Hokkaido Mathematical Journal Vol. 38 (2009) p. 683–700

Riesz decomposition for superbiharmonic functions in the unit ball

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(Received September 10, 2008; Revised April 14, 2009)

Abstract. For a superbiharmonic function u in the unit ball with the growth condition of spherical means, we show that u is represented as the sum of a generalized Riesz potential and a biharmonic function. This representation is referred to as Riesz decomposition for superbiharmonic functions.

The superharmonic case is treated similarly.

Key words: superbiharmonic functions, spherical means, Riesz decomposition.

1. Introduction

A superharmonic function on $\mathbf{R}^n$ is represented locally as the sum of a Riesz potential and a harmonic function. This representation is referred to as Riesz decomposition (see e.g. Armitage-Gardiner [2], Heyman-Kennedy [8] and Mizuta [12]). Our aim in this paper is to establish Riesz decomposition for superbiharmonic functions on the unit ball $\mathbf{B}$.

A function u on an open set $\Omega \subset \mathbf{R}^n$ is called biharmonic if $u \in C^4(\Omega)$ and $(-\Delta)^2 u = 0$ on Ω, where Δ denotes the Laplacian and $(-\Delta)^2 u = -\Delta(-\Delta u)$. We say that a locally integrable function u on Ω is superbiharmonic in Ω if

(1) $\mu = (-\Delta)^2 u$ is a nonnegative measure on Ω, that is,

$$\int_\Omega u(x)(-\Delta)^2\varphi(x)dx \geq 0 \quad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega);$$

(2) u is lower semicontinuous on Ω;

(3) every point of Ω is a Lebesgue point for u, that is,

$$u(x) = \lim_{r\to 0} \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} u(y)\ dS(y)$$

2000 Mathematics Subject Classification : Primary 31B30, 31B05, 31B15.

for every $x \in \Omega$, where ω_n is the surface area of a unit sphere and $S(x,r)$ is the sphere centered at x with radius r.

If $(-\Delta)^2 T \geq 0$ on $\mathbf{R}^n$ in the sense of distribution, then one can find a superbiharmonic function u on $\mathbf{R}^n$ such that

$$T = u \quad \text{in the sense of distribution.}$$

This is an easy consequence of Riesz decomposition theorem (see expression (2.1) below).

We denote by $\mathcal{H}^2(\Omega)$ and $\mathcal{SH}^2(\Omega)$ the space of biharmonic functions on Ω and the space of superbiharmonic functions on Ω. For fundamental properties of biharmonic functions, we refer the reader to Nicolesco [15] and Aronszajn, Creese and Lipkin [3].

Consider the Riesz kernel of order 4 defined by

$$\mathcal{R}_4(x) = \begin{cases} \dfrac{|x|^{4-n}}{2(4-n)(2-n)\omega_n} & \text{if } n \neq 2, 4, \\ \dfrac{(-1)^{n/2}}{4\omega_n}|x|^{4-n}\log\left(\dfrac{1}{|x|}\right) & \text{if } n = 2 \text{ or } 4. \end{cases}$$

Then we know (see Hayman and Korenblum [9]) that

$$(-\Delta)^2 \mathcal{R}_4 = \delta_0,$$

where δ_y denotes the Dirac measure at y, so that $\mathcal{R}_4$ is superbiharmonic in $\mathbf{R}^n$.

We denote by $B(x,r)$ the open ball centered at x with radius r, whose boundary is written as $S(x,r) = \partial B(x,r)$. We use the notation $\mathbf{B}$ to denote the unit ball $B(0,1)$. For a Borel measurable function u on $\mathbf{R}^n$, we define the spherical mean by

$$M(u,x,r) = \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} u \, dS.$$

If $x = 0$, then we write simply $B(r) = B(0,r)$, $S(r) = S(0,r)$ and $M(u,r) = M(u,0,r)$.

Recently, the second and the third authors studied the Riesz decomposition for superbiharmonic functions u on $\mathbf{R}^n$ such that $M(u,2r)-4M(u,r)$ is bounded for $r>1$. In fact, they showed the following result ([11, Theorems 1.1, 1.2]).

Theorem A *Let u be a superbiharmonic function on $\mathbf{R}^n$ such that $M(u,2r)-4M(u,r)$ is bounded for $r>1$. Set $\mu=(-\Delta)^2u$.*

(1) *If $n \le 4$, then u is biharmonic in $\mathbf{R}^n$.*
(2) *If $n \ge 5$, then*

$$u(x)=\int_{\mathbf{R}^n}\mathcal{R}_4(x-y)d\mu(y)+h(x) \quad \textit{for} \;\; x\in\mathbf{R}^n,$$

where h is a biharmonic function on $\mathbf{R}^n$.

Our aim in this paper is to extend Theorem A to the unit ball $\mathbf{B}$. For this purpose, we introduce a generalized kernel function $K_{2,L}(x,y)$ such that

$$(-\Delta)^2K_{2,L}(\cdot,y)=\delta_y$$

for fixed $y\in\mathbf{B}$ and

$$u(x)=\int_{\mathbf{B}}K_{2,L}(x,y)d\mu(y)+h_L(x)$$

for all $x\in\mathbf{B}$, when u is a superbiharmonic function on $\mathbf{B}$ satisfying a growth condition near the boundary $\partial\mathbf{B}$, where $\mu=(-\Delta)^2u\ge 0$, L is an integer determined by the growth condition on u and h_L is biharmonic in $\mathbf{B}$. Riesz decomposition for superbiharmonic functions on the unit disk was studied by Abkar-Hedenmalm [1]. They showed that under certain condition near the unit circle, superbiharmonic function is represented as the sum of a biharmonic Green potential and a biharmonic function. For related results, we also refer to the papers by Futamura, Kishi and Mizuta [4], Ishikawa, Nakai and Tada [10] and Nakai and Tada [13], [14].

We would like to express our hearty thanks to referee for his valuable comments.

2. Preliminaries and statement of result

Throughout this paper, let C denote various constants independent of the variables in question.

We denote a point of the n-dimensional Euclidean space $\mathbf{R}^n$ by $x = (x_1, x_2, \ldots, x_n)$. We write

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for the inner product of $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

For a multi-index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and a point $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$
$$\lambda! = \lambda_1! \lambda_2! \cdots \lambda_n!,$$
$$x^\lambda = {x_1}^{\lambda_1} {x_2}^{\lambda_2} \cdots {x_n}^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

Following the book by Hayman-Kennedy [8], we consider the remainder term in the Taylor expansion of $\mathcal{R}_4(\cdot - y)$ given by

$$\mathcal{R}_{4,L}(x,y) = \begin{cases} \mathcal{R}_4(x-y) - \sum_{|\lambda| \le L} \frac{x^\lambda}{\lambda!} \big(D^\lambda \mathcal{R}_4\big)(-y) & \text{when } |y| \ge 1/2, \\ \mathcal{R}_4(x-y) & \text{when } |y| < 1/2, \end{cases}$$

where L is an integer; if $L \le -1$, then we set $\mathcal{R}_{4,L}(x,y) = \mathcal{R}_4(x-y)$. Here note that

$$(-\Delta)^2 \mathcal{R}_{4,L}(\cdot, y) = \delta_y.$$

Then, if u is superbiharmonic in a neighborhood of $\overline{B(R)}$, then Riesz decomposition theorem implies that

$$u(x) = \int_{B(R)} \mathcal{R}_{4,L}(x,y)d\mu(y) + h_{R,L}(x) \tag{2.1}$$

for every $x \in B(R)$, where $\mu = (-\Delta)^2 u$, L is an integer and $h_{R,L} \in \mathcal{H}^2(B(R))$. This implies that superbiharmonic functions are continuous if $n = 2, 3$.

For $x \in \mathbf{B}$ and $y \in \mathbf{B} \setminus \{0\}$, we have

$$|x-y|^2 = |x - \tilde{y} + t\tilde{y}|^2 = |x-\tilde{y}|^2 + s = |x-\tilde{y}|^2(1 + s/|x-\tilde{y}|^2)$$

where $\tilde{y} = y/|y|$, $t = 1 - |y|$ and $s = t^2 + 2t(x-\tilde{y}) \cdot \tilde{y}$. For a real number γ, consider the binomial expansion of $(1+a+b)^\gamma$, that is,

$$(1+a+b)^\gamma = \sum_{m=0}^{\infty} \binom{\gamma}{m} (a+b)^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{\gamma}{m} \binom{m}{k} a^k b^{m-k}.$$

The double series converges absolutely for $|a| + |b| < 1$. Hence we have the following result.

Lemma 2.1 *Let γ be as above. If $\sqrt{2}(1-|y|) < |x-y| < \sqrt{2}|x-\tilde{y}|$, then*

$$|x-y|^{2\gamma} = \sum_{\ell} \Bigg(\sum_{\{m : \ell/2 \le m \le \ell\}} a_{m,\ell} |x-\tilde{y}|^{2\gamma - 2m} (x \cdot \tilde{y} - 1)^{2m-\ell} \Bigg) t^\ell,$$

where $a_{m,\ell} = a_{m,\ell;\gamma} = \binom{\gamma}{m}\binom{m}{\ell-m} 2^{2m-\ell}$.

Now we define a generalized kernel function $K_{2,L}(x,y)$. First, if $n \neq 2, 4$, then we set

$$K_{2,L}(x,y) = \begin{cases} \dfrac{1}{2(4-n)(2-n)\omega_n} |x-y|^{4-n} & \text{for } y \in B(1/2), \\[2ex] \dfrac{1}{2(4-n)(2-n)\omega_n} \left\{ |x-y|^{4-n} - \displaystyle\sum_{\ell=0}^{L} \varphi_\ell(x,\tilde{y})(1-|y|)^\ell \right\} & \\ & \text{for } y \in \mathbf{B} \setminus B(1/2), \end{cases}$$

where

$$\varphi_\ell(x,\tilde{y}) = \sum_{\{m:\ell/2\le m\le \ell\}} a_{m,\ell}|x-\tilde{y}|^{4-n-2m}(x\cdot\tilde{y}-1)^{2m-\ell}$$

with $a_{m,\ell} = a_{m,\ell;(4-n)/2}$. Note that $\varphi_\ell(\cdot,\tilde{y})$ is biharmonic in $\mathbf{B}$ in this case.

Next, we deal with the case $n = 2$ or 4. We have

$$\begin{aligned}\log\frac{1}{|x-y|} &= \log\frac{1}{|x-\tilde{y}|} - \frac{1}{2}\log\left(1+\frac{s}{|x-\tilde{y}|^2}\right)\\ &= \log\frac{1}{|x-\tilde{y}|} - \frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\frac{s}{|x-\tilde{y}|^2}\right)^m\\ &= \log\frac{1}{|x-\tilde{y}|} + \sum_{\ell=1}^{\infty}\left\{\sum_{\ell/2\le m\le\ell} b_{m,\ell}|x-\tilde{y}|^{-2m}(x\cdot\tilde{y}-1)^{2m-\ell}\right\}t^\ell,\end{aligned}$$

where $b_{m,\ell} = \frac{(-1)^m}{m}\binom{m}{\ell-m}2^{2m-\ell-1}$. Then we set

$$K_{2,L}(x,y) = \begin{cases} \dfrac{(-1)^{n/2}}{4\omega_n}|x-y|^{4-n}\log\dfrac{1}{|x-y|} & \text{for } y\in B(1/2),\\ \dfrac{(-1)^{n/2}}{4\omega_n}|x-y|^{4-n}\left\{\log\dfrac{|x-\tilde{y}|}{|x-y|} - \displaystyle\sum_{\ell=1}^{L}\varphi_\ell(x,\tilde{y})(1-|y|)^\ell\right\} & \\ & \text{for } y\in\mathbf{B}\setminus B(1/2),\end{cases}$$

where

$$\varphi_\ell(x,\tilde{y}) = \sum_{\ell/2\le m\le\ell} b_{m,\ell}|x-\tilde{y}|^{-2m}(x\cdot\tilde{y}-1)^{2m-\ell}.$$

Lemma 2.2 *Let $L\ge 0$ and $0<r<r'<1$. Then*

$$|K_{2,L}(x,y)|\le C(1-|y|)^{L+1}$$

whenever $x\in B(0,r)$ and $y\in D(r') = \{z\in\mathbf{B}: |z|>(\sqrt{2}-1)(\sqrt{2}+r')\}$.

Proof. To show this when $n\ne 2,4$, for fixed $x\in\mathbf{B}$ and $\xi\in\partial\mathbf{B}$ consider

$$f_1(t) = |x-\xi+t\xi|^{2-n}$$

and

$$f_2(t) = |x - \xi + t\xi|^{4-n}.$$

Here note that

$$\varphi_\ell(x,\xi) = \frac{f_2^{(\ell)}(0)}{\ell!} = \frac{4-n}{\ell}\left\{\frac{f_1^{(\ell-1)}(0)}{(\ell-1)!}(x-\xi)\cdot\xi + \frac{f_1^{(\ell-2)}(0)}{(\ell-2)!}\right\}.$$

For $t < |x-\xi|$, we see from [8, Lemma 4.1 of chapter 4] that

$$\left|\frac{f_1^{(\ell)}(0)}{\ell!}\right| \le A_\ell |x-\xi|^{2-n-\ell},$$

so that

$$|\varphi_\ell(x,\xi)| \le B_\ell |x-\xi|^{4-n-\ell},$$

where $A_\ell = (n+\ell-3)(n+\ell-4)\cdots(\ell+1)/(n-3)!$ and $B_\ell = 2|n-4|(n+\ell-4)(n+\ell-5)\cdots(\ell+1)/(n-3)!$. Applying Taylor's theorem, we have for $t < (\sqrt{2}-1)|x-\xi|$

$$\begin{aligned}
\left|f_2(t) - \sum_{\ell=0}^{L}\frac{f_2^{(\ell)}(0)}{\ell!}t^\ell\right| &= \left|\sum_{\ell=L+1}^{\infty}\frac{f_2^{(\ell)}(0)}{\ell!}t^\ell\right| \\
&\le \sum_{\ell=L+1}^{\infty} B_\ell |x-\xi|^{4-n-\ell} t^\ell \\
&= |x-\xi|^{3-n-L} t^{L+1} \sum_{k=0}^{\infty} B_{L+k+1}\left(\frac{t}{|x-\xi|}\right)^k \\
&\le |x-\xi|^{3-n-L} t^{L+1} \sum_{k=0}^{\infty} B_{L+k+1}\left(\sqrt{2}-1\right)^k \\
&\le C|x-\xi|^{3-n-L} t^{L+1}.
\end{aligned}$$

Thus the present lemma with $n \ne 2, 4$ follows.

In the same way as above, we give a proof in case $n = 4$. □

Remark 2.3 In view of Hayman-Korenblum [9], biharmonic Green's function of $\mathbf{B}$ is given by

$$G_2(x,y) = \frac{1}{2(4-n)(2-n)\omega_n}\Big\{|x-y|^{4-n} - (|x||x^*-y|)^{4-n} - \frac{n-4}{2}(|x||x^*-y|)^{2-n}(1-|x|^2)(1-|y|^2)\Big\}$$

when $n \geq 5$, where $x^* = x/|x|^2$. Noting that $|x-y|^2 = (|x||x^*-y|)^2 - (1-|x|^2)(1-|y|^2)$, we have the expansion

$$|x-y|^{4-n} = (|x||x^*-y|)^{4-n} - \sum_m \binom{\frac{n-4}{2}}{m}(|x||x^*-y|)^{4-n-2m}(1-|x|^2)^m(1-|y|^2)^m.$$

Unfortunately, each term on the right sum might not be biharmonic in $\mathbf{B}$ as a function of x (except for $m=1$).

Now we are ready to state our main theorem.

Theorem 2.4 *Let $u \in \mathcal{SH}^2(\mathbf{B})$ and $\mu = (-\Delta)^2 u$.*

(1) *If $\lim_{r\to 1} M(u,r) < \infty$, then*

$$u(x) = \int_{\mathbf{B}} K_{2,2}(x,y)\; d\mu(y) + h(x) \quad \text{for } x \in \mathbf{B}, \tag{2.2}$$

where $h \in \mathcal{H}^2(\mathbf{B})$.

(2) *If $\limsup_{r\to 1}(1-r)^s M(u,r) < \infty$ for some $s > 0$, then*

$$u(x) = \int_{\mathbf{B}} K_{2,L}(x,y)\; d\mu(y) + h_L(x) \quad \text{for } x \in \mathbf{B}, \tag{2.3}$$

where $h_L \in \mathcal{H}^2(\mathbf{B})$ and $L > s+2$.

Remark 2.5 Let u be a biharmonic function on $\mathbf{B}$. By the Almansi expansion, there exist harmonic functions u_1 and u_2 on $\mathbf{B}$ such that $u(x) = u_1(x) + |x|^2 u_2(x)$. Then we see that $M(u,r) = u_1(0) + u_2(0)r^2$, so that $M(u,r)$ is bounded.

Remark 2.6 In view of the paper by Futamura-Mizuta [6], we see that if u is a superbiharmonic function on $\mathbf{B}$ such that

$$\liminf_{r\to 1}(1-r)^{-1}M(|u|,r) < \infty,$$

then

$$u(x) = \int_{\mathbf{B}} G_2(x,y)\ d\mu(y) + (1-|x|^2)h(x) \quad \text{for } x \in \mathbf{B},$$

where h is harmonic in $\mathbf{B}$.

Remark 2.7 One sees from [1, Proposition 2.3] and [6, Lemma 4.2] that the limit

$$\lim_{r\to 1} M(u,r)$$

exists in $(-\infty, +\infty]$, when u is a superbiharmonic function on $\mathbf{B}$.

3. Spherical means for superbiharmonic functions

First we collect some fundamental properties of $K_{2,L}(x,y)$.

Lemma 3.1 *The following hold*:

(1) $K_{2,L}(\cdot,y)$ *is biharmonic in* $\mathbf{B}\setminus\{y\}$ *for each fixed* $y \in \mathbf{B}$.
(2) $K_{2,L}(\cdot,y)$ *is superbiharmonic in* $\mathbf{B}$ *and* $(-\Delta)^2 K_{2,L}(\cdot,y) = \delta_y$ *for each fixed* $y \in \mathbf{B}$.
(3) $K_{2,L}(x,y) = O((1-|y|)^{L+1})$ *as* $|y| \to 1$ *for fixed* $x \in \mathbf{B}$.

Lemma 3.1 gives the following Lemma.

Lemma 3.2 *Let* $u \in \mathcal{SH}^2(\mathbf{B})$ *and* $\mu = (-\Delta)^2 u$. *Suppose*

$$\int_{\mathbf{B}} (1-|y|)^{L+1} d\mu(y) < \infty$$

for some integer L. *Then* u *is of the form*

$$u(x) = \int_{\mathbf{B}} K_{2,L}(x,y) d\mu(y) + h_L(x),$$

where $h_L \in \mathcal{H}^2(\mathbf{B})$.

For $0 < t \le r$, set

$$g(t,r) = \mathcal{R}_4(re_1) - \mathcal{R}_4(te_1) + \frac{1}{2n}\left(t^2 \Delta \mathcal{R}_4(re_1) - r^2 \Delta \mathcal{R}_4(te_1)\right),$$

where $e_1 = (1, 0, \ldots, 0) \in \partial \mathbf{B}$, that is,

$$g(t,r) = \begin{cases} -\dfrac{1}{4\omega_2}\left\{ r^2 \log \dfrac{1}{r} - t^2 \log \dfrac{1}{t} + t^2\left(\log \dfrac{1}{r} - 1\right) - r^2\left(\log\dfrac{1}{t} - 1\right)\right\} & \text{if } n = 2, \\ \dfrac{1}{4\omega_4}\left\{ \log\dfrac{1}{r} - \log\dfrac{1}{t} - \dfrac{1}{4}\left(t^2 r^{-2} - r^2 t^{-2}\right)\right\} & \text{if } n = 4, \\ \dfrac{1}{2(4-n)(2-n)\omega_n}\left\{ r^{4-n} - t^{4-n} + \dfrac{4-n}{n}\left(t^2 r^{2-n} - r^2 t^{2-n}\right)\right\} & \text{otherwise.} \end{cases}$$

Note that $g(t, r)$ is strictly decreasing as a function of t for fixed $r > 0$ (cf. [5, Lemma 4.4]).

Lemma 3.3 *Let* $u \in \mathcal{SH}^2(\mathbf{B})$ *and* $\mu = (-\Delta)^2 u$. *Then there exist positive constants* a, b *such that*

$$M(u,r) = \int_{B(r)\setminus B(1/2)} g(|y|, r)\ d\mu(y) + H(r) + a + br^2$$

for $1/2 < r < 1$, *where*

$$H(r) = \mathcal{R}_4(re_1)\mu(B(1/2)) + \frac{1}{2n}\Delta\mathcal{R}_4(re_1)\int_{B(1/2)} |y|^2 d\mu(y).$$

Proof. For $0 < r_1 < r_2 < 1$, expression (2.1) implies that

$$u(x) = \int_{B(r_j)} \mathcal{R}_{4,2}(x,y) d\mu(y) + h_j(x) \quad (x \in B(r_j)),$$

where $h_j \in \mathcal{H}^2(B(r_j)), j = 1, 2$. Then, by use of [5, Lemma 4.4], we find

$$M(u,r) = \frac{1}{\omega_n r^{n-1}} \int_{S(r)} \left(\int_{B(r_j)} \mathcal{R}_{4,2}(x,y) d\mu(y) \right) dS(x) + M(h_j, r)$$
$$= \int_{B(r)\setminus B(1/2)} g(|y|, r) d\mu(y) + H(r) + a_j + b_j r^2$$

for $1/2 < r < r_1$. Hence it follows that

$$a_1 + b_1 r^2 = a_2 + b_2 r^2 \quad \text{for} \quad 0 < r < r_1$$

which implies $a_1 = a_2$ and $b_1 = b_2$. Hence the proof is completed. □

Lemma 3.4 *There exists a constant $C_1 \geq 1$ such that*

$$C_1^{-1}(r-t)^3 \leq g(t,r) \leq C_1 (r-t)^3 \quad \textit{for} \quad \frac{1}{2} \leq t \leq r < 1.$$

Proof. Fix r such that $1/2 \leq r < 1$. Note that

$$\left(t^{n-1} g'(t,r)\right)' = \begin{cases} \dfrac{t}{\omega_2}(\log(1/t) - \log(1/r)) & \text{if } \ n = 2, \\[2ex] \dfrac{t^{n-1}}{(n-2)\omega_n}(t^{2-n} - r^{2-n}) & \text{if } \ n \neq 2. \end{cases}$$

Then we see that

$$\left(t^{n-1} g'(t,r)\right)' \approx r - t \quad \text{for} \quad \frac{1}{2} \leq t \leq r.$$

Here we write $f_1 \approx f_2$ for two positive functions f_1 and f_2, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. Since $g(r,r) = g'(r,r) = 0$, we have for $1/2 \leq t \leq r$

$$g(t,r) = \int_t^r s^{1-n} \left(\int_s^r \left(u^{n-1} g'(u,r)\right)' du \right) ds \approx \int_t^r s^{1-n} \left(\int_s^r (r-u)\ du \right) ds$$
$$= \int_t^r \left(\int_t^u s^{1-n}\ ds \right) (r-u)\ du \approx \int_t^r (u-t)(r-u)\ du = \frac{1}{6}(r-t)^3.$$

Thus Lemma 3.4 follows. □

Lemma 3.5 *Let $u \in \mathcal{SH}^2(\mathbf{B})$ and $\mu = (-\Delta)^2 u$.*

(1) *If $\lim_{r\to1} M(u,r) < \infty$, then*

$$\int_{\mathbf{B}} (1-|y|)^3 d\mu(y) < \infty. \tag{3.1}$$

(2) *If $\limsup_{r\to1}(1-r)^s M(u,r) < \infty$ for some $s > 0$, then*

$$\int_{\mathbf{B}} (1-|y|)^{s+3} \left(\log\left(\frac{e}{1-|y|}\right)\right)^{-\gamma} d\mu(y) < \infty \tag{3.2}$$

for each $\gamma > 1$. In particular,

$$\int_{\mathbf{B}} (1-|y|)^{\beta} d\mu(y) < \infty$$

for each $\beta > s+3$.

Proof. By lemmas 3.3 and 3.4, we obtain

$$M(u,r) \geq C_1^{-1} \int_{B(r)\setminus B(1/2)} (r-|y|)^3 d\mu(y) + O(1) \quad \text{as} \quad r \to 1.$$

First we assume that $\lim_{r\to1} M(u,r) < \infty$. Then we see that

$$\lim_{r\to1} \int_{B(r)\setminus B(1/2)} (r-|y|)^3 d\mu(y) < \infty,$$

which implies (3.1) by Fatou's theorem.

Next we assume that $\limsup_{r\to1}(1-r)^s M(u,r) < \infty$ for some $s > 0$. Then we have

$$\limsup_{r\to1}(1-r)^s \int_{B(r)\setminus B(1/2)} (r-|y|)^3 d\mu(y) < \infty,$$

which implies that

$$\alpha = \sup_{r\in[5/6,1)} (1-r)^{s+3} \mu(A(r)) < \infty,$$

where $A(r) = \{x : r - 2(1-r) \le |x| < r - (1-r)/2\}$. Hence we establish

$$\begin{aligned}
&\int_{\mathbf{B}\setminus B(5/8)} (1-|y|)^{s+3}\left(\log\left(\frac{e}{1-|y|}\right)\right)^{-\gamma} d\mu(y) \\
&\quad = \sum_{j=3}^{\infty} \int_{A(1-2^{-j})} (1-|y|)^{s+3}\left(\log\left(\frac{e}{1-|y|}\right)\right)^{-\gamma} d\mu(y) \\
&\quad \le 3^{s+3}(\log 2)^{-\gamma} \sum_{j=3}^{\infty} 2^{-j(s+3)}(j-1)^{-\gamma}\mu(A(1-2^{-j})) \\
&\quad \le 3^{s+3}(\log 2)^{-\gamma}\alpha \sum_{j=3}^{\infty} (j-1)^{-\gamma} < \infty
\end{aligned}$$

for $\gamma > 1$. This gives (3.2) readily. □

4. Proof of Theorem 2.4

In this section we complete the proof of Theorem 2.4. First suppose $\lim_{r\to 1} M(u,r) < \infty$. By Lemma 3.5,

$$\int_{\mathbf{B}} (1-|y|)^3 d\mu(y) < \infty.$$

In view of Lemma 3.2 with $L = 2$, the conclusion follows.

Next suppose $\limsup_{r\to 1}(1-r)^s M(u,r) < \infty$ for some $s > 0$. Then we obtain by Lemma 3.5

$$\int_{\mathbf{B}} (1-|y|)^{\beta} d\mu(y) < \infty$$

for $\beta > s+3$. Thus, by use of Lemma 3.2 with $L > s+2$, we have the required expression. □

5. The superharmonic case

In this section, along the same lines as in the preceding discussions, we give a representation theorem for superharmonic functions, which is proved easier than before.

Recall that

$$\mathcal{R}_{2,0}(x,y) = \begin{cases} \mathcal{R}_2(x-y) - \mathcal{R}_2(-y) & \text{if } |y| \geq 1/2, \\ \mathcal{R}_2(x-y) & \text{if } |y| < 1/2, \end{cases}$$

where

$$\mathcal{R}_2(x) = \begin{cases} \dfrac{|x|^{2-n}}{(n-2)\omega_n} & \text{if } \ n \neq 2, \\ \dfrac{1}{\omega_2} \log\left(\dfrac{1}{|x|}\right) & \text{if } \ n = 2. \end{cases}$$

Let u be a superharmonic function on $\mathbf{B}$ and set $\mu = (-\Delta)u$. Then we see that for $0 < r < R < 1$,

$$u(x) = \int_{B(R)} \mathcal{R}_{2,0}(x,y)\ d\mu(y) + h_R(x) \quad (x \in B(R)),$$

where h_R is harmonic in $B(R)$. As in Lemma 3.3, we find a constant a such that

$$\begin{aligned} M(u,r) &= \int_{B(r)} M(r, \mathcal{R}_{2,0}(\cdot, y))\ d\mu(y) + a \\ &= \int_{B(r)\setminus B(1/2)} g(|y|, r)\ d\mu(y) + \mathcal{R}_2(re_1)\mu(B(1/2)) + a \end{aligned}$$

for $1/2 < r < 1$, where $g(t,r) = \mathcal{R}_2(re_1) - \mathcal{R}_2(te_1)$.

Lemma 5.1 *There exists a constant $C_2 \geq 1$ such that*

$$C_2^{-1}(r-t) \leq -g(t,r) \leq C_2(r-t) \quad \textit{for } \ \frac{1}{2} \leq t \leq r < 1.$$

If u is superharmonic in $\mathbf{B}$, then $M(u,r)$ is nonincreasing on the interval $(0,1)$, so that we consider a lower estimate for $M(u,r)$.

Lemma 5.2 *Let u be superharmonic in $\mathbf{B}$ and $\mu = (-\Delta)u$.*

(a) *If $\lim_{r\to 1} M(u,r) > -\infty$, then*

$$\int_{\mathbf{B}} (1-|y|)d\mu(y) < \infty.$$

(b) *If* $\liminf_{r\to 1}(1-r)^s M(u,r) > -\infty$ *for some* $s>0$, *then*

$$\int_{\mathbf{B}} (1-|y|)^{s+1}\left(\log\left(\frac{e}{1-|y|}\right)\right)^{-\gamma} d\mu(y) < \infty$$

for each $\gamma > 1$. *In particular,*

$$\int_{\mathbf{B}} (1-|y|)^{\beta} d\mu(y) < \infty$$

for each $\beta > s+1$.

We consider a new kernel $K_{1,L}(x,y)$. When $n \neq 2$, we set

$$K_{1,L}(x,y) = \begin{cases} \dfrac{1}{(n-2)\omega_n}|x-y|^{2-n} & \text{for } y \in B(1/2), \\ \dfrac{1}{(n-2)\omega_n}\left\{|x-y|^{2-n} - \displaystyle\sum_{\ell=0}^{L} \varphi_\ell(x,\tilde{y})(1-|y|)^\ell\right\} & \\ & \text{for } y \in \mathbf{B}\setminus B(1/2), \end{cases}$$

where

$$\varphi_\ell(x,\tilde{y}) = \sum_{\ell/2 \le m \le \ell} \binom{\frac{2-n}{2}}{m}\binom{m}{\ell-m} 2^{2m-\ell}|x-\tilde{y}|^{2-n-2m}(x\cdot\tilde{y}-1)^{2m-\ell}.$$

When $n=2$, we set

$$K_{1,L}(x,y) = \begin{cases} \dfrac{1}{\omega_2}\log\left(\dfrac{1}{|x-y|}\right) & \text{for } y \in B(1/2), \\ \dfrac{1}{\omega_2}\left\{\log\left(\dfrac{|x-\tilde{y}|}{|x-y|}\right) - \displaystyle\sum_{\ell=1}^{L} \varphi_\ell(x,\tilde{y})(1-|y|)^\ell\right\} & \\ & \text{for } y \in \mathbf{B}\setminus B(1/2), \end{cases}$$

where

$$\varphi_\ell(x, \tilde{y}) = \frac{1}{2} \sum_{\ell/2 \le m \le \ell} \frac{(-1)^m}{m} \binom{m}{\ell - m} 2^{2m-\ell} |x - \tilde{y}|^{-2m} (x \cdot \tilde{y} - 1)^{2m-\ell}.$$

As in Lemma 3.1, we have the next Lemma.

Lemma 5.3 *The following hold:*

(1) $K_{1,L}(\cdot, y)$ *is harmonic in* $\mathbf{B} \setminus \{y\}$ *for fixed* $y \in \mathbf{B}$.
(2) $K_{1,L}(\cdot, y)$ *is superharmonic in* $\mathbf{B}$ *and* $(-\Delta) K_{1,L}(\cdot, y) = \delta_y$ *for fixed* $y \in \mathbf{B}$.
(3) $K_{1,L}(x, y) = O((1 - |y|)^{L+1})$ *as* $|y| \to 1$ *for fixed* $x \in \mathbf{B}$.

Lemma 5.3 gives the following Lemma.

Lemma 5.4 *Let* u *be superharmonic in* $\mathbf{B}$ *and* $\mu = (-\Delta)u$. *If*

$$\int_{\mathbf{B}} (1 - |y|)^{L+1} d\mu(y) < \infty,$$

then u *is of the form*

$$u(x) = \int_{\mathbf{B}} K_{1,L}(x, y) d\mu(y) + h_L(x),$$

where h_L *is harmonic in* $\mathbf{B}$.

Now we give the Riesz decomposition theorem in the harmonic case.

Theorem 5.5 *Let* u *be superharmonic in* $\mathbf{B}$ *and* $\mu = (-\Delta)u$.

(1) *If* $\lim_{r \to 1} M(u, r) > -\infty$, *then* u *is of the form*

$$u(x) = \int_{\mathbf{B}} K_{1,0}(x, y)\ d\mu(y) + h(x) \quad \text{for} \ \ x \in \mathbf{B}, \tag{5.1}$$

where h *is harmonic in* $\mathbf{B}$.
(2) *If* $\liminf_{r \to 1} (1 - r)^s M(u, r) > -\infty$ *for some* $s > 0$, *then*

$$u(x) = \int_{\mathbf{B}} K_{1,L}(x, y)\ d\mu(y) + h_L(x) \quad \text{for} \ \ x \in \mathbf{B},$$

where $L > s$ *and* h_L *is harmonic in* $\mathbf{B}$.

Remark 5.6 In (5.1), $K_{1,0}(x,y)$ can be replaced by Green's function $G(x,y)$ for $\mathbf{B}$. It is well-known that u is a superharmonic function on $\mathbf{B}$ which is bounded below, then

$$u(x) = \int_{\mathbf{B}} G(x,y)\ d\mu(y) + h(x) \quad \text{for} \ \ x \in \mathbf{B},$$

where h is harmonic in $\mathbf{B}$. Theorem 5.5 gives a representation for a superharmonic function which is not bounded below.

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