

(without cut-offs) let  $(\varphi, \chi_{\pm})$  be any solution with  $B\varphi \in X^{k,b}[0, T]$ ,  $B\chi_{\pm} \in X_{\pm}^{l,b_1}[0, T]$ . Consider e.g. the case  $l + 1 < k < l + 2$  and  $b = 1/2+$ ,  $b_1 = 1/2+$ . Let  $(\tilde{\varphi}, \tilde{\chi}_{\pm})$  be any extension with  $B\tilde{\varphi} \in X^{k,b}$ ,  $B\tilde{\chi}_{\pm} \in X_{\pm}^{l,b_1}$ . Then we have by the same estimates as above:

$$\begin{aligned} & \left\| \int_0^t e^{i(t-s)\Delta} B^{-1} \left( (\nabla\varphi(s) \times \nabla(\chi_+(s) + \chi_-(s))) \cdot e \right) ds \right\|_{X^{k,b}[0, T]} \\ & \leq \left\| \psi_T(t) \int_0^t e^{i(t-s)\Delta} \right. \\ & \quad \times B^{-1} \left( (\psi_{2T}(s)\nabla\tilde{\varphi}(s) \times \psi_{2T}(s)\nabla(\tilde{\chi}_+(s) + \tilde{\chi}_-(s))) \cdot e \right) ds \left. \right\|_{X^{k,b}} \\ & \leq cT^{\Theta-} \|B\tilde{\varphi}\|_{X^{k,1/2}} (\|B\tilde{\chi}_+\|_{X_+^{l,1/2}} + \|B\tilde{\chi}_-\|_{X_-^{l,1/2}}). \end{aligned}$$

Thus

$$\begin{aligned} & \left\| \int_0^t e^{i(t-s)\Delta} B^{-1} \left( (\nabla\varphi(s) \times \nabla(\chi_+(s) + \chi_-(s))) \cdot e \right) ds \right\|_{X^{k,b}[0, T]} \\ & \leq cT^{\Theta-} \|B\varphi\|_{X^{k,1/2}[0, T]} (\|B\chi_+\|_{X_+^{l,1/2}[0, T]} + \|B\chi_-\|_{X_-^{l,1/2}[0, T]}). \end{aligned}$$

Similarly we can treat this term in the other cases using the  $Y$ -spaces and also the integral term in the second integral equation. A standard argument implies uniqueness for the original system in  $[0, T]$ .

The claim that  $B\varphi$  belongs to  $C^0([0, T], H^k)$  and  $B\chi$  to  $C^0([0, t], H^l)$  follows directly from the embeddings  $X^{k,b}[0, T] \subset C^0([0, T], H^k)$  and  $X_{\pm}^{l,b_1}[0, T] \subset C^0([0, T], H^l)$  for  $b > 1/2$  and  $b_1 > 1/2$ . If  $b = 1/2$  (or similarly  $b_1 = 1/2$ ) this follows from the fact that the nonlinearity  $B^{-1}((\psi_{2T}\nabla\varphi) \times (\psi_{2T}\nabla\chi_{\pm})) \cdot e$  belongs to  $Y^k$  for  $B\varphi \in X^{k,1/2}$  and  $B\chi_{\pm} \in X_{\pm}^{l,1/2}$  (cf. estimate above). This implies by [5], Lemma 2.2:

$$\begin{aligned} & \int_0^t e^{i(t-s)\Delta} B((\psi_{2T}\nabla\varphi) \times (\psi_{2T}\nabla(\chi_+ + \chi_-))) \cdot e \, ds \\ & \qquad \qquad \qquad \in C^0(\mathbf{R}, H^k(\mathbf{R}^3)), \end{aligned}$$

which by the integral equation implies  $B\varphi \in C^0([0, T], H^k(\mathbf{R}^3))$ .

The additional claim for  $l \leq 0$  follows easily by replacing in the application of Lemma 2.4, Lemma 2.6 and Lemma 2.8  $\|B\chi_{\pm}\|_{X_{\pm}^{l,1/2}}$  by  $\|\chi_{\pm}\|_{\dot{X}_{\pm}^{l+1,1/2}}$  and in the application of Lemma 2.5 and Lemma 2.7  $\|D\tilde{\varphi}D\varphi\|_{X_{\pm}^{l+2,-1/2(+)}}$  by  $\|D\tilde{\varphi}D\varphi\|_{\dot{X}_{\pm}^{l+2,-1/2(+)}}$  and in Lemma 2.9  $\|D\tilde{\varphi}D\varphi\|_{Y_{\pm}^{l+2}}$  by  $\|D\tilde{\varphi}D\varphi\|_{\dot{Y}_{\pm}^{l+2}}$ .  $\square$

**Remark** The case  $k = 1, l = -1$  especially shows that, given data  $\varphi_0, \chi_0$  with  $B\varphi_0 \in H^1(\mathbf{R}^3)$  and  $\chi_0, B^{-1}\chi_1 \in L^2(\mathbf{R}^3)$ , there exists a unique local solution  $(\varphi, \chi)$  of problem (4), (5), (3) on  $[0, T]$ ,

$$T = T(\|B\varphi_0\|_{H^1}, \|\chi_0\|_{L^2}, \|B^{-1}\chi_1\|_{L^2}),$$

with  $B\varphi \in X^{1,1/2}[0, T]$  and  $\chi, B^{-1}\chi_t \in X_+^{0,1/2+}[0, T] + X_-^{0,1/2+}[0, T]$ . Moreover  $B\varphi \in C^0([0, T], H^1(\mathbf{R}^3))$  and  $\chi, B^{-1}\chi_t \in C^0([0, T], L^2(\mathbf{R}^3))$ .

Combining the last remark with Proposition 1.1 we immediately get

**Theorem 2.2** *Let  $\varphi, \chi_0, \chi_1$  be given with*

$$\|B\varphi_0\|_{H^1} + \|\chi_0\|_{L^2} + \|B^{-1}\chi_1\|_{L^2} < \epsilon_0$$

where  $\epsilon_0$  is a sufficiently small constant (depending only on  $e \in \mathbf{R}^3$  and a Sobolev embedding constant). Then the Cauchy problem (4), (5), (3) has a unique global solution  $(\varphi, \chi)$  with

$$B\varphi \in X^{1,1/2}, \quad \chi, B^{-1}\chi_t \in X_+^{0,1/2+} + X_-^{0,1/2+}.$$

Moreover

$$B\varphi \in C^0(\mathbf{R}, H^1(\mathbf{R}^3)), \quad \chi, B^{-1}\chi_t \in C^0(\mathbf{R}, L^2(\mathbf{R}^3)).$$

Using the refinements of the nonlinear estimates given in Corollary 2.1, Corollary 2.2, Corollary 2.3, Corollary 2.4, Corollary 2.5 and Corollary 2.6 we get the following variant of Theorem 2.2.

**Theorem 2.3** *Assume  $k \geq 1, l \geq -1, l + 1 \leq k \leq l + 2$  and*

$$B\varphi_0 \in H^k(\mathbf{R}^3), \quad \chi_0, B^{-1}\chi_1 \in H^{l+1}(\mathbf{R}^3).$$

Then there exists  $1 \geq T > 0, T = T(\|B\varphi_0\|_{H^1}, \|\chi_0\|_{L^2}, \|B^{-1}\chi_1\|_{L^2})$ , such that problem (4), (5), (3) has a unique solution  $(\varphi, \chi)$  with

$$B\varphi \in X^{k,1/2}[0, T], \quad \chi, B^{-1}\chi_t \in X_+^{l+1,b_1}[0, T] + X_-^{l+1,b_1}[0, T],$$

where  $b_1 = 1/2+$ , if  $l + 1 < k \leq l + 2$ , and  $b_1 = 1/2$ , if  $k = l + 1$ . This solution satisfies

$$B\varphi \in C^0([0, T], H^k(\mathbf{R}^3)), \quad B\chi, B^{-1}\chi_t \in C^0([0, T], H^{l+1}(\mathbf{R}^3)).$$

*Proof.* One has to modify the usual contraction argument in the proof of Theorem 2.1 combining the following fundamental estimates, which e.g. in the case  $l + 1 < k \leq l + 2$  read as follows:

$$\|B^{-1}(D\varphi D\chi)\|_{X^{1,-1/2}} \leq cT^\Theta \|D\varphi\|_{X^{1,1/2}} \|\chi\|_{X^\pm_{0,1/2}} \tag{38}$$

$$\|B^{-1}(D\varphi D\chi)\|_{Y^1} \leq cT^\Theta \|D\varphi\|_{X^{1,1/2}} \|\chi\|_{X^\pm_{0,1/2}} \tag{39}$$

$$\begin{aligned} &\|B^{-1}(D\varphi D\chi)\|_{X^{k,-1/2}} \\ &\leq cT^\Theta (\|D\varphi\|_{X^{1,1/2}} \|\chi\|_{X^\pm_{l+1,1/2}} + \|D\varphi\|_{X^{k,1/2}} \|\chi\|_{X^\pm_{0,1/2}}) \end{aligned} \tag{40}$$

$$\|B(D\bar{\varphi}_1 D\varphi_2)\|_{X^\pm_{1,(-1/2)+}} \leq cT^\Theta \|D\varphi_1\|_{X^{1,1/2}} \|D\varphi_2\|_{X^{1,1/2}} \tag{41}$$

$$\begin{aligned} &\|B(D\bar{\varphi}_1 D\varphi_2)\|_{X^\pm_{l+1,(-1/2)+}} \\ &\leq cT^\Theta (\|D\varphi_1\|_{X^{1,1/2}} \|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}} \|D\varphi_2\|_{X^{1,1/2}}). \end{aligned} \tag{42}$$

Here (38), (39), (40), (41) and (42) follow from Lemma 2.6 (+remark), Lemma 2.8 (+remark), Corollary 2.3, Lemma 2.5 and Corollary 2.6, respectively.

In the limiting case  $k = l + 1$  we only have to replace (42) by

$$\begin{aligned} &\|B(D\bar{\varphi}_1 D\varphi_2)\|_{X^\pm_{l+1,-1/2}} \\ &\leq cT^\Theta (\|D\varphi_1\|_{X^{1,1/2}} \|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}} \|D\varphi_2\|_{X^{1,1/2}}), \end{aligned}$$

which follows from Corollary 2.4, and to add

$$\begin{aligned} &\|B(D\bar{\varphi}_1 D\varphi_2)\|_{Y^\pm_{l+1}} \\ &\leq cT^\Theta (\|D\varphi_1\|_{X^{1,1/2}} \|D\varphi_2\|_{X^{k,1/2}} + \|D\varphi_1\|_{X^{k,1/2}} \|D\varphi_2\|_{X^{1,1/2}}), \end{aligned}$$

coming from Corollary 2.6.

We omit the proof and just refer to [8], Theorem 1.1, where a detailed proof can be found.  $\square$

Combining Theorem 2.3 with Proposition 1.1 we can also show global well-posedness for smoother data, namely

**Theorem 2.4** *Assume  $k \geq 1, l \geq -1, l + 1 \leq k \leq l + 2$  and*

$$B\varphi_0 \in H^k(\mathbf{R}^3), \quad \chi_0, B^{-1}\chi_1 \in H^{l+1}(\mathbf{R}^3)$$

with

$$\|B\varphi_0\|_{H^1} + \|\chi_0\|_{L^2} + \|B^{-1}\chi_1\|_{L^2} < \epsilon_0,$$

where  $\epsilon_0$  is sufficiently small, dependent only on  $e \in \mathbf{R}^3$  and a Sobolev embedding constant. Then the Cauchy problem (4), (5), (3) has a unique global solution  $(\varphi, \chi)$  with

$$B\varphi \in X^{k,1/2}, \quad \chi, B^{-1}\chi_t \in X_+^{l+1,b_1} + X_-^{l+1,b_1},$$

where  $b_1 = 1/2+$ , if  $l + 1 < k \leq l + 2$ , and  $b_1 = 1/2$ , if  $k = l + 1$ . This solution satisfies

$$B\varphi \in C^0(\mathbf{R}, H^k(\mathbf{R}^3)), \quad \chi, B^{-1}\chi_t \in C^0(\mathbf{R}, H^{l+1}(\mathbf{R}^3)).$$

### 3. Local existence in 2 + 1 dimensions

**Lemma 3.1** *In space dimension  $n = 2$  the following estimate holds under the assumptions of Lemma 2.4:*

$$\begin{aligned} \|B^{-1+\epsilon}(D\varphi D\chi)\|_{X^{k-\epsilon,(-1/2)+}} &\leq cT^\Theta \|B^\epsilon D\varphi\|_{X^{k-\epsilon,1/2}} \|B^{-\delta} D\chi\|_{X_\pm^{l+\delta,1/2}} \end{aligned}$$

with  $\Theta > 0$ , if  $0 < \epsilon < 1$  and  $\delta > 0$ .

**Remark** If  $l < 0$ , we can replace  $\|B^{-\delta} D\chi\|_{X_\pm^{l+\delta,1/2}}$  by  $\|\chi\|_{\dot{X}_\pm^{l+1,1/2}}$ .

*Proof.* We follow the proof of Lemma 2.4 and have to give the estimate

$$\begin{aligned} S := \left| \int \frac{\widehat{v}_1 \widehat{v}_2 |\xi_1|^{-1+\epsilon} \langle \xi_1 \rangle^{k-\epsilon}}{|\xi|^{-\delta} \langle \xi \rangle^{l+\delta} |\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}}} \right| \\ \leq cT^\Theta \|v\|_{L_{xt}^2} \|v_1\|_{L_{xt}^2} \|v_2\|_{L_{xt}^2}. \end{aligned}$$

**Region A:**  $|\xi_1| \leq (1/2)|\xi_2|$  ( $\implies |\xi| \sim |\xi_2|$ ).

*Case 1:*  $|\xi_1| \geq 1, |\xi_2| \geq 1$ .

The same calculation as in Lemma 2.4 gives the desired estimate.

*Case 2:*  $|\xi_1| \leq 1, |\xi_2| \geq 1$ .

We have

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1+\epsilon}}{\langle \xi_2 \rangle^{l+k} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-\langle \sigma_2 \rangle^{1/2}}}.$$

a.  $l + k \leq 0$ .

Using (16) we get

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|(\langle\sigma\rangle + \langle\sigma_1\rangle + \langle\sigma_2\rangle)^{(-k-l)/2}}{|\xi_1|^{1-\epsilon}\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}.$$

Remark that  $-k-l < 1$ , so that Lemma 2.3 can be applied with  $m = 1 - \epsilon$  and gives  $2(a + a_1 + a_2) + m = k + l - 4 - \epsilon - \geq 3 - \epsilon - > 2$ , because  $k \geq 0$ ,  $l \geq -1$ , thus the desired estimate follows.

b.  $l + k \geq 0$ .

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{|\xi_1|^{1-\epsilon}\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}.$$

Using Lemma 2.3 with  $m = 1 - \epsilon$  gives the desired result.

Case 3:  $|\xi_1| \leq 1$ ,  $|\xi_2| \leq 1$  and w.l.o.g.  $\delta \leq \epsilon$ .

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2||\xi_1|^{-1+\epsilon}}{|\xi_2|^{\epsilon-\delta}\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}} \\ &\leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{|\xi_1|^{1-\delta}\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}. \end{aligned}$$

Using Lemma 2.3 with  $m = 1 - \delta$  gives the result.

**Region B:**  $(1/2)|\xi_2| \leq |\xi_1| \leq 2|\xi_2|$  ( $\Rightarrow |\xi| \leq 3|\xi_1|, 3|\xi_2|$ ).

We have

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle\xi\rangle^{-l-\delta}|\xi|^\delta}{|\xi_1|\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}.$$

Case 1:  $|\xi| \leq 1$ .

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{|\xi_1|^{1-\delta}\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}.$$

This can easily be handled by Lemma 2.3 with  $m = 1 - \delta$ .

Case 2:  $|\xi| \geq 1$  ( $\Rightarrow |\xi_1| \geq 1/3$ ).

$$S \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|\langle\xi\rangle^{-l}}{\langle\xi_1\rangle\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2-\langle\sigma_2\rangle^{1/2}}}.$$

a.  $l \leq 0$ .

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \xi_1 \rangle^{1+l} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}.$$

Because  $1 + l \geq 0$  this can easily be handled by Lemma 2.2 or Lemma 2.3.

b.  $l \geq 0$ .

We get

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \xi_1 \rangle \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}},$$

which can be treated by Lemma 2.2.

**Region C:**  $|\xi_1| \geq 2|\xi_2|$  ( $\Rightarrow |\xi| \sim |\xi_1|$ ).

We get

$$S \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1+\epsilon+\delta} \langle \xi_1 \rangle^{k-\epsilon-l-\delta}}{|\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}.$$

*Case 1:*  $|\xi_1| \geq 1, |\xi_2| \geq 1$ .

This case can be handled like the 3-dimensional case in Lemma 2.4.

*Case 2:*  $|\xi_1| \geq 1, |\xi_2| \leq 1$ .

We have by (16):

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{k-l-1}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(k-l-1)/2}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

Because  $k > l + 2$  we can apply Lemma 2.3 with  $m = \epsilon$  and compute  $2(a + a_1 + a_2) + m = 2(1/2 + 1/2 + 1/2 - (k - l - 1)/2) + \epsilon > 2 + \epsilon$ , so that the claimed estimate follows.

*Case 3:*  $|\xi_1| \leq 1, |\xi_2| \leq 1$  and w.l.o.g.  $\delta \leq 1 - \epsilon$ .

$$\begin{aligned} S &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1+\epsilon+\delta}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_2|^{1-\delta} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

An application of Lemma 2.3 with  $m = 1 - \delta$  gives the desired estimate.

□



**Lemma 3.2** *Let  $n = 2$ . Under the assumptions of Lemma 2.5 we have*

$$\|B^{2-\delta}(D\bar{\varphi}D\varphi)\|_{X_{\pm}^{l+\delta,-1/2+}} \leq cT^{\Theta} \|B^{\epsilon}D\varphi\|_{X^{k-\epsilon,1/2}}^2$$

with  $\Theta > 0$  for  $0 < \delta < 1$ ,  $0 < \epsilon < 1$ .

*Proof.* Arguing as in Lemma 2.5 we have to show

$$W := \left| \int \frac{\widehat{v}_1 \widehat{v}_2 \langle \xi \rangle^{l+\delta} |\xi|^{2-\delta}}{|\xi_1|^{\epsilon} \langle \xi_1 \rangle^{k-\epsilon} |\xi_2|^{\epsilon} \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}} \right| \leq cT^{\Theta} \|v\|_{L_{xt}^2} \|v_1\|_{L_{xt}^2} \|v_2\|_{L_{xt}^2}.$$

**Region A:**  $|\xi_2|/2 \leq |\xi_1| \leq 2|\xi_2|$  ( $\Rightarrow |\xi| \leq 3|\xi_1|, 3|\xi_2|$ ).

*Case 1:*  $|\xi_1| \geq 1$  ( $\Rightarrow |\xi_2| \geq 1/2$ ).

Using the assumption  $k \geq (l+2)/2$  we get

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi \rangle^{l+2}}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi \rangle^{l+2-2k}}{\langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}} \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{\langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}}. \end{aligned}$$

Lemma 2.3 gives the claimed estimate.

*Case 2:*  $|\xi_1| \leq 1$  ( $\Rightarrow |\xi_2| \leq 2 \Rightarrow |\xi| \leq 3$ ).

Using  $2-\delta-\epsilon > 0$  and  $|\xi| \leq 3$  we get the estimate

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi|^{2-\delta}}{|\xi_1|^{\epsilon} |\xi_2|^{\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi|^{2-\delta-\epsilon}}{|\xi_2|^{\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_2|^{\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}}. \end{aligned}$$

Lemma 2.3 gives the claimed estimate.

**Region B:**  $|\xi_1| \geq 2|\xi_2|$  ( $\Rightarrow |\xi| \sim |\xi_1|$ ) (and similarly  $|\xi_2| \geq 2|\xi_1|$ ).

We get

$$W \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{l+\delta} |\xi_1|^{2-\delta-\epsilon}}{\langle \xi_1 \rangle^{k-\epsilon} |\xi_2|^{\epsilon} \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}}$$

$$\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{l+2-k}}{|\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}}$$

Case 1:  $|\xi_2| \geq 1$ .

$$W \leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{l+2-k}}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}}$$

This is exactly the integral treated in Lemma 2.5 in the case  $n = 3$ .

Case 2:  $|\xi_2| \leq 1$ .

Assuming w.l.o.g.  $k \leq l + 2$  and using (16) we get the estimate

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{l+2-k}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq \int \frac{|\widehat{v}_1 \widehat{v}_2| (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(l+2-k)/2}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

The exponents in the denominator are nonnegative, because  $k > l + 1$ . Thus we apply Lemma 2.3 with e.g.  $a = 1/2 - (l + 2 - k)/2 > 0$ ,  $a_1 = a_2 = 1/2$ ,  $m = \epsilon$ , so that  $2(a + a_1 + a_2) + m > 2 + \epsilon > 2$ .  $\square$

The following variant of Lemma 3.2 is also true:

**Lemma 3.3** *Let  $n = 2$ . Under the assumptions of Lemma 2.5 we have*

$$\begin{aligned} \|D\bar{\varphi}D\varphi\|_{\dot{X}_\pm^{l+2,(-1/2)+}} &\sim \|B^{2-\delta}D\bar{\varphi}D\varphi\|_{\dot{X}_\pm^{l+\delta,(-1/2)+}} \\ &\leq cT^\Theta \|B^\epsilon D\varphi\|_{\dot{X}^{k-\epsilon,1/2}}^2 \end{aligned}$$

with  $\Theta > 0$  for  $0 < \epsilon < 1$ .

*Proof.* The proof of Lemma 3.2 is modified as follows. We have to estimate

$$W := \left| \int \frac{\widehat{v}_1 \widehat{v}_2 |\xi|^{l+2}}{|\xi_1|^\epsilon \langle \xi_1 \rangle^{k-\epsilon} |\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2-\langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \right|.$$

**Region A:**  $|\xi_2|/2 \leq |\xi_1| \leq 2|\xi_2|$  ( $\Rightarrow |\xi| \leq 3|\xi_1|, 3|\xi_2|$ ).

Case 1:  $|\xi_1| \geq 1$  ( $\Rightarrow |\xi_2| \geq 1/2$ ).

This case is treated exactly as in Lemma 3.2.

Case 2:  $|\xi_1| \leq 1$  ( $\Rightarrow |\xi_2| \leq 2 \Rightarrow |\xi| \leq 3$ ).

Using  $l + 2 - \epsilon > 0$  and  $|\xi| \leq 3$  we get the bound

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi|^{l+2}}{|\xi_1|^\epsilon |\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi|^{l+2-\epsilon}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2|}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}, \end{aligned}$$

which can be estimated by Lemma 2.3.

**Region B:**  $|\xi_1| \geq 2|\xi_2|$  ( $\Rightarrow |\xi| \sim |\xi_1|$ ) (and similarly  $|\xi_2| \geq 2|\xi_1|$ ).

Using  $l + 2 - \epsilon > 0$  we get the bound

$$\begin{aligned} W &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{l+2-\epsilon}}{\langle \xi_1 \rangle^{k-\epsilon} |\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}_1 \widehat{v}_2| \langle \xi_1 \rangle^{l+2-k}}{|\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

This is exactly the integral treated in the proof of Lemma 3.2, Region B. Thus the claimed estimate follows.  $\square$

In order to treat the limiting cases  $k = l + 1$  and  $k = l + 2$  we also need the following results:

**Lemma 3.4** *Let  $n = 2$ ,  $l \geq -1$ ,  $l + 1 \leq k \leq l + 2$ , and let  $\varphi, \chi$  be supported in  $\{|t| \leq cT\}$ . Then the following estimate holds:*

$$\|B^{-1+\epsilon}(D\varphi D\chi)\|_{X^{k-\epsilon, -1/2}} \leq cT^\Theta \|B^\epsilon D\varphi\|_{X^{k-\epsilon, 1/2}} \|B^{-\delta} D\chi\|_{X_\pm^{l+\delta, 1/2}}$$

with  $\Theta > 0$  for  $0 < \epsilon < 1$ ,  $\delta > 0$ .

**Remark** For  $l < 0$  we can replace  $\|B^{-\delta} D\chi\|_{X_\pm^{l+\delta, 1/2}}$  by  $\|\chi\|_{X_\pm^{l+1, 1/2}}$ .

*Proof.* We repeat the proof of Lemma 3.1 replacing  $\langle \sigma_1 \rangle^{1/2-}$  by  $\langle \sigma_1 \rangle^{1/2}$ . We only have to remark that the limit case  $k = l + 2$  is allowed in Region C, Case 1 and Case 2, because the power of the  $\sigma$ -modules in the denominator remains nonnegative in this case.  $\square$

**Lemma 3.5** *Let  $n = 2, l \geq -1, k \geq (l + 2)/2, k = l + 1$  and  $\text{supp } \varphi \subset \{|t| \leq cT\}$ . Then*

$$\|B^{2-\delta}(D\bar{\varphi}D\varphi)\|_{X_{\pm}^{l+\delta,-1/2}} \leq cT^{\Theta} \|B^{\epsilon}D\varphi\|_{X^{k-\epsilon,1/2}}^2$$

with  $\Theta > 0$  for  $0 < \delta < 1, 0 < \epsilon < 1$ .

*Proof.* We repeat the proof of Lemma 3.2 with  $\langle \sigma \rangle^{1/2-}$  replaced by  $\langle \sigma \rangle^{1/2}$ . The condition  $k < l + 1$  was only used in Region B, Cases 1 and 2 to produce nonnegative exponents of the  $\sigma$ -modules in the denominator, which is satisfied now also for  $k = l + 1$ . □

**Remark** The estimate of Lemma 3.3 remains true for  $k = l + 1$  in the following form:

$$\|D\bar{\varphi}D\varphi\|_{\dot{X}_{\pm}^{l+2,-1/2}} \leq cT^{\Theta} \|B^{\epsilon}D\varphi\|_{X^{k-\epsilon,1/2}}^2$$

with  $\Theta > 0$  for  $0 < \epsilon < 1$ .

This follows similarly as Lemma 3.5.

**Lemma 3.6** *Assume  $n = 2, l \geq -1, k = l + 2$ , and let  $\varphi, \chi$  be supported in  $\{|t| \leq cT\}$ . Then*

$$\|B^{-1+\epsilon}(D\varphi D\chi)\|_{Y^{k-\epsilon}} \leq cT^{\Theta} \|B^{\epsilon}D\varphi\|_{X^{k-\epsilon,1/2}} \|B^{-\delta}D\chi\|_{X_{\pm}^{l+\delta,1/2}}$$

with  $\Theta > 0$  for  $0 < \epsilon < 1, \delta > 0$ .

**Remark** For  $l < 0$  we can replace  $\|B^{-\delta}D\chi\|_{X_{\pm}^{l+\delta,1/2}}$  by  $\|\chi\|_{\dot{X}^{l+1,1/2}}$ .

*Proof.* Arguing as in the proof of Lemma 3.1 we now have to give the following estimate (cf. the proof of Lemma 2.8):

$$\begin{aligned} \tilde{S} &:= \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1+\epsilon} \langle \xi_1 \rangle^{k-\epsilon}}{|\xi|^{-\delta} \langle \xi \rangle^{l+\delta} |\xi_2|^{\epsilon} \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}} \\ &\leq cT^{\Theta} \|v\|_{L_{xt}^2} \|w_1\|_{L_x^2} \|v_2\|_{L_{xt}^2}. \end{aligned}$$

The only case where the strict inequality  $k < l + 2$  was used in the proof of Lemma 3.1 was Region C, Case 1 and 2. In all other regions we define  $\widehat{v}_1 := \langle \sigma_1 \rangle^{-1/2-} \widehat{w}_1$ , so that  $\|v_1\|_{L_{xt}^2} \leq c\|w_1\|_{L_x^2}$ , and  $\tilde{S}$  reads as follows:

$$\tilde{S} = \int \frac{|\widehat{v}_1 \widehat{v}_2| |\xi_1|^{-1+\epsilon} \langle \xi_1 \rangle^{k-\epsilon}}{|\xi|^{-\delta} \langle \xi \rangle^{l+\delta} |\xi_2|^{\epsilon} \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2-} \langle \sigma_2 \rangle^{1/2}}.$$

This is exactly the integral treated in the proof of Lemma 3.1, so that the result in these regions follows. It remains to consider Region C, Case 1 and 2 in the proof of Lemma 3.1. Similarly as there we get in Region C, Case 1 (with  $k = l + 2$ ):

$$\tilde{S} \leq c \int \frac{|\widehat{v}\widehat{w}_1\widehat{v}_2|\langle \xi_1 \rangle}{\langle \xi_2 \rangle^k \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}}.$$

This integral was already treated in the proof of Lemma 2.8. In Region C, Case 2 by use of (17) we arrive at

$$\begin{aligned} \tilde{S} &\leq c \int \frac{|\widehat{v}\widehat{w}_1\widehat{v}_2|\langle \xi_1 \rangle}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{v}\widehat{w}_1\widehat{v}_2|(\langle \sigma \rangle + \langle \sigma_2 \rangle + \langle \sigma_1 \rangle \phi_{\{c_1|\sigma_1| \leq |\xi_1|^2 \leq c_2|\sigma_1|\}})^{1/2}}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^{1/2}}. \end{aligned}$$

The two terms coming from  $\langle \sigma \rangle$  and  $\langle \sigma_2 \rangle$  in the numerator are treated by defining  $\widehat{v}_1$  as before by Lemma 2.3 with e.g.  $a = 0$ ,  $a_1 = 1/2-$ ,  $a_2 = 1/2$ ,  $m = \epsilon$ , so that  $2(a + a_1 + a_2) + m = 2 + \epsilon - > 2$ , whereas the term coming from  $\langle \sigma_1 \rangle$  is treated by defining  $\widehat{v}_1 := \langle \sigma_1 \rangle^{-1/2} \widehat{w}_1 \phi_{\{c_1|\sigma_1| \leq |\xi_1|^2 \leq c_2|\sigma_1|\}}$  so that  $\|v_1\|_{L^2_{xt}} \leq c\|w_1\|_{L^2_x}$ . Thus we are left with

$$\int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{|\xi_2|^\epsilon \langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}},$$

which can be handled by Lemma 2.3. □

Finally we get

**Lemma 3.7** *Let  $n = 2$ ,  $l \geq -1$ ,  $k \geq (l + 2)/2$ ,  $k = l + 1$  and  $\text{supp } \varphi \subset \{|t| \leq cT\}$ . Then*

$$\|B^{2-\delta}(D\bar{\varphi}D\varphi)\|_{Y^\pm_{l+\delta}} \leq cT^\Theta \|B^\epsilon D\varphi\|_{X^{k-\epsilon, 1/2}}^2$$

with  $\Theta > 0$  for  $0 < \delta < 1$ ,  $0 < \epsilon < 1$ .

*Proof.* We follow the proof of Lemma 3.2 and have to show

$$\begin{aligned} \tilde{W} &:= \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|\langle \xi \rangle^{l+\delta} |\xi|^{2-\delta}}{|\xi_1|^\epsilon \langle \xi_1 \rangle^{k-\epsilon} |\xi_2|^\epsilon \langle \xi_2 \rangle^{k-\epsilon} \langle \sigma \rangle \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \\ &\leq cT^\Theta \|w\|_{L^2_x} \|v_1\|_{L^2_{xt}} \|v_2\|_{L^2_{xt}}. \end{aligned}$$

In Region A, Case 1 of the proof of Lemma 3.2 we define  $\widehat{v} := \langle \sigma \rangle^{-1/2-\widehat{w}}$ ,

so that  $\|v\|_{L^2_{xt}} \leq c\|w\|_{L^2_x}$ , and we get as in Lemma 3.2 the estimate

$$\tilde{W} \leq c \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}},$$

which can easily be handled by Lemma 2.3.

Similarly, in Region A, Case 2 we arrive at

$$\tilde{W} \leq c \int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{|\xi_1|^\epsilon \langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}},$$

which can be controlled by Lemma 2.3 again.

In Region B, Case 1 we get for  $k = l + 1$  using (18):

$$\begin{aligned} \tilde{W} &\leq c \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|\langle\xi_1\rangle}{\langle\xi_2\rangle^k \langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \\ &\leq c \int \frac{|\widehat{w}\widehat{v}_1\widehat{v}_2|(\langle\sigma_1\rangle + \langle\sigma_2\rangle + \langle\sigma\rangle\phi_{\{c_1|\sigma|\leq|\xi|^2\leq c_2|\sigma|\}})^{1/2}}{\langle\xi_2\rangle^k \langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}. \end{aligned}$$

The two terms coming from  $\langle\sigma_1\rangle$  and  $\langle\sigma_2\rangle$  in the numerator are treated by defining  $\widehat{v}$  as before and using Lemma 2.2, whereas the term coming from  $\langle\sigma\rangle$  is treated by defining  $\widehat{v} := \langle\sigma\rangle^{-1/2}\widehat{w}\phi_{\{c_1|\sigma|\leq|\xi|^2\leq c_2|\sigma|\}}$ , so that  $\|v\|_{L^2_{xt}} \leq c\|w\|_{L^2_x}$ , leading to

$$\int \frac{|\widehat{v}\widehat{v}_1\widehat{v}_2|}{\langle\xi_2\rangle^k \langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}},$$

which again can be handled by Lemma 2.2 (remark that  $k \geq 1/2$ ).

In Region B, Case 2 we arrive at the corresponding integrals where  $\langle\xi_2\rangle^k$  is replaced by  $|\xi_2|^\epsilon$ . This can be treated by use of Lemma 2.3.  $\square$

**Remark** The following variant of Lemma 3.7 is also true, as follows similarly from the proof of Lemma 3.3:

Let  $n = 2$ ,  $l \geq -1$ ,  $k \geq (l + 2)/2$ ,  $k = l + 1$  and  $\text{supp } \varphi \subset \{|t| \leq cT\}$ . Then

$$\|(D\bar{\varphi}D\varphi)\|_{\dot{Y}^{l+2}_\pm} \leq cT^\Theta \|B^\epsilon D\varphi\|_{X^{k-\epsilon, 1/2}}^2$$

with  $\Theta > 0$  for  $0 < \epsilon < 1$ .

These results can now be used to prove a local existence and uniqueness result as in the 3 + 1-dimensional case.

**Theorem 3.1** *In space dimension  $n = 2$  assume  $l \geq -1$ ,  $l+1 \leq k \leq l+2$ ,  $k \geq (l+2)/2$ ,  $0 < \epsilon$ ,  $\delta < 1$ , and*

$$B^{1+\epsilon}\varphi_0 \in H^{k-\epsilon}(\mathbf{R}^2), B^{1-\delta}\chi_0 \in H^{l+\delta}(\mathbf{R}^2), B^{-\delta}\chi_1 \in H^{l+\delta}(\mathbf{R}^2).$$

*Then there exists  $1 \geq T = T(\|B^{1+\epsilon}\varphi_0\|_{H^{k-\epsilon}}, \|B^{1-\delta}\chi_0\|_{H^{l+\delta}}, \|B^{-\delta}\chi_1\|_{H^{l+\delta}}) > 0$ , such that the problem (1), (2), (3) has a unique solution  $(\varphi, \chi)$  with*

$$B^{1+\epsilon}\varphi \in X^{k-\epsilon, b}[0, T], \\ B^{1-\delta}\chi, B^{-\delta}\chi_t \in X_+^{l+\delta, b_1}[0, T] + X_-^{l+\delta, b_1}[0, T].$$

*Here  $b = 1/2+$ ,  $b_1 = 1/2+$ , if  $l+1 < k < l+2$ ,  $b = 1/2$ ,  $b_1 = 1/2+$ , if  $k = l+2$ , and  $b = 1/2+$ ,  $b_1 = 1/2$ , if  $k = l+1$ . This solution satisfies*

$$B^{1+\epsilon}\varphi \in C^0([0, T], H^{k-\epsilon}(\mathbf{R}^2)), \\ B^{1-\delta}\chi, B^{-\delta}\chi_t \in C^0([0, T], H^{l+\delta}(\mathbf{R}^2)).$$

*If  $l < 0$  we can replace  $B^{1-\delta}\chi_0, B^{-\delta}\chi_1 \in H^{l+\delta}$  by  $\chi_0 \in \dot{H}^{l+1}, \chi_1 \in \dot{H}^l$ , and  $B^{1-\delta}\chi, B^{-\delta}\chi_t \in X_+^{l+\delta, b_1}[0, T] + X_-^{l+\delta, b_1}[0, T]$  by  $\chi \in \dot{X}_+^{l+1, b_1}[0, T] + \dot{X}_-^{l+1, b_1}[0, T]$ ,  $\chi_t \in \dot{X}_+^{l, b_1}[0, T] + \dot{X}_-^{l, b_1}[0, T]$ , and we have  $\chi \in C^0([0, T], \dot{H}^{l+1}(\mathbf{R}^2)), \chi_t \in C^0([0, T], \dot{H}^l(\mathbf{R}^2))$ .*

**Remark** If this theorem would be true for  $\epsilon = 0$ , we would have local existence and uniqueness for data  $B\varphi_0 \in H^1(\mathbf{R}^2)$ ,  $\chi_0 \in L^2(\mathbf{R}^2)$ ,  $B^{-1}\chi_1 \in L^2(\mathbf{R}^2)$ . Using the a-priori bounds for  $\|B\varphi\|_{H^1} + \|\chi\|_{L^2} + \|B^{-1}\chi_t\|_{L^2}$  under a smallness assumption on  $\|B\varphi_0\|_{L^2}$  (cf. Chapter 1), this would imply global existence in these spaces under this smallness assumption.

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