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Center manifold approach to discrete integrable systems related to eigenvalues and singular values

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Abstract. The existence of center manifolds is closedly related to local behavior of dynamical systems. In this paper we consider center manifolds both of the discrete Toda equation and the discrete Lotka-Volterra system. Their solutions converge to eigenvalues and singular values of certain structured matrices. A free parameter plays a key role to show the existence of a center manifold of the discrete Lotka-Volterra system. A monotone convergence of the solution of the discrete Lotka-Volterra system is proved with the help of the existence of a center manifold. In contrast, a center manifold of the discrete Toda equation does not always exist.

Key words: center manifold, integrable systems, asymptotic behavior.

1. Introduction

The theory of integrable dynamical systems has an unexpected relationship to numerical algorithms. Some integrable systems of Lax form describe continuous flows of efficient numerical algorithms for eigenvalues [19, 4, 20]. We know, for example, a relationship between the finite nonperiodic Toda equation and the classical QR algorithm [14]. Each step of the QR algorithm for the exponential of a symmetric tridiagonal matrix L lies on the continuous flow of the Toda equation. Time-1 evolution of Toda is just one step of QR. Global convergence of the flow follows from that of QR, and vice versa. Note that the matrix exponential $\exp L$ has real, simple, positive eigenvalues. Local convergence of the Toda flow is discussed in [2] by using the center manifold theory.

A skillful time discretization of some integrable systems enables us to formulate numerical algorithms for computing eigenvalues [15]. A time discretization of the Toda equation is known as the quotient difference (qd) algorithm of Rutishauser [17]. A new singular value computing algorithm named the discrete Lotka-Volterra [10, 11] is also formulated starting from the finite Lotka-Volterra system by the same manner. Let us give a brief

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review on this aspect.

Let us consider the semi-infinite Toda equation

$$\begin{cases} \dfrac{dV_k}{dt} = V_k(J_k - J_{k+1}), & V_0 = 0, \\ \dfrac{dJ_k}{dt} = V_{k-1} - V_k, & k = 0,\, 1,\, 2,\, \ldots, \end{cases} \tag{1}$$

where $V_k = V_k(t)$ and $U_k = U_k(t)$, t and k are the time and spatial variables, respectively. We write $V_k(n\delta)$ as $V_k^{(n)}$ for some positive δ and integer n. Then $V_k^{(n+1)} = V_k(n\delta + \delta)$ and so on. Let Δ_n denote the difference with respect to the discrete time variable n. Namely, $\Delta_n f^{(n)} := (f^{(n+1)} - f^{(n)})/\delta$. A discrete time dynamical system

$$\begin{cases} \Delta_n V_k^{(n)} = V_k^{(n+1)} J_k^{(n+1)} - V_k^{(n)} J_{k+1}^{(n)}, & V_0^{(n)} = 0, \\ \Delta_n J_k^{(n)} = V_{k-1}^{(n+1)} - V_k^{(n)}, & k = 0,\, 1,\, 2,\, \ldots \end{cases} \tag{2}$$

is essentially given in [7]. This system is clearly a time discretization of the semi-infinite Toda equation (1), since it goes to (1) in the continuous limit $\delta \to 0$ with $n\delta = t$. Moreover it has a class of determinant solutions which is a discrete analogue of that of the continuous Toda equation. Let us introduce a set of variables $\{q_k^{(n)}\}$ and $\{e_k^{(n)}\}$ by $J_k^{(n)} = (1 - q_k^{(n)})/\delta$, $V_k^{(n)} = e_k^{(n)}/\delta^2$. Then the discrete Toda equation (2) leads to the recurrence relation of the qd algorithm [17]

$$e_k^{(n+1)} = \frac{q_{k+1}^{(n)} e_k^{(n)}}{q_k^{(n+1)}}, \quad q_k^{(n+1)} = q_k^{(n)} - e_{k-1}^{(n+1)} + e_k^{(n)}. \tag{3}$$

It is possible [16] to compute, for example, eigenvalues of a tridiagonal matrix T in the finite case where $k = 1,\, 2,\, \ldots m$ with $e_0^{(n)} \equiv 0$ and $e_m^{(n)} \equiv 0$ for $n = 0,\, 1,\, \ldots$. Thus the qd algorithm is equivalent to the discrete Toda equation. There is no free parameter in (3). The recurrence relation (3) has a subtraction and it is numerically endangered because of the possibility that the denominator $q_k^{(n+1)}$ may vanish. To guarantee the convergence to eiginvalues one should suppose that $q_k^{(0)} > 0$ and $e_k^{(0)} > 0$ for all k. This implies that the eigenvalue of T are all real, simple and positive [17]. In this case we say that the variable $q_k^{(n)}$ of the qd algorithm converges to eigenvalue as $n \to \infty$.

The dLV algorithm is formulated as follows. Let us consider the dynamical system

$$\frac{du_k}{dt} = u_k(u_{k+1} - u_{k-1}), \quad u_0(t) = 0, \quad k = 1, 2, \ldots. \tag{4}$$

This system is sometimes called the semi-infinite Lotka-Volterra system having Hankel determinant solutions [5]. In infinite case where $k = 0, \pm 1, \pm 2, \ldots$ and $u_0(t)$ is not specified, (4) is known as a spatial discretization of the celebrated KdV equation having soliton solutions [13]. The finite ($k = 1, 2, \ldots, m$) Lotka-Volterra system is essentially equivalent to Chu's dynamical system [3] whose solution converges to squares of singular values of a bidiagonal matrix. There are two ways to discretize (4) which keep the integrability. One is Hirota's method [8]. The other is an approach through spectral transformations of orthogonal polynomials [18]. The discrete Lotka-Volterra system is then

$$\Delta_n u_k^{(n)} = u_k^{(n)} u_{k+1}^{(n)} - u_k^{(n+1)} u_{k-1}^{(n+1)}, \quad u_0^{(n)} = 0, \quad k = 1, 2, \ldots. \tag{5}$$

Singular value problem for any rectangular matrix is reduced to that for an upper bidiagonal matrix of the form [6]

$$B := \begin{pmatrix} b_1 & b_2 & & \\ & b_3 & \ddots & \\ & & \ddots & b_{2m-2} \\ 0 & & & b_{2m-1} \end{pmatrix}, \quad b_k > 0.$$

It is shown in [10] that the solution of the discrete Lotka-Volterra system linearly converges to the square of singular values of B providing that the initial values and boundary values are set as $u_{2k-1}^{(0)} = b_{2k-1}^2/(1 + \delta u_{2k-2}^{(0)})$, $u_{2k}^{(0)} = b_{2k}^2/(1 + \delta u_{2k-1}^{(0)})$ and $u_0^{(n)} = 0$, $u_{2m}^{(n)} = 0$, respectively. Then the following convergence is proved

$$\lim_{n\to\infty} u_{2k-1}^{(n)} = \sigma_k^2, \quad \lim_{n\to\infty} u_{2k}^{(n)} = 0 \tag{6}$$

for any $\delta > 0$, where σ_k are singular values of B such that $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$. Therefore a new numerical algorithm for computing singular values is designed in terms of the discrete Lotka-Volterra system providing that the singular values of B are all simple and positive. This algorithm is named the dLV algorithm. There is a free parameter $\delta > 0$ in the recurrence relation (5). Recently the mdLVs (modified dLV with shift) algorithm which is a dLV algorithm with a stable shift is presented in [12]. The mdLVs has locally a quadratic convergence rate and a relative accuracy higher than the existing bidiagonal singular value computation algorithms.

The existence of center manifolds is closely related to local convergence and numerical stability of dynamical systems [1]. In this paper we consider center manifolds both of the discrete Toda equation, or equivalently the recurrence relation of the qd algorithm, and of the discrete Lotka-Volterra system. Center manifols of the discrete Toda equation and the discrete Lotka-Volterra system are discussed in §2 and §3, respectively. The free positive parameter δ plays an important role to show the existence of center manifold of the discrete Lotka-Volterra system. In contrast, center manifold of the discrete Toda equation does not always exist. Asymptotic behavior of solutions of the discrete Toda equation and the discrete Lotka-Volterra system are then investigated in §4 with the help of the center manifold theory. A monotone convergence of the solution of the discrete Lotka-Volterra system is then proved with the help of the existence of center manifold. It is shown in this paper that the dLV algorithm has a convergence property better than the qd algorithm.

2. Center manifold and the discrete Toda equation

In this section, we discuss a center manifold related to the finite discrete Toda equation

$$\begin{cases} q_k^{(n+1)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, & k = 1, 2, \ldots, m, \\ q_k^{(n+1)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, & k = 1, 2, \ldots, m-1, \end{cases} \tag{7}$$
$$e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \quad n = 0, 1, \ldots,$$

where $q_k^{(n)}$, $e_k^{(n)}$ denote the value of q_k, e_k at the discrete time n, respectively. The discrete Toda equation (7) generates the time evolution from n to $n+1$ of $\{q_k^{(n)}, e_k^{(n)}\}$. It is shown in [17] that $q_k^{(n)}$ and $e_k^{(n)}$ converge to a certain nonzero positive constant c_k and 0, respectively, as n tends to infinity if $q_k^{(0)} > 0$ and $e_k^{(0)} > 0$.

Let $\bar{q}_k^{(n)}$ be the difference between the discrete Toda variable $q_k^{(n)}$ and its equilibrium point c_k, namely, $\bar{q}_k^{(n)} := q_k^{(n)} - c_k$. Then we have the next lemma concerning the evolution from n to $n+1$ of $\{\bar{q}_k^{(n)}, e_k^{(n)}\}$.

Lemma 2.1 *Let $q_k^{(n)} = \bar{q}_k^{(n)} + c_k$ in the discrete Toda equation (7). If $|\bar{q}_k^{(n+1)}| < c_k$ for $k = 1, 2, \ldots, m$, then $\bar{q}_k^{(n+1)}$, $e_k^{(n+1)}$ are expressed in the form*

$$\bar{q}_k^{(n+1)} = -\alpha_{k-1} e_{k-1}^{(n)} + \bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}, \quad k = 1, 2, \ldots, m, \tag{8}$$

$$e_k^{(n+1)} = \alpha_k e_k^{(n)} + \tilde{b}_k, \quad k = 1, 2, \ldots, m-1, \tag{9}$$

respectively, where $\tilde{b}_0 \equiv 0$ and $\alpha_k := c_{k+1}/c_k$. Here $\tilde{b}_k = \tilde{b}_k(\bar{q}^{(n)}, e^{(n)})$, $k = 1, 2, \ldots, m-1$ denote certain functions of $\bar{q}^{(n)} := (\bar{q}_1^{(n)}, \bar{q}_2^{(n)}, \ldots, \bar{q}_m^{(n)})^\top \in \boldsymbol{R}^m$ and $e^{(n)} := (e_1^{(n)}, e_2^{(n)}, \ldots, e_{m-1}^{(n)})^\top \in \boldsymbol{R}^{m-1}$. The function $\tilde{b}_k$ and its first order derivative $\nabla_{(\bar{q}^{(n)}, e^{(n)})} \tilde{b}_k$ are zero at the origin $(\bar{q}^{(n)}, e^{(n)}) = (0, 0)$, namely,

$$\tilde{b}_k(0, 0) = 0, \quad \nabla_{(\bar{q}^{(n)}, e^{(n)})} \tilde{b}_k(0, 0) = 0, \quad k = 1, 2, \ldots, m-1, \tag{10}$$

where $\nabla_{(\bar{q}^{(n)}, e^{(n)})} := (\partial/\partial \bar{q}_1^{(n)}, \partial/\partial e_1^{(n)}, \ldots, \partial/\partial \bar{q}_{m-1}^{(n)}, \partial/\partial e_{m-1}^{(n)}, \partial/\partial \bar{q}_m^{(n)})^\top$.

Proof. By substituting $q_k^{(n)} = \bar{q}_k^{(n)} + c_k$ into (7), it follows that

$$\bar{q}_k^{(n+1)} = \bar{q}_k^{(n)} + e_k^{(n)} - e_{k-1}^{(n+1)}, \tag{11}$$

$$e_k^{(n+1)} = \frac{e_k^{(n)}}{c_k} (c_{k+1} + \bar{q}_{k+1}^{(n)}) \Big(1 + \frac{\bar{q}_k^{(n+1)}}{c_k}\Big)^{-1}. \tag{12}$$

Let us assume that $|\bar{q}_k^{(n+1)}| < c_k$. Note here that $(1 + \bar{q}_k^{(n+1)}/c_k)^{-1} = 1 + \sum_{j=1}^{\infty} (-\bar{q}_k^{(n+1)}/c_k)^j$. Then from (12) we derive (9) with $\tilde{b}_k$ as follows:

$$\tilde{b}_k = \frac{e_k^{(n)}}{c_k} \Big\{ \bar{q}_{k+1}^{(n)} + (c_{k+1} + \bar{q}_{k+1}^{(n)}) \sum_{j=1}^{\infty} \Big(-\frac{\bar{q}_k^{(n+1)}}{c_k} \Big)^j \Big\}. \tag{13}$$

Obviously, (9) and (11) lead to (8). By combining (13) with (8), we have the recurrence relation between $\tilde{b}_k$ and $\tilde{b}_{k-1}$,

$$\tilde{b}_k = \frac{e_k^{(n)}}{c_k} \Big\{ \bar{q}_{k+1}^{(n)} + (c_{k+1} + \bar{q}_{k+1}^{(n)}) \sum_{j=1}^{\infty} \Big(\frac{e_{k-1}^{(n)}}{c_{k-1}} - \frac{\bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}}{c_k} \Big)^j \Big\}. \tag{14}$$

When $k = 1$ in (14) with $e_0^{(0)} = 0$ and $\tilde{b}_0 = 0$, we can regard $\tilde{b}_1$ as the function of $\bar{q}^{(n)}$ and $e^{(n)}$ such that $\tilde{b}_1 = 0$ at the origin $(\bar{q}^{(n)}, e^{(n)}) = (0, 0)$. Namely, $\tilde{b}_1 = \tilde{b}_1(\bar{q}^{(n)}, e^{(n)})$ and $\tilde{b}_1(0, 0) = 0$. Eq. (14) also implies that $\tilde{b}_k =$

$\tilde{b}_k(\bar{q}^{(n)},\, e^{(n)})$ and $\tilde{b}_k(0,\, 0) = 0$ if $\tilde{b}_{k-1} = \tilde{b}_{k-1}(\bar{q}^{(n)},\, e^{(n)})$ and $\tilde{b}_{k-1}(0,\, 0) = 0$ for some k. Hence we see that $\tilde{b}_k = \tilde{b}_k(\bar{q}^{(n)},\, e^{(n)})$ and $\tilde{b}_k(0,0) = 0$ for any k. Derivatives of $\tilde{b}_k$ are given as

$$\begin{aligned}\frac{\partial \tilde{b}_k}{\partial \bar{q}_s^{(n)}} = \frac{e_k^{(n)}}{c_k}\Big\{\rho_{k+1}\sum_{j=0}^{\infty}\Big(\frac{e_{k-1}^{(n)}}{c_{k-1}} - \frac{\bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}}{c_k}\Big)^j \\ + \frac{c_{k+1} + \bar{q}_{k+1}^{(n)}}{c_k}\Big(\frac{\partial \tilde{b}_{k-1}}{\partial \bar{q}_s^{(n)}} - \rho_k\Big) \\ \times \sum_{j=1}^{\infty} j\Big(\frac{e_{k-1}^{(n)}}{c_{k-1}} - \frac{\bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}}{c_k}\Big)^{j-1}\Big\},\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial \tilde{b}_k}{\partial e_s^{(n)}} = \frac{\rho_k}{c_k}\bar{q}_{k+1}^{(n)} + \frac{c_{k+1} + \bar{q}_{k+1}^{(n)}}{c_k} \\ \times\Big\{\rho_k\sum_{j=1}^{\infty}\Big(\frac{e_{k-1}^{(n)}}{c_{k-1}} - \frac{\bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}}{c_k}\Big)^j \\ + e_k^{(n)}\Big(\frac{\rho_{k-1}}{c_{k-1}} - \frac{\rho_k}{c_k} + \frac{1}{c_k}\frac{\partial \tilde{b}_{k-1}}{\partial e_s^{(n)}}\Big) \\ \times \sum_{j=1}^{\infty} j\Big(\frac{e_{k-1}^{(n)}}{c_{k-1}} - \frac{\bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}}{c_k}\Big)^{j-1}\Big\}\end{aligned} \tag{16}$$

where $\rho_k = 1$ (if $s = k$) or $\rho_k = 0$ (otherwise). Obviously, $\nabla_{(\bar{q}^{(n)},e^{(n)})}\tilde{b}_1(0,\, 0) = 0$. Suppose that $\nabla_{(\bar{q}^{(n)},e^{(n)})}\tilde{b}_{k-1}(0,\, 0) = 0$ for some k. Then it follows immediately from (15) and (16) with $\tilde{b}_{k-1}(0,\, 0) = 0$ that $\nabla_{(\bar{q}^{(n)},e^{(n)})}\tilde{b}_k(0,\, 0) = 0$. Therefore it is concluded that $\nabla_{(\bar{q}^{(n)},e^{(n)})}\tilde{b}_k(0,\, 0) = 0$ for $k = 1,\, 2,\, \ldots,\, m-1$. □

Let us introduce a new variable $r^{(n)} := (r_1^{(n)},\, r_2^{(n)},\, \ldots,\, r_m^{(n)})^\top$, $r_k^{(n)} := -c_k(c_{k-1}-c_k)^{-1}e_{k-1}^{(n)} + \bar{q}_k^{(n)} + c_k(c_k-c_{k+1})^{-1}e_k^{(n)}$ where $c_1 > c_2 > \cdots > c_m > 0$. Then the recurrence relation between $\{r^{(n)},\, e^{(n)}\}$ and $\{r^{(n+1)},\, e^{(n+1)}\}$, derived from (8) and (9), satisfies the following lemma.

Lemma 2.2 *Let* $A := I \in \boldsymbol{R}^{m\times m}$*,* $B := \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{m-1}) \in \boldsymbol{R}^{(m-1)\times(m-1)}$ *and* $a := (a_1,\, a_2,\, \ldots,\, a_m)^\top \in \boldsymbol{R}^m$, $b := (b_1,\, b_2,\, \ldots,\, b_{m-1})^\top \in \boldsymbol{R}^{m-1}$*. Then* (8) *and* (9) *yield the following system*

$$\begin{cases} r^{(n+1)} = Ar^{(n)} + a(r^{(n)},\, e^{(n)}), \\ e^{(n+1)} = Be^{(n)} + b(r^{(n)},\, e^{(n)}), \end{cases} \tag{17}$$

where a, b and their Jacobi matrices $Da = \nabla_{(r^{(n)}, e^{(n)})} a^\top$, $Db = \nabla_{(r^{(n)}, e^{(n)})} b^\top$, at the origin $(r^{(n)},\, e^{(n)}) = (0,\, 0)$, satisfy

$$\begin{cases} a(0,\, 0) = 0, \quad Da(0,\, 0) = 0, \\ b(0,\, 0) = 0, \quad Db(0,\, 0) = 0. \end{cases} \tag{18}$$

Proof. Obviously $\bar{q}^{(n)}$ can be expressed by $r^{(n)}$ and $e^{(n)}$. We may regard $\tilde{b}_k = \tilde{b}_k(\bar{q}^{(n)},\, e^{(n)})$ as a function of $r^{(n)}$ and $e^{(n)}$, namely, $\tilde{b}_k(\bar{q}^{(n)},\, e^{(n)}) = b_k(r^{(n)},\, e^{(n)})$. By substituting $\tilde{b}_k = b_k$, for $k = 1, 2, \ldots, m-1$, into (9), we have $e^{(n+1)} = Be^{(n)} + b(r^{(n)},\, e^{(n)})$. Let us recall that $\tilde{b}_k(\bar{q}^{(n)},\, e^{(n)}) = 0$ at $(\bar{q}^{(n)},\, e^{(n)}) = (0,\, 0)$ in Lemma 2.1. Since $r^{(n)} = 0$ if $\bar{q}^{(n)} = 0$ and $e^{(n)} = 0$, we also see that $b_k(0,\, 0) = \tilde{b}_k(0,\, 0) = 0$ for $k = 1, 2, \ldots, m-1$. Note here that

$$\begin{cases} \dfrac{\partial \tilde{b}_k(0,\, 0)}{\partial \bar{q}_s^{(n)}} = \displaystyle\sum_{j=1}^{m} \frac{\partial r_j^{(n)}}{\partial \bar{q}_s^{(n)}} \frac{\partial b_k(0,\, 0)}{\partial r_j^{(n)}} = \sum_{j=1}^{m} \varrho_j \frac{\partial b_k(0,\, 0)}{\partial r_j^{(n)}} = 0, \\ \dfrac{\partial \tilde{b}_k(0,\, 0)}{\partial e_s^{(n)}} = \dfrac{\partial b_k(0,\, 0)}{\partial e_s^{(n)}} = 0, \end{cases} \tag{19}$$

where $\varrho_j = 1$ (if $j = s$) or $\varrho_j = 0$ (otherwise). Hence it follows that $b(0,\, 0) = 0$ and $Db(0,\, 0) = 0$.

On the other hand, the relationship between $r^{(n+1)}$ and $r^{(n)}$ is derived from

$$\begin{aligned} r_k^{(n+1)} &= -\frac{c_k e_{k-1}^{(n+1)}}{c_{k-1} - c_k} + \bar{q}_k^{(n+1)} + \frac{c_k e_k^{(n+1)}}{c_k - c_{k+1}} \\ &= -\frac{c_k(\alpha_{k-1} e_{k-1}^{(n)} + \tilde{b}_{k-1})}{c_{k-1} - c_k} + (-\alpha_{k-1} e_{k-1}^{(n)} + \bar{q}_k^{(n)} + e_k^{(n)} - \tilde{b}_{k-1}) \\ &\quad + \frac{c_k(\alpha_k e_k^{(n)} + \tilde{b}_k)}{c_k - c_{k+1}} \\ &= -\frac{c_k e_{k-1}^{(n)}}{c_{k-1} - c_k} + \bar{q}_k^{(n)} + \frac{c_k e_k^{(n)}}{c_k - c_{k+1}} + c_k \sum_{j=0}^{1} \frac{(-1)^{j+1}}{c_{k+j-1} - c_{k+j}} \tilde{b}_{k+j-1} \end{aligned}$$

$$= r_k^{(n)} + c_k \sum_{j=0}^{1} \frac{(-1)^{j+1}}{c_{k+j-1} - c_{k+j}} b_{k+j-1}. \tag{20}$$

Let $a_k := c_k \sum_{j=0}^{1} (-1)^{j+1} (c_{k+j-1} - c_{k+j})^{-1} b_{k+j-1}$ for $k = 1, 2, \ldots, m$. Then (20) leads to $r_k^{(n+1)} = r_k^{(n)} + a_k$ with the function a_k satisfying

$$\begin{cases} a_k(0,\, 0) = c_k \displaystyle\sum_{j=0}^{1} \frac{(-1)^{j+1}}{c_{k+j-1} - c_{k+j}} b_{k+j-1}(0,\, 0) = 0, \\ \dfrac{\partial a_k(0,\, 0)}{\partial r_s^{(n)}} = c_k \displaystyle\sum_{j=0}^{1} \frac{(-1)^{j+1}}{c_{k+j-1} - c_{k+j}} \frac{\partial b_{k+j-1}(0,\, 0)}{\partial r_s^{(n)}} = 0, \\ \dfrac{\partial a_k(0,\, 0)}{\partial e_s^{(n)}} = c_k \displaystyle\sum_{j=0}^{1} \frac{(-1)^{j+1}}{c_{k+j-1} - c_{k+j}} \frac{\partial b_{k+j-1}(0,\, 0)}{\partial e_s^{(n)}} = 0. \end{cases} \tag{21}$$

Therefore we have $r^{(n+1)} = Ae^{(n)} + a(r^{(n)},\, e^{(n)})$ with $a(0,\, 0) = 0$ and $Da(0,\, 0) = 0$. $\square$

By using the center manifold theory [1] with the help of Lemmas 2.1 and 2.2, we have the following theorem for $\psi_{Toda}^{(n)} \colon (r^{(n)},\, e^{(n)}) \mapsto (r^{(n+1)},\, e^{(n+1)})$ derived from the transformation $(q^{(n)},\, e^{(n)}) \mapsto (r^{(n)},\, e^{(n)})$ in the discrete Toda equation (7).

Theorem 2.1 *Let* $q_k^{(n)} = c_k(c_{k-1} - c_k)^{-1} e_{k-1}^{(n)} + r_k^{(n)} - c_k(c_k - c_{k+1})^{-1} e_k^{(n)} + c_k$ *in the discrete Toda equation* (7)*. If* $|q_k^{(n+1)} - c_k| < c_k$ *for* $k = 1,\, 2,\, \ldots,\, m$*, then the map* $\psi_{Toda}^{(n)} \colon (r^{(n)},\, e^{(n)}) \mapsto (r^{(n+1)},\, e^{(n+1)})$ *is given as* (17)*. There also exists a center manifold* $h_{Toda} \colon \boldsymbol{R}^m \to \boldsymbol{R}^{m-1}$ *for* $\psi_{Toda}^{(n)}$*.*

This may suggest that h_{Toda} for $\psi_{Toda}^{(n)}$ does not always exist. The center manifold h_{Toda} itself for $\psi_{Toda}^{(n)}$ exists if $|q_k^{(n+1)} - c_k| < c_k$ at some $n = n_0$. We here find it dubious that $|q_k^{(n+2)} - c_k| < c_k$ even if $|q_k^{(n+1)} - c_k| < c_k$. Namely, the existence of h_{Toda} for $\psi_{Toda}^{(n)}$ is not always guaranteed even if n holds $n > n_0$.

3. Center manifold and the discrete Lotka-Volterra system

Let us consider the discrete Lotka-Volterra system

$$u_k^{(n+1)}(1 + \delta u_{k-1}^{(n+1)}) = u_k^{(n)}(1 + \delta u_{k+1}^{(n)}), \quad k = 1,\, 2,\, \ldots,\, 2m-1, \tag{22}$$

$$u_0^{(n)} \equiv 0, \quad u_{2m}^{(n)} \equiv 0, \quad n = 0, 1, \ldots,$$

where the discrete step-size δ is some positive constant and $u_k^{(n)}$ denotes the value of u_k at the discrete time n. The discrete Lotka-Volterra system (22) is a time discretization of the continuous Lotka-Volterra system which describes the dynamics of preys and predators. In [10], the discrete Lotka-Volttera system (22) is shown to be applicable to the singular value computation of bidiagonal matrices.

In this section, we discuss a center manifold related to the discrete Lotka-Volttera system (22) by the same process as in §2. Note here that $u_{2k-1}^{(n)} \to t_k$ and $u_{2k}^{(n)} \to 0$ as $n \to \infty$ where t_k is some nonzero positive constant. Let us introduce a new variable $\bar{u}_{2k-1}^{(n)} := u_{2k-1}^{(n)} - t_k$. Then we obtain the following lemma for the recurrence relation between $\{\bar{u}_{2k-1}^{(n+1)}, u_{2k}^{(n+1)}\}$ and $\{\bar{u}_{2k-1}^{(n)}, u_{2k}^{(n)}\}$.

Lemma 3.1 *Let $u_{2k-1}^{(n)} = \bar{u}_{2k-1}^{(n)} + t_k$ in the discrete Lotka-Volterra system (22). If $|\bar{u}_{2k-1}^{(n+1)}| < \delta^{-1}$ for $k = 1, 2, \ldots, m$ and $|u_{2k}^{(n+1)}| < \delta^{-1}$ for $k = 1, 2, \ldots, m-1$, then $\{\bar{u}_{2k-1}^{(n+1)}\}_{k=1,2,\ldots,m}$ and $\{u_{2k}^{(n+1)}\}_{k=1,2,\ldots,m-1}$ take the form*

$$\bar{u}_{2k-1}^{(n+1)} = -\delta t_k \beta_{k-1} u_{2k-2}^{(n)} + \bar{u}_{2k-1}^{(n)} + \delta t_k u_{2k}^{(n)} + \tilde{f}_k(\bar{u}^{(n)}, u^{(n)}), \tag{23}$$

$$u_{2k}^{(n+1)} = \beta_k u_{2k}^{(n)} + \tilde{g}_k(\bar{u}^{(n)}, u^{(n)}), \tag{24}$$

respectively, where $\beta_k := (1 + \delta t_{k+1})/(1 + \delta t_k)$. Here $\tilde{f}_k = \tilde{f}_k(\bar{u}^{(n)}, u^{(n)})$ and $\tilde{g}_k = \tilde{g}_k(\bar{u}^{(n)}, u^{(n)})$ denote certain functions of $\bar{u}^{(n)} := (\bar{u}_1^{(n)}, \bar{u}_3^{(n)}, \ldots, \bar{u}_{2m-1}^{(n)})^\top \in \boldsymbol{R}^m$ and $u^{(n)} := (u_2^{(n)}, u_4^{(n)}, \ldots, u_{2m-2}^{(n)})^\top \in \boldsymbol{R}^{m-1}$. The functions $\tilde{f}_k$ and $\tilde{g}_k$ and their first derivatives $\nabla_{(\bar{u}^{(n)}, u^{(n)})}\tilde{f}_k$ and $\nabla_{(\bar{u}^{(n)}, u^{(n)})}\tilde{g}_k$ also satisfy

$$\tilde{f}_k(0, 0) = 0, \quad \nabla_{(\bar{u}^{(n)},u^{(n)})}\tilde{f}_k(0, 0) = 0, \tag{25}$$

$$\tilde{g}_k(0, 0) = 0, \quad \nabla_{(\bar{u}^{(n)},u^{(n)})}\tilde{g}_k(0, 0) = 0, \tag{26}$$

where

$$\nabla_{(\bar{u}^{(n)},u^{(n)})} := \Big(\frac{\partial}{\partial \bar{u}_1^{(n)}}, \frac{\partial}{\partial u_2^{(n)}}, \ldots, \frac{\partial}{\partial \bar{u}_{2m-3}^{(n)}}, \frac{\partial}{\partial u_{2m-2}^{(n)}}, \frac{\partial}{\partial \bar{u}_{2m-1}^{(n)}}\Big)^\top.$$

Proof. Let $\nabla := \nabla_{(\bar{u}^{(n)}, u^{(n)})}$, for simplicity. Let us assume that

$$\begin{cases} u_{2k-2}^{(n+1)} = \beta_{k-1} u_{2k-2}^{(n)} + \tilde{g}_{k-1}(\bar{u}^{(n)}, u^{(n)}), \\ \tilde{g}_{k-1}(0, 0) = 0, \quad \nabla \tilde{g}_{k-1}(0, 0) = 0 \end{cases} \tag{27}$$

for some k. Then $u_{2k-2}^{(n+1)}$ can be expressed by $\bar{u}^{(n)}$ and $u^{(n)}$. Let $u_{2k-2}^{(n+1)} = g_{k-1}^*(\bar{u}^{(n)}, u^{(n)})$. By substituting $u_{2k-1}^{(n)} = \bar{u}_{2k-1}^{(n)} + t_k$, $u_{2k-1}^{(n+1)} = \bar{u}_{2k-1}^{(n+1)} + t_k$ and $u_{2k-2}^{(n+1)} = g_{k-1}^*$ into the discrete Lotka-Volterra system (22), we have

$$\bar{u}_{2k-1}^{(n+1)} = (1+\delta u_{2k}^{(n)})(\bar{u}_{2k-1}^{(n)}+t_k)\left(1+\delta g_{k-1}^*\right)^{-1} - t_k, \tag{28}$$

where $g_{k-1}^* = \beta_{k-1} u_{2k-2}^{(n)} + \tilde{g}_{k-1}$. Note that $(1+\delta g_{k-1}^*)^{-1} = 1+\sum_{j=1}^{\infty}(-\delta g_{k-1}^*)^j$ if $|g_{k-1}^*| < \delta^{-1}$. Then, for a sufficiently small δ, (28) can be transformed to (23) with $\tilde{f}_k$ as follows:

$$\begin{cases} \tilde{f}_k = -\delta t_k \tilde{g}_{k-1} - \delta \eta_k \displaystyle\sum_{j=1}^{\infty} (-\delta g_{k-1}^*)^j, \\ \eta_k := (u_{2k}^{(n)} - g_{k-1}^*)(\bar{u}_{2k-1}^{(n)} - \delta t_k g_{k-1}^*). \end{cases} \tag{29}$$

Obviously, $g_{k-1}^*(0, 0) = \tilde{g}_{k-1}(0, 0) = 0$. Let $\bar{u}^{(n)} = 0$ and $u^{(n)} = 0$ in (29). Then it follows from $\tilde{g}_{k-1}(0, 0) = 0$ and $g_{k-1}^*(0, 0) = 0$ that $\eta_k(0, 0) = 0$ and $\tilde{f}_k(0, 0) = 0$. Here the first derivatives of $\tilde{f}_k$, η_k and $\tilde{g}_k$ are given as

$$\begin{cases} \nabla f_k = -\delta t_k \nabla \tilde{g}_{k-1} + \delta(\nabla \eta_k - \delta \eta_k \nabla g_{k-1}^*) \displaystyle\sum_{j=1}^{\infty} (-\delta g_{k-1}^*)^j, \\ \nabla \eta_k = (\bar{u}_{2k-1}^{(n)} - \delta t_k g_{k-1}^*)(\nabla u_{2k}^{(n)} - \nabla g_{k-1}^*) \\ \qquad\quad + (u_{2k}^{(n)} - g_{k-1}^*)(\nabla \bar{u}_{2k-1}^{(n)} - \delta t_k \nabla g_{k-1}^*), \\ \nabla g_{k-1}^* = \beta_{k-1} \nabla u_{2k-2}^{(n)} + \nabla \tilde{g}_{k-1}. \end{cases} \tag{30}$$

Since $\eta_k(0, 0) = 0$, $g_{k-1}^*(0, 0) = 0$ and $\nabla \tilde{g}_{k-1}(0, 0) = 0$, we also see from (30) that $\nabla g_{k-1}^*(0, 0) = (0, 0, \ldots, 0, \beta_{k-1}, 0, 0, \ldots, 0)^\top$, $\nabla \eta_k(0, 0) = 0$ and $\nabla \tilde{f}_k(0, 0) = 0$. Hence, for some k, (23) is derived from the discrete Lotka-Volterra system (22) and $\tilde{f}_k$ satisfies (25) under the assumption (27).

Similarly, for a suitable δ, $u_{2k}^{(n+1)}$ is written as (24) with

$$\begin{cases} \tilde{g}_k = -\delta\gamma_k\xi_k \sum_{j=0}^{\infty}(-\delta\gamma_k f_k^*)^j, \\ \xi_k := u_{2k}^{(n)}(\beta_k f_k^* - \bar{u}_{2k+1}^{(n)}), \\ f_k^* := -\delta t_k \beta_{k-1} u_{2k-2}^{(n)} + \bar{u}_{2k-1}^{(n)} + \delta t_k u_{2k}^{(n)} + \tilde{f}_k, \end{cases} \tag{31}$$

where $\gamma_k = (1 + \delta t_k)^{-1}$. By combining $\tilde{f}_k(0, 0) = 0$ with (31), it turns out that $f_k^*(0, 0) = 0$, $\xi_k(0, 0) = 0$ and $\tilde{g}_k(0, 0) = 0$. Moreover the first derivatives

$$\begin{cases} \nabla\tilde{g}_k = -\delta\gamma_k(\nabla\xi_k - j\delta\gamma_k\xi_k\nabla f_k^*)\sum_{j=0}^{\infty}(-\delta\beta_k f_k^*)^j, \\ \nabla\xi_k = \nabla u_{2k}^{(n)}(\beta_k f_k^* - \bar{u}_{2k+1}^{(n)}) + u_{2k}^{(n)}(\beta_k\nabla f_k^* - \nabla\bar{u}_{2k+1}^{(n)}), \\ \nabla f_k^* = -\delta t_k\beta_{k-1}\nabla u_{2k-2}^{(n)} + \nabla\bar{u}_{2k-1}^{(n)} + \delta t_k\nabla u_{2k}^{(n)} + \nabla\tilde{f}_k \end{cases} \tag{32}$$

imply that $\nabla f_k^*(0, 0) = (0, 0, \ldots, 0, -\delta t_k\beta_{k-1}, \delta t_k, 1, 0, 0, \ldots, 0)^\top$, $\nabla\xi_k(0, 0) = 0$ and $\nabla\tilde{g}_k(0, 0) = 0$. Therefore, the assumption (27) leads to (23)–(26) for some k.

When $k = 1$ in (22), we have $\bar{u}_1^{(n+1)} = \bar{u}_1^{(n)} + \delta t_1 u_2^{(n)} + \tilde{f}_1$ with $\tilde{f}_1 = \delta\bar{u}_1^{(n)}u_2^{(n)}$. Obviously, $\tilde{f}_1(0, 0) = 0$ and $\nabla\tilde{f}_1(0, 0) = 0$. The discrete Lotka-Volterra system (22) with $k = 1$ also yields that $u_2^{(n+1)} = \beta_1 u_2^{(n)} + \tilde{g}_1$ and

$$\begin{cases} \tilde{g}_1 = -\delta\gamma_1\xi_1 \sum_{j=0}^{\infty}(-\delta\gamma_1 f_1^*)^j, \\ \xi_1 := u_2^{(n)}(\beta_1 f_1^* - \bar{u}_3^{(n)}), \\ f_1^* := \bar{u}_1^{(n)} + \delta t_1 u_2^{(n)} + \tilde{f}_1, \end{cases} \tag{33}$$

if $\delta < |f_1^*|^{-1}$. We may regard (33) as (31) with $k = 1$. Hence, we derive $\tilde{g}_1(0, 0) = 0$ and $\nabla\tilde{g}_1(0, 0) = 0$ from $\tilde{f}_1(0, 0) = 0$ and $\nabla\tilde{f}_1(0, 0) = 0$, immediately. Without loss of generality, $u_2^{(n+1)}$ and $\tilde{g}_1$ satisfy (27) with $k = 2$. □

Moreover, we describe a lemma derived from (23) and (24) in Lemma 3.1.

Lemma 3.2 *Let $t_1 > t_2 > \cdots > t_m$. Then from* (23) *and* (24) *it follows that*

$$\begin{cases} v^{(n+1)} = Fv^{(n)} + f(v^{(n)}, u^{(n)}), \\ u^{(n+1)} = Gu^{(n)} + g(v^{(n)}, u^{(n)}), \end{cases} \tag{34}$$

where $v^{(n)} := (v_1^{(n)}, v_3^{(n)}, \ldots, v_{2m-1}^{(n)})^\top \in \boldsymbol{R}^m$, $v_{2k-1}^{(n)} := -t_k(1+\delta t_k)(t_{k-1}-t_k)^{-1}u_{2k-2}^{(n)} + \bar{u}_{2k-1}^{(n)} + t_k(1+\delta t_k)(t_k - t_{k+1})^{-1}u_{2k}^{(n)}$ *and* $F := \mathrm{diag}(1, 1, \ldots, 1) \in \boldsymbol{R}^{m\times m}$, $G := \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{m-1}) \in \boldsymbol{R}^{(m-1)\times(m-1)}$. *The functions* f, g *and their Jacobi matrices* $Df = \nabla_{(v^{(n)},u^{(n)})}f^\top$, $Dg = \nabla_{(v^{(n)},u^{(n)})}g^\top$ *are zero at the origin, namely,* $f(0, 0) = 0$, $g(0, 0) = 0$ *and* $Df(0, 0) = 0$, $Dg(0, 0) = 0$.

Proof. There exist some functions $\hat{f}_k$ and $\hat{g}_k$ such that $\hat{f}_k(v^{(n)}, u^{(n)}) = \tilde{f}_k(\bar{u}^{(n)}, u^{(n)})$ and $\hat{g}_k(v^{(n)}, u^{(n)}) = \tilde{g}_k(\bar{u}^{(n)}, u^{(n)})$. This is because $\bar{u}^{(n)}$ can be expressed by using $v^{(n)}$ and $u^{(n)}$. By the transformation $(\bar{u}^{(n)}, u^{(n)}) \mapsto (v^{(n)}, u^{(n)})$ in (23) and (24), we derive

$$\begin{cases} v_{2k-1}^{(n+1)} = v_{2k-1}^{(n)} + \hat{f}_k + \displaystyle\sum_{j=0}^{1} \frac{(-1)^{j+1}t_k(1+\delta t_k)}{t_{k+j-1} - t_{k+j}}\hat{g}_{k+j-1}, \\ u_{2k}^{(n+1)} = \beta_k u_{2k}^{(n)} + \hat{g}_k. \end{cases} \tag{35}$$

Let $f_k(v^{(n)}, u^{(n)})$ and $g_k(v^{(n)}, u^{(n)})$ defined by

$$\begin{cases} f_k := \hat{f}_k + \displaystyle\sum_{j=0}^{1} \frac{(-1)^{j+1}t_k(1+\delta t_k)}{t_{k+j-1} - t_{k+j}}\hat{g}_{k+j-1}, \\ g_k := \hat{g}_k. \end{cases} \tag{36}$$

Let $f := (f_1, f_2, \ldots, f_m)^\top$ and $g := (g_1, g_2, \ldots, g_{m-1})^\top$. Then we have (34). Obviously, $(v^{(n)}, u^{(n)}) = 0$ if $(\bar{u}^{(n)}, u^{(n)}) = 0$. Hence we see from (25) and (26) that

$$\begin{cases} \hat{f}_k(0, 0) = \tilde{f}_k(0, 0) = 0, \\ \nabla_{(v^{(n)},u^{(n)})}\hat{f}_k(0, 0) = \nabla_{(\bar{u}^{(n)},u^{(n)})}\tilde{f}_k(0, 0) = 0, \\ \hat{g}_k(0, 0) = \tilde{g}_k(0, 0) = 0, \\ \nabla_{(v^{(n)},u^{(n)})}\hat{g}_k(0, 0) = \nabla_{(\bar{u}^{(n)},u^{(n)})}\tilde{g}_k(0, 0) = 0. \end{cases} \tag{37}$$

Consequently, it follows from (36) and (37) that $f(0, 0) = 0$, $g(0, 0) = 0$ and $Df(0, 0) = 0$, $Dg(0, 0) = 0$. □

Lemmas 3.1 and 3.2 with the center manifold theory state to the following theorem for $\psi_{LV}^{(n)}\colon (v^{(n)}, u^{(n)}) \mapsto (v^{(n+1)}, u^{(n+1)})$ associated with the discrete Lotka-Volterra system (22).

Theorem 3.1 *Let* $u_{2k-1}^{(n)} = t_k(1+\delta t_k)(t_{k-1}-t_k)^{-1}u_{2k-2}^{(n)} + v_{2k-1}^{(n)} - t_k(1+\delta t_k)(t_k - t_{k+1})^{-1}u_{2k}^{(n)} + t_k$ *in the discrete Lotka-Volterra system* (22)*. If* $|\bar{u}_{2k-1}^{(n+1)}| < \delta^{-1}$ *for* $k = 1, 2, \ldots, m$ *and* $|u_{2k}^{(n+1)}| < \delta^{-1}$ *for* $k = 1, 2, \ldots, m-1$*, then the map* $\psi_{LV}^{(n)} \colon (v^{(n)}, u^{(n)}) \mapsto (v^{(n+1)}, u^{(n+1)})$ *is given as* (34)*. There also exists a center manifold* $h_{LV} \colon \boldsymbol{R}^m \to \boldsymbol{R}^{m-1}$ *for* $\psi_{LV}^{(n)}$*.*

It is obvious from [11] that $0 < \max_k |u_k^{(n)}| < M$ for some positive M. Namely, there exists a discrete step-size δ such that $\min_k |\bar{u}_{2k-1}^{(n+1)}| < \delta^{-1}$ and $\min_k |u_{2k}^{(n+1)}| < \delta^{-1}$ at any n. The center manifold for $\psi_{LV}^{(n)}$ exists certainly at any n if the discrete step-size δ, a free parameter of the discrete Lotka-Volterra system, is suitably chosen. This is greatly different from the center manifold for $\psi_{Toda}^{(n)}$ related to the discrete Toda equation (7).

4. Asymptotic behavior

In this section, we investigate the asymptotic behavior of the solution of the discrete Toda equation (7) and the discrte Lotka-Volterra system (22) with the help of the center manifold theorems in [1].

Let us assume $\mathcal{T} \colon \boldsymbol{R}^{2m-1} \to \boldsymbol{R}^{2m-1}$ has the following form

$$\mathcal{T}(x, y) = (\mathcal{A}x + \zeta(x, y), \mathcal{B}y + \chi(x, y)), \tag{38}$$

where $x \in \boldsymbol{R}^m$ and $y \in \boldsymbol{R}^{m-1}$. Let $\mathcal{A}$ and $\mathcal{B}$ be square matrices such that each eigenvalue of $\mathcal{A}$ has modulus 1 and each eigenvalues of $\mathcal{B}$ has modulus less than 1. Let ζ and χ be $\boldsymbol{C}^2$ functions such that

$$\zeta(0, 0) = 0, \quad D\zeta(0, 0) = 0, \tag{39}$$

$$\chi(0, 0) = 0, \quad D\chi(0, 0) = 0, \tag{40}$$

where $D\zeta$ and $D\chi$ denote Jacobi matrices of ζ and χ, respectively. Obviously, there exists a center manifold h for $\mathcal{T}$. In general, it is not easy to find center manifolds for maps exactly. The following theorem is useful in finding such manifolds approximately.

Theorem 4.1 (Carr) *Let* $\phi \colon \boldsymbol{R}^{\ell_1} \to \boldsymbol{R}^{\ell_2}$ *be a* $\boldsymbol{C}^1$ *map with* $\phi(0) = 0$ *and* $D\phi(0) = 0$*. Let* M *be the operator on* ϕ *given as*

$$M\phi(x) = \phi\big(\mathcal{A}x + \zeta(x, \phi(x))\big) - \mathcal{B}\phi(x) - \chi(x, \phi(x)). \tag{41}$$

If $M\phi(x) = O(|x|^p)$ *as* $x \to 0$ *for some* $p > 1$*, then a center manifold* h *satisfies* $h(x) = \phi(x) + O(|x|^p)$ *as* $x \to 0$*.*

Let us recall the functions a, b and f, g shown in previous sections. Note here that $a(r^{(n)}, 0) = 0$, $b(r^{(n)}, 0) = 0$ and $f(v^{(n)}, 0) = 0$, $g(v^{(n)}, 0) = 0$. Let $\zeta(x, y)$ and $\chi(x, y)$ satisfy $\zeta(x, 0) = 0$ and $\chi(x, 0) = 0$, respectively. Then it is obvious that $\phi(x) = 0$ is a solution of $M\phi(x) = 0$. Hence, for sufficiently small x, $\phi(x) = 0$ is a good approximation of the center manifold h.

Let us consider the discrete system

$$\begin{cases} x^{(n+1)} = \mathcal{A}x^{(n)} + \zeta(x^{(n)}, y^{(n)}), \\ y^{(n+1)} = \mathcal{B}y^{(n)} + \chi(x^{(n)}, y^{(n)}). \end{cases} \tag{42}$$

From viewpoint of the center manifold theory, (42) is the same type of discrete system as (17) and (34). The evolution from n to $n+1$ in (42) is also equivalent to that by the map $\mathcal{T}\colon (x^{(n)}, y^{(n)}) \mapsto (x^{(n+1)}, y^{(n+1)})$. To investigate the asymptotic behavior of solution of (42) for small $x^{(n)}$ can be simplified by the following two theorems.

Theorem 4.2 (Carr) *The asymptotic behavior of small solution of* (42) *is governed by the flow on the center manifold* h *for* $\mathcal{T}$ *which is given by*

$$z^{(n+1)} = \mathcal{A}z^{(n)} + \zeta(z^{(n)}, h(z^{(n)})). \tag{43}$$

Theorem 4.3 (Carr) *The stability of the zero solution of* (42) *is equivalent to that of* (43)*. In particular, suppose that* $(x^{(n)}, y^{(n)})$ *is a solution of* (42) *with the sufficiently small initial value* $(x^{(0)}, y^{(0)})$ *and zero solution of* (43) *is stable. Then there exists a solution* $z^{(n)}$ *of* (43) *such that* $|x^{(n)} - z^{(n)}| \leq \kappa\varepsilon^n$ *and* $|y^{(n)} - h(z^{(n)})| \leq \kappa\varepsilon^n$ *at any* n *where* κ *and* ε *are positive constants with* $\varepsilon < 1$*.*

Let $h = 0$ in (43). Then we have $\zeta(z^{(n)}, 0) = 0$ and $z^{(n+1)} = z^{(n)}$. Namely, the zero solution of (43) is stable. Hence we see from Theorem 4.3 that the solution $(x^{(n)}, y^{(n)})$ with sufficiently small $(x^{(0)}, y^{(0)})$ of (42) monotonically converges. On the other hand, it is not clear whether we may apply the center manifold theory to the asymptotic analysis of the discrete Toda equation (7) in the case where c_k is extremely small. Regarding the discrete Lotka-Volterra system (22), the existance of the associated center manifold can be guaranteed by a suitable choice of δ. In concluding, we establish the following theorem for the discrete Lotka-Volterra system (22).

Theorem 4.4 *The solution* $(u_{2k-1}^{(n)}, u_{2k}^{(n)})$ *of the Lotka-Volterra system* (22) *monotonically tends to* $(t_k,\, 0)$ *for* $n \geq n^*$*, if* $(u_{2k-1}^{(n_*)} - t_k,\, u_{2k}^{(n_*)})$ *is sufficiently small.*

With respect to the discrete Toda equation (7), we also have a similar theorem except for the case where extremely small $q_k^{(n)} \approx c_k$ emerges in the denominator.

5. Concluding remarks

In this paper, we analyse two discrete integrable systems, the discrete Toda equation and the discrete Lotka-Volterra system, from the viewpoint of the center manifold theory. We discuss the existence of the center manifolds for the associated maps with the discrete Toda equation and the discrete Lotka-Volterra system in §2 and §3, respectively. In §4, we investigate the asymptotic behavior, which is not able to be realized from an approach to global convergence in [17] and [10, 11], of the solutons of both integrable systems by using the center manifold theorems.

As a result, the solution of the discrete Lotka-Volterra system is shown to have more desirable convergence than that of the discrete Toda equation. Moreover, the property shown in Theorem 4.4 is convenient to design an efficient numerical algorithm and for finite arithmetic on computer. This is because the residual to equilibrium decreases monotonically if n is sufficiently large. In other words, the residual does not grow suddenly. The dLV algorithm based on the discrete Lotka-Volterra system is accordingly a highly credible algorithm for singular values.

The center manifold theory is applicable to some discrete systems in numerical algorithms, for example, the discrete Lotka-Volterra system with variable step-size [9, 18] appeared in the mdLVs algorithm [12]. This aspect will be discussed in a separate paper.

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