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On spacelike surfaces in Anti de Sitter 3-space from the contact viewpoint

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Abstract. We study differential geometry of spacelike surfaces in Anti de Sitter 3-space from the contact viewpoint. We define the timelike Anti de Sitter Gauss images and timelike Anti de Sitter height functions on spacelike surfaces and investigate the geometric meanings of singularities of these mappings. We consider the contact of spacelike surfaces with models (so-called AdS-great-hyperboloids) as an application of Legendrian singularity theory.

Key words: Anti de Sitter 3-space, TAdS-Gauss image, AdS-G-K curvature, Legendrian singularities.

1. Introduction

Recently, there appeared several articles of differential geometry on submanifolds in Lorentzian space forms as applications of singularity theory [7], [8], [9], [10], [11], [12], [13], [15]. Minkowski space is a flat Lorentzian space form and de Sitter space is the Lorentzian space form with positive constant curvature. The Lorentzian space form with the negative constant curvature is called Anti de Sitter space which is a vacuum solution of the Einstein equation. However, there are very few researches on differential geometry of submanifolds in Anti de Sitter space as applications of singularity theory so far as we know. In this paper we study the differential geometry on spacelike surfaces in Anti de Sitter 3-space from the view point of the theory of Legendrian singularities.

On the other hand, hypersurfaces in hyperbolic space have been studied in [14]. The basic notions and tools for the study of the differential geometry of hypersurfaces in hyperbolic space have been established. Especially, the hyperbolic Gauss indicatrix of a hypersurface in hyperbolic space has

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been explicity described and the contact of hypersurfaces with model hypersurfaces has been systematically studied as an application of singulary theory to the hyperbolic Gauss indicatrix. Our aim in this paper is to develop the analogous study for spacelike surfaces in Anti de Sitter 3-space. In Section 2 we first show the basic notions on semi-Euclidean 4-space with index 2 and contact geometry. Especially we give the Legendrian duality theorem (Theorem 2.1) between Anti de Sitter 3-spaces, which is the key to see the view of the whole. In Section 3 we develop local differential geometry of spacelike surfaces in Anti de Sitter 3-space and introduce the notion of timelike Anti de Sitter Gauss images of spacelike surfaces in Anti de Sitter 3-space. Corresponding to this notion we define the Anti de Sitter Gauss Kronecker (briefly, AdS-G-K) curvature and consider the geometric meaning of this curvature. One of our conclusions asserts that the AdS-G-K curvature describes the contact of spacelike surfaces with models (i.e., AdS-great-hyperboloids). We introduce the notion of timelike height functions in Section 4, named AdS-height function, which is useful to show that the TAdS-Gauss image has a singular point if and only if the AdS-G-K curvature vanishes at such point. In Section 5, we apply mainly the theory of Legendrian singularities for the study of the contact of spacelike surfaces with AdS-great-hyperboloids. In Section 6 we give the generic classification of singularities of TAdS-Gauss images.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. The basic notations and the duality theorem

In this section we prepare basic notions on semi-Euclidean 4-space with index 2 and contact geometry.

Let $\mathbb{R}^4 = \{(x_1, \ldots, x_4) | x_i \in \mathbb{R}\ (i = 1, \ldots, 4)\}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \ldots, x_4)$ and $\boldsymbol{y} = (y_1, \ldots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *smei-Euclidean 4-space with index 2* and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a vector $\boldsymbol{n} \in \mathbb{R}^4_2$ and a real number c, we define the *hyperplane with pseudo-normal* $\boldsymbol{n}$ by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We call $HP(\boldsymbol{n}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane of index 2* or a *null hyperplane* if $\boldsymbol{n}$ is *timelike, spacelike or null* respectively.

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*) by

$$H^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}.$$

For any $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \in \mathbb{R}^4_2$. We define a vector $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ by

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{X}_i = (x^i_1, x^i_2, x^i_3, x^i_4)$. We can easily check that

$$\langle \boldsymbol{X}, \boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 \rangle = \det(\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3),$$

so that $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ is pseudo-orthogonal to any $\boldsymbol{X}_i$ (for $i = 1, 2, 3$).

In this paper we stick to spacelike surfaces in Anti de Sitter 3-space H^3_1. Typical spacelike surfaces in H^3_1 are given by the intersection of H^3_1 with a Lorentz hyperplane in $\mathbb{R}^4_2$:

$$AH(\boldsymbol{n}, c) = H^3_1 \cap HP(\boldsymbol{n}, c),$$

where $\|\boldsymbol{n}\| > |c|$. We say that $AH(\boldsymbol{n}, c)$ is a *AdS-hyperboloid* in the Anti de Sitter 3-space. In particular, we call $AH(\boldsymbol{n}, 0)$ the *AdS-great-hyperboloid.*

On the other hand, we now give a brief review on contact manifolds and Legendrian submanifolds. For some detailed results on contact geometry, please refer to [1], [3]. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α is a *contact form.* Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around p such that

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^{m} p_i dx_i$$

Moreover, let $\pi : PT^*M \longrightarrow M$ be the projective cotangent bundle. This fibration is a Legendrian fibration with the canonical contact structure K. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*M \longrightarrow PT^*M$ and differential map $d\pi : TPT^*M \longrightarrow TM$ of π. For any $X \in TPT^*M$, there exits an element $\alpha \in T^*M$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x M$, the property $\alpha(V) = 0$ does not depend on the choice of the representative of the class $[\alpha]$. Thus we can define the canonical contact structure on PT^*M by

$$K = \{X \in TPT^*M \mid \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighborhood $(U, (x_1, \ldots, x_n))$ on M, we have a trivialization

$$PT^*U \cong U \times P(\mathbb{R}^{n-1})^*$$

and we call $((x_1, \ldots, x_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,\xi)}$ if and only if $\sum_{i=1}^{n} \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i}$. This means that the contact form α on the affine coordinates $U_j = \{(\boldsymbol{x}, [\xi]) \mid \xi_j \neq 0\} \subset PT^*U$ is given by $\alpha =$

$\sum_{i=1}^{n}(\xi_i/\xi_j)dx_i$. An immersion $i : L \longrightarrow PT^*M$ is said to be a *Legendrian immersion* if $\dim L = n-1$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ a *Legendrian map* and the set $W(i) = \text{image } \pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

In [6], [7] S. Izumiya have shown the basic duality theorem which is the fundamental tool for the study of hypersurfaces in Minkowski pseudo-spheres. In this paper we consider the similar duality theorem in H^3_1. We now consider the following double fibrations:

(1) $H^3_1 \times H^3_1 \supset \Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$,
(2) $\pi_1 : \Delta \longrightarrow H^3_1$, $\pi_2 : \Delta \longrightarrow H^3_1$,
(3) $\theta_1 = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle \mid \Delta$, $\theta_2 = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle \mid \Delta$.

Here,

$$\pi_1(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}, \quad \pi_2(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w},$$
$$\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_1dv_1 - w_2dv_2 + w_3dv_3 + w_4dv_4,$$
$$\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_1dw_1 - v_2dw_2 + v_3dw_3 + v_4dw_4.$$

The basic duality theorem in this paper is the following theorem:

Theorem 2.1 *Under the same notations as the above paragraph, each* $(\Delta, \theta_i^{-1}(0))$ $(i = 1, 2)$ *is a contact manifold and both of* π_i $(i = 1, 2)$ *are Legendrian fibrations.*

The proof of the theorem is almost the same as Proposition 2.2 in [7], so that we omit it.

3. The local differential geometry of spacelike surfaces in Anti de Sitter 3-space

In this section we introduce the local differential geometry of spacelike surfaces in Anti de Sitter 3-space.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be spacelike if $\boldsymbol{X}_i$ $(i = 1, 2)$ are spacelike. Throughout the remainder in this paper we assume that M is an spacelike surface in H_1^3. Since $\langle \boldsymbol{X}, \boldsymbol{X} \rangle \equiv -1$, we have

$$\langle \boldsymbol{X}, \boldsymbol{X}_{u_i} \rangle \equiv 0 \quad (\text{for } i = 1, 2),$$

where $u = (u_1, u_2) \in U$. We define a vector $\boldsymbol{e}(u)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\left\| \boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u) \right\|}.$$

By definition, we have $\langle \boldsymbol{e}, \boldsymbol{X}_{u_i} \rangle \equiv \langle \boldsymbol{e}, \boldsymbol{X} \rangle \equiv 0$. Since $\boldsymbol{X}$ is timelike and $\boldsymbol{X}_{u_i}$ $(i = 1, 2)$ are spacelike, $\boldsymbol{e}$ is timelike. Therefore $\langle \boldsymbol{e}, \boldsymbol{e} \rangle \equiv -1$. We now define a map

$$\mathbb{T} : U \longrightarrow H_1^3$$

by $\mathbb{T}(u) = \boldsymbol{e}(u)$ which is called the *timelike Anti de Sitter Gauss image* (briefly, *TAdS-Gauss image*) of $\boldsymbol{X}$(or M).

We now consider the geometric meanings of the TAdS-Gauss image of a spacelike surface. We have the following proposition.

Proposition 3.1 *Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a spacelike surface in Anti de Sitter 3-space. If the TAdS-Gauss image $\mathbb{T}$ is constant, then the spacelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-great-hyperboloid.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}_2^4 \mid \langle \boldsymbol{y}, \boldsymbol{e} \rangle = 0\}$. Since $\mathbb{T} = \boldsymbol{e}$ is constant, the set $V = HP(\boldsymbol{e}, 0)$ is a Lorentz hyperplane. We also have $\langle \boldsymbol{X}, \boldsymbol{e} \rangle \equiv 0$, so $\boldsymbol{X}(U) = M \subset V \cap H_1^3$. □

It is easy to show that $\mathbb{T}_{u_i}$ $(i = 1, 2)$ are tangent vectors of M. Therefore we have a linear transformation $W_p = -d\mathbb{T}(u) : T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter shape operator* (briefly, *AdS-shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We denote the eigenvalue of W_p by $k_i(p)$ $(i = 1, 2)$.

The *Anti de Sitter Gauss-Kronecker curvature* (briefly, *AdS-G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be

$$K_{AdS}(u) = \det W_p = k_1(p) \cdot k_2(p).$$

We say that a point $p = \boldsymbol{X}(u)$ is an *Anti de Sitter parabolic point* (or, briefly an *AdS-parabolic point*) of $\boldsymbol{X} : U \longrightarrow H_1^3$ if $K_{AdS}(u) = 0$. We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $W_p = k(p)id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic. Then we have the following proposition.

Proposition 3.2 *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k(p)$ is constant k. Under this condition, we have the following classification.*

(1) *If $k \neq 0$ then M is a part of a AdS-hyperboloid $HP(\boldsymbol{n}, -1) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{X} + \frac{1}{k}\boldsymbol{e}$ is a constant timelike vector.*
(2) *If $k = 0$ then M is a part of a AdS-great-hyperboloid $HP(\boldsymbol{n}, 0) \cap H^3_1$, where $\boldsymbol{n} = \boldsymbol{e}$ is a constant timelike vector.*

The proof is also given by direct calculations, so that we omit it.

Since $\boldsymbol{X}_{u_1}$ and $\boldsymbol{X}_{u_2}$ are spacelike vectors, we first introduce the Riemannian metric $ds^2 = \sum_{i,j=1}^{2} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *Anti de Sitter second fundamental invariant* by $h_{ij}(u) = \langle -\mathbb{T}_{u_i}(u), \boldsymbol{X}_{u_j}(u) \rangle$ for any $u \in U$. We can also show the following results by exactly the same arguments as those of [14].

Proposition 3.3 *With the above notation, we have the following Anti de Sitter Weingarten formula*:

$$\mathbb{T}_{u_i} = -\sum_{j=1}^{2} h_i^j \boldsymbol{X}_{u_j},$$

where $(h_i^j) = (h_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. □

As a corollary of the above proposition, we have an explicit expression for the AdS-G-K curvature by Riemannian metric and the Anti de Sitter second fundamental invariant.

Corollary 3.4 *With the same notation as in the above Proposition, we have the AdS-G-K curvature as follows*:

$$K_{AdS}(u) = \frac{\det(h_{ij}(u))}{\det(g_{\alpha\beta}(u))}. \qquad \square$$

Since ds^2 is a Riemannian metric, we have the sectional curvature K_I of M, which we call an intrinsic Gaussian curvature. By B. O'Neil [19] (Page 107 Corollary 20), we remark that $K_{AdS} = -1 - K_I$.

4. The timelike Anti de Sitter height function

In this section we define a family of functions on a spacelike surface in Anti de Sitter 3-space which is useful for the study of singularities of TAdS-Gauss image.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. We define a family of functions

$$H : U \times H^3_1 \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. We call H *a timelike Anti de Sitter height function* (or, a *AdS-height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 4.1 *Let $M = \boldsymbol{X}(U)$ be a spacelike surface in H^3_1 and $H : U \times H^3_1 \longrightarrow \mathbb{R}$ be a AdS-height function. Then we have the following assertions*:

(1) $H(u, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = 0$ *(for $i = 1, 2$) if and only if $\boldsymbol{v} = \pm \boldsymbol{e}(u) = \pm \mathbb{T}(u)$*;
(2) *Let $\boldsymbol{v}_0 = \boldsymbol{e}(u_0)$, then $\det\mathrm{Hess}(h_{v_0})(u_0) = 0$ if and only if $K_{AdS}(u_0) = 0$.*

Proof. (1) Since $\{\boldsymbol{X}, \boldsymbol{e}, \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is a basis of the vector space $T_p\mathbb{R}^4_2$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that $\boldsymbol{v} = \lambda \boldsymbol{X} + \eta \boldsymbol{e} + \alpha_1 \boldsymbol{X}_{u_1} + \alpha_2 \boldsymbol{X}_{u_2}$. Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle \boldsymbol{X}(u), \boldsymbol{v} \rangle = 0$. Since $0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{X}_{u_i}, \boldsymbol{v} \rangle = \sum_{j=1}^2 g_{ij}\alpha_i$ and (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore we have $\boldsymbol{v} = \eta \boldsymbol{e}$. Then from a straight forward calculation, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v_0})(u_0) = \big(\langle \boldsymbol{X}_{u_i u_j}(u_0), \mathbb{T}(u_0) \rangle\big) = \big(- \langle \boldsymbol{X}_{u_i}(u_0), \mathbb{T}_{u_j}(u_0) \rangle\big).$$

By the AdS-Weingarten formula, we have

$$-\langle \boldsymbol{X}_{u_i}, \mathbb{T}_{u_j} \rangle = \sum_{\alpha=1}^{2} h_i^\alpha \langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j} \rangle = \sum_{\alpha=1}^{2} h_i^\alpha g_{\alpha j} = h_{ij}.$$

Therefore we have

$$K_{AdS} = \frac{\det(h_{i,j})}{\det(g_{\alpha\beta})} = \frac{\det\mathrm{Hess}(h_{v_0})(u_0)}{\det(g_{\alpha\beta}(u_0))}.$$

Then we complete the proof. □

As an application of the above proposition, we have the following.

Corollary 4.2 *Let $H : U \times H^3_1 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be a AdS-height function on spacelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{T}$ be the TAdS-Gauss image, $p = \boldsymbol{X}(u)$. Then the following conditions are equivalent*:

(1) *There exists $\boldsymbol{v} \in H^3_1$, such that $p \in M$ is a degenerate singular point of AdS-height function h_v;*
(2) *There exists $\boldsymbol{v} \in H^3_1$, such that $p \in M$ is a singular point of TAdS-Gauss image $\mathbb{T}$;*
(3) *$K_{AdS}(u) = 0$.* □

5. Contact with AdS-great-hyperboloids

In this section we consider the geometric meaning of the singularities of the TAdS-Gauss image of spacelike surface $M = \boldsymbol{X}(U)$ in H^3_1. We consider the contact of spacelike surfaces with AdS-great-hyperboloids. We now briefly review the theory of contact due to Montaldi [18]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [18], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 5.1 *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \boldsymbol{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\boldsymbol{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

For the definition of the $\mathcal{K}$-equivalent, See Martinet [17]. We now consider a function $\mathcal{H} : H^3_1 \times H^3_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in H^3_1$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have the AdS-great-hyperboloid $\mathfrak{h}^{-1}_{v_0}(0) = H^3_1 \cap HP(\boldsymbol{v}_0, 0) = AH(\boldsymbol{v}_0, 0)$. For any $u_0 \in U$, we consider the timelike vector $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Then we have

$$\mathfrak{h}_{v_0} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ \big(\boldsymbol{X} \times id_{H^3_1}\big)(u_0, \boldsymbol{v}_0) = H(u_0, \mathbb{T}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbb{T}(u_0)) = 0,$$

for $i = 1, 2$. This means that the AdS-great-hyperboloid $AH(\boldsymbol{v}_0, 0)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}_0, 0)$ the *tangent AdS-great-hyperboloid* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write $AH(\boldsymbol{X}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be timelike vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1, 0)$ and $HP(\boldsymbol{v}_2, 0)$ are equal. Therefore, AdS-great-hyperboloids $AH(\boldsymbol{v}_1, 0) = AH(\boldsymbol{v}_2, 0)$. Then we have the following simple lemma.

Lemma 5.2 *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a spacelike surface. Consider two points $u_1, u_2 \in U$. Then*

$$\mathbb{T}(u_1) = \mathbb{T}(u_2) \;\; \textit{if and only if} \;\; AH(\boldsymbol{X}, u_1) = AH(\boldsymbol{X}, u_2).$$

We now consider the contact of M with tangent AdS-great-hyperboloid at $p \in M$ as an application of Legendrian singularity theorey. We introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersion (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has, in the second neighborhood, a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs.

Proposition 5.3 *Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$*

and $\pi \circ i'$ respectively are dense. Then i and i' are Legemndrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.

This result had been firstly pointed out by Zakalyukin [22]. The assumption in the above proposition is a generic condition for i and i'. In particular, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We first give a brief review on Legendrian singularity theory [1]. Here we only consider the local properties. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family of hypersurfaces* $f_x^{-1}(0)_{x \in (\mathbb{R}^n, 0)}$ if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k,\, x_1, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ and $f_x(q) = F(q, x)$. In this case we have a smooth $(n-1)$-dimensional submanifold,

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion germ. Then we call F a *generating family* of $\Phi_F(\Sigma_*(F))$.

We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathcal{M}_n = \{h \in \mathcal{E}_n \mid h(\mathbf{0}) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are $P-\mathcal{K}$-*equivalent* if there exists a diffeomorphsim germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphim defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a $\mathcal{K}$-*versal deformation* of $f = F \mid \mathbb{R}^k \times \{\mathbf{0}\}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}\bigg| \mathbb{R}^k \times \{\mathbf{0}\}, \dots, \frac{\partial F}{\partial x_n}\bigg| \mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

The main result in the theory of Arnold [1] and Zakalyukin [21] is the following:

Theorem 5.4 *Let* $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ *be Morse families. Then*

(1) Φ_F *and* Φ_G *are Legendrian equivalent if and only if* F *and* G *are* $P-\mathcal{K}$-*equivalent*;
(2) Φ_F *is Legendrian stable if and only if* F *is a* $\mathcal{K}$-*versal deformation of* $f = F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably $P - \mathcal{K}$-equivalences under this situation (cf. [1]). By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition 5.3 and Theorem 5.4, we have the following classification result of Legendrian stable germs (cf. [5]). For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring* of f by $Q(f) = \mathcal{E}_n / f^*(\mathcal{M}_p)\mathcal{E}_n$.

Proposition 5.5 *Let* $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ *be Morse families. Suppose that* Φ_F *and* Φ_G *are Legendrian stable. Then the following conditions are equivalent*:

(1) $(W(\Phi_F), \mathbf{0})$ *and* $(W(\Phi_G), \mathbf{0})$ *are diffeomorphic as germs*;
(2) Φ_F *and* Φ_G *are Legendrian equivalent*;
(3) $Q(f)$ *and* $Q(g)$ *are isomorphic as* $\mathbb{R}$-*algebras, where* $f = F \mid \mathbb{R}^k \times \{\mathbf{0}\}$ *and* $g = G \mid \mathbb{R}^k \times \{\mathbf{0}\}$.

Now we can apply the above arguments to our situation. We first can show the following proposition:

Proposition 5.6 *The AdS-height function* $H : U \times H^3_1 \longrightarrow \mathbb{R}$ *is a Morse family of hypersurfaces* $(h_v)^{-1}(0)_{v \in H^3_1}$.

Proof. For any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in H^3_1$, we have $v_1 \neq 0$ or $v_2 \neq 0$.

Without loss of the generality, we might assume that $v_1 > 0$, then $v_1 = \sqrt{1+v_3^2+v_4^2-v_2^2}$. It follows that

$$H(u,\boldsymbol{v}) = -x_1(u)\sqrt{1+v_3^2+v_4^2-v_2^2} - x_2(u)v_2 + x_3(u)v_3 + x_4(u)v_4$$

where $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$. We have to prove the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2}\right)$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2}, \boldsymbol{v}\rangle & x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ \langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_1u_2}, \boldsymbol{v}\rangle & x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ \langle \boldsymbol{X}_{u_2u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2u_2}, \boldsymbol{v}\rangle & x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A = \begin{pmatrix} x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix},$$

does not vanish at $(u,\boldsymbol{v}) \in \Delta^* H^{-1}(\boldsymbol{0})$. In this case, $\boldsymbol{v} = \mathbb{T}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix}, \quad \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det A = & -\frac{v_1}{v_1}\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4) \\ & -\frac{v_3}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3). \end{aligned}$$

On the other hand, we have

$$\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} = \big(-\det(\boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1\ \boldsymbol{b}_3\ \boldsymbol{b}_4),\\ \det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3)\big)$$

Therefore we have

$$\begin{aligned}\det A &= \left\langle \left(-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1} \right), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \right\rangle \\ &= -\frac{1}{v_1} \big\langle \mathbb{T}, \|\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\| \boldsymbol{e} \big\rangle \\ &= \frac{\|\boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2}\|}{v_1} \neq 0. \end{aligned}$$ □

We now define a mapping

$$\mathcal{L} : U \longrightarrow \Delta$$

by $\mathcal{L}(u) = (\boldsymbol{X}(u), \mathbb{T}(u))$. Since $\langle \boldsymbol{X}(u), \mathbb{T}(u) \rangle = \langle d\boldsymbol{X}(u), \mathbb{T}(u) \rangle = 0$, the mapping $\mathcal{L}$ is a Legendrian immersion. By the above argument we can show that H is a generating family of $\mathcal{L}(U) \subset \Delta$.

We have the tools for study of the contact of spacelike surfaces with AdS-great-hyperboloids. Let $\mathbb{T}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{v}_i)$ (for $i = 1, 2$) be TAdS-Gauss image germs of spacelike surface germs $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$. We say that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (H^3_1, \boldsymbol{v}_1) \longrightarrow (H^3_1, \boldsymbol{v}_2)$ such that $\Phi \circ \mathbb{T}_1 = \mathbb{T}_2 \circ \phi$. Suppose the regular set of $\mathbb{T}_i$ is dense in (U, u_i) for each $i = 1, 2$. It follows from Proposition 5.3 that $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs $\mathcal{L}^1 : (U, u_1) \longrightarrow (\Delta, \mathbf{z}_1)$ and $\mathcal{L}^2 : (U, u_2) \longrightarrow (\Delta, \mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are $P-\mathcal{K}$-equivalent by Theorem 5.4. Here, $H_i : (U \times H^3_1, (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the corresponding AdS-height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v_i} = H_i(u, \boldsymbol{v}_i)$; then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{X}_i(u)$. By Theorem 5.1,

$$K\big(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1\big) = K\big(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2\big)$$

if and only if h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q(\boldsymbol{X}, u_0)$ the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q(\boldsymbol{X}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{X}(u), \mathbb{T}(u_0)\rangle\rangle_{C^\infty_{u_0}(U)}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 5.7 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be spacelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{L}^i : (U, u_i) \longrightarrow (\Delta, \boldsymbol{z}_i)$ are Legendrian stable. Then the following conditions are equivalent*:

(1) *TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent*;
(2) *H_1 and H_2 are $P-\mathcal{K}$-equivalent*;
(3) *h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent*;
(4) $K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2)$;
(5) *$Q(\boldsymbol{X}_1, u_1)$ and $Q(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. By the previous arguments (mainly from Theorem 5.1), it has already been shown that conditions (3) and (4) are equivalent. Other assertions follow from Proposition 5.5. □

For a spacelike surface germ

$$\boldsymbol{X} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{X}(u_0)),$$

we call $\boldsymbol{X}^{-1}(AH(\mathbb{T}(u_0), 0), u_0)$ the *tangent AdS-great-hyperboloidic indicatrix germ* of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 5.8 *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be spacelike surface germs such that their AdS-parabolic sets have no interior points as subspaces of U. If TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent, then*

$$K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2).$$

In this case, $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1), 0), u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2), 0), u_2)$ are dif-

feomorphic as set germs.

Proof. The AdS-parabolic set is the set of singular points of the TAdS-Gauss image. So the corresponding Legendrian embedding $\mathcal{L}^i$ satisfy the hypothesis of Proposition 5.3. If TAdS-Gauss image germs $\mathbb{T}_1$ and $\mathbb{T}_2$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^1$ and $\mathcal{L}^2$ are Legendrian equivalent, so that H_1 and H_2 are $P-\mathcal{K}$-equivalent. Therefore, h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. By Theorem 5.7, this condition is equivalent to the condition that $K(\boldsymbol{X}_1(U), AH(\boldsymbol{X}_1,u_1),\boldsymbol{v}_1) = K(\boldsymbol{X}_2(U), AH(\boldsymbol{X}_2,u_2),\boldsymbol{v}_2)$.

Moreover, we have $\boldsymbol{X}_i^{-1}(AH(\mathbb{T}_i(u_i),0),u_i) = (h_{i,v_i}^{-1}(0),u_i)$. It follows from this fact that $\boldsymbol{X}_1^{-1}(AH(\mathbb{T}_1(u_1),0),u_1)$ and $\boldsymbol{X}_2^{-1}(AH(\mathbb{T}_2(u_2),0),u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalent preserves the zero level sets. □

From the above proposition, the diffeomorphism type of the tangent AdS-great-hyperboloidic indicatrix germ is an invariant of $\mathcal{A}$-classification of the TAdS-Gauss image germ of $\boldsymbol{X}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for a function germ. The local ring of a function is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We denote

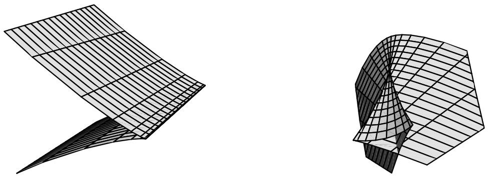
$$\text{AdS-ord}(\boldsymbol{X},u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle h_{v_0}, \partial h_{v_0}/\partial u_i\rangle_{C^\infty_{u_0}(U)}},$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Usually AdS-ord$(\boldsymbol{X},u_0)$ is called the $\mathcal{K}$-codimension of h_{v_0}. However, we call it the *order of contact with tangent AdS-great-hyperboloid* at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$\text{AdS-corank}(\boldsymbol{X},u_0) = 2 - \text{rankHess}(h_{v_0})(u_0),$$

where $\boldsymbol{v}_0 = \mathbb{T}(u_0)$.

By Proposition 4.1, $\boldsymbol{X}(u_0)$ is an AdS-parabolic point if and only if AdS-corank $(\boldsymbol{X},u_0)\geq 1$. On the other hand, a function germ $f:(\mathbb{R}^{n-1},\boldsymbol{a})\longrightarrow\mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2\pm\cdots\pm u_{n-2}^2+u_{n-1}^{k+1}$. If AdS-corank$(\boldsymbol{X},u_0)=1$, the AdS-height function h_{v_0} has the A_k-type singularity at u_0 and is generic. In this case we have AdS-



Cuspital edge Swallowtail

Figure 1

ord$(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [P19–P52, 2]). This is the reason why we call AdS-ord$(\boldsymbol{X}, u_0)$ the order of contact with the AdS-great-hyperboloid at $\boldsymbol{X}(u_0)$.

6. Classification of singularities of TAdS-Gauss images

In this section we consider the generic singularities of TAdS-Gauss images. We have almost the same arguments as those of [14], so that we omit the details. We now consider the space of spacelike embeddings $\mathrm{Emb}_S(U, H^3_1)$ with the Whitney C^∞-topology. By the classification of stable Legendrian singularities of $n = 3$ and the transversality theorem of [14] (Proposition 7.1), we have the following theorem.

Theorem 6.1 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_S(U, H^3_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$ the following conditions hold.*

(1) *The AdS-parabolic set $K^{-1}_{AdS}(0)$ is a regular curve. We call such a curve the AdS-parabolic curve.*
(2) *The TAdS-Gauss image $\mathbb{T}$ along the AdS-parabolic curve is a cuspidal edge except at isolated points. At such the point $\mathbb{T}$ is the swallowtail.*

Here, a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ is called a cuspidal edge if it is $\mathcal{A}$-equivalent to the germ (u_1, u_2^2, u_2^3) and a swallowtail if it is $\mathcal{A}$-equivalent to the germ $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$.

The assertion of Theorem 6.1 can be interpreted as saying that the Legendrian embedding $\mathcal{L}$ of the TAdS-Gauss image $\mathbb{T}$ of $\boldsymbol{X}$ is Legendrian stable at each point. Following the terminology of Whitney [20], we say that a spacelike surface $\boldsymbol{X} : U \longrightarrow H^3_1$ has the *excellent TAdS-Gauss image* $\mathbb{T}$ if $\mathcal{L}$ is a stable Legendrian immersion germ at each point. In this case, the TAdS-

Gauss image $\mathbb{T}$ has only cuspidal edges and swallowtails as singularities. Theorem 6.1 assert that a spacelike surface with the excellent TAdS-Gauss image is generic in the space of all spacelike surfaces in H^3_1.

We now consider the geometric meanings of cuspidal edges and swallowtails of the TAdS-Gauss image. We have the following results analogous to the results of Banchoff *et al.* [2].

Theorem 6.2 *Let $\mathbb{T} : (U, u_0) \longrightarrow (H^3_1, \boldsymbol{v}_0)$ be the excellent TAdS-Gauss image germ of a spacelike surface $\boldsymbol{X}$ and $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ be the AdS-height function germ at $\boldsymbol{v}_0 = \mathbb{T}(u_0)$. Then we have the following.*

(1) *The point u_0 is an AdS-parabolic point of $\boldsymbol{X}$ if and only if* AdS-corank$(\boldsymbol{X}, u_0) = 1$.
(2) *If u_0 is an AdS-parabolic point of $\boldsymbol{X}$, then h_{v_0} has the A_k-type singularity for $k = 2, 3$.*
(3) *Suppose that u_0 is an AdS-parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent*:
   (a) $\mathbb{T}$ *has the cuspidal edge at u_0*;
   (b) *h_{v_0} has the A_2-type singularity*;
   (c) AdS-order$(\boldsymbol{X}, u_0) = 2$;
   (d) *the tangent AdS-great-hyperboloidic indicatrix germ is an ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called an ordinary cusp if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$.*
(4) *Suppose that u_0 is an AdS-parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent*:
   (a) $\mathbb{T}$ *has the swallowtail at u_0*;
   (b) *h_{v_0} has the A_3-type singularity*;
   (c) AdS-order$(\boldsymbol{X}, u_0) = 3$;
   (d) *the tangent AdS-great-hyperboloidic indicatrix germ is an point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called a tachnodal if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$.*
   (e) *for each $\varepsilon > 0$, there exit two points $u_1, u_2 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2$, neither of u_1 nor u_2 is an AdS-parabolic point and the tangent AdS-great-hyperboloids to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal.* □

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