Hokkaido Mathematical Journal Vol. 38 (2009) p. 511–517

Finiteness theorem for topological contact equivalence of map germs

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(Received June 6, 2008; Revised September 11, 2008)

Abstract. Let $P^k(n,2)$ be the set of all real polynomial map germs $f = (f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ with degree of $f_1, f_2 \leq k$. The main result of this paper shows that the set of equivalence classes of $P^k(n,2)$, with respect to topological contact equivalence, is finite.

Key words: topological contact equivalence, finiteness theorem.

1. Introduction

A classification question (or, in other words, equivalence relation) in Singularity Theory is called tame if it satisfies the following "finiteness property": The set of equivalence classes of polynomial map-germs in $P^k(n,p)$, i.e., polynomial map-germs of degree less than or equal to k from $\mathbb{R}^n$ to $\mathbb{R}^p$, with respect to this equivalence relation is finite. For example, the question of topological classification (or, in other words, the topological equivalence) of polynomial map-germs is not tame. Nakai [6] proved that the space $P^4(3,2)$ has "moduli" with respect to the topological equivalence. On the other hand the problem of topological classification of polynomial functions on $\mathbb{R}^n$ has this finiteness property. It was proved by Fukuda (see [2]). Moreover, the problem of topological classification of the polynomial map-germs from $\mathbb{R}^2$ to $\mathbb{R}$ ($p \geq 2$) has the finiteness property (see [1], [9]). This paper is devoted to the question of topological contact equivalence. The smooth contact equivalence of singularities was discovered by J. Mather ([4], [5]) and it is closely related to the smooth equivalence. The topological contact equivalence was defined by Nishimura (see [7] and [8]) and it is related to

2000 Mathematics Subject Classification : 32S15, 32S05.
[*1]Research supported under CNPq grant no 300985/93-2.
[*2]Research supported by FAPESP grant 2007/01274-6.
[*3]Research supported under CNPq grant no 300393/2005-9.

the topological equivalence. But, this equivalence relation is a bit weaker: the topological equivalence implies the topological contact equivalence, but not vice-versa.

Here we prove that topological contact equivalence is tame if we consider the polynomial map-germs from $\mathbb{R}^n$ to $\mathbb{R}^2$. In [8] Nishimura relates topological contact equivalence of so-called "topologically contact finite" (somewhat generic case) map-germs from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the homotopy type of the restriction of representatives of them to the complement of the origin. In fact, this idea works in much more general set-up. We prove that if the zero-sets of the map-germs are the same, the map-germs are topologically contact equivalent if the restrictions of representatives them to the complement of the zero set, considered as the maps to $\mathbb{R}^2$, are homotopic. We show that the set of homotopy types of the semialgebraic maps between two given semialgebraic sets with the complexity bounded from above by some value k is finite. This statement implies the finiteness property. In addition, we prove a contact topological analog of the conical structure theorem of Fukuda ([2]). We show that any map-germ from $\mathbb{R}^n$ to $\mathbb{R}^2$ is topologically contact equivalent to a conical map.

2. Basic definitions and main results

Definition 2.1 We say two map germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ are *topologically contact equivalent* if there exist two germs of homeomorphisms $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc}
(\mathbb{R}^n, 0) & \xrightarrow{(id, f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\
\downarrow h & & \downarrow H & & \downarrow h \\
(\mathbb{R}^n, 0) & \xrightarrow[(id, g)]{} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow[\pi_n]{} & (\mathbb{R}^n, 0)
\end{array}$$

where $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity mapping and $\pi_n : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ is the canonical projection.

Definition 2.2 Let $X \subset \mathbb{R}^n$ be a semialgebraic set. An ε-tube of X is the set defined by

$$X^\varepsilon = \{x \in \mathbb{R}^n \mid d(x, X) \leq \varepsilon\}.$$

We define an *ε-shell of X* as the boundary of X^ε, i.e., the ε-shell of X is given by

$$\partial X^\varepsilon = \{x \in \mathbb{R}^n \mid d(x, X) = \varepsilon\}.$$

Definition 2.3 A mapping $f : \mathbb{R}^n \to \mathbb{R}^2$ is called *pseudo-conical with respect to $X \subset \mathbb{R}^n$* if for all sufficiently small $\varepsilon > 0$ we have $f(\partial X^\varepsilon) \subset S^1_\varepsilon$, where S^1_ε is the ε-sphere centered at $0 \in \mathbb{R}^2$.

Note that the family of the ε-shells is topologically trivial for small ε. It means that there exist a positive number $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists a semialgebraic homeomorphism

$$h_\varepsilon : \partial X^\varepsilon \to \partial X^{\varepsilon_0}.$$

Definition 2.4 A mapping $f : \mathbb{R}^n \to \mathbb{R}^2$ is called *topologically conical with respect to $X \subset \mathbb{R}^n$* if for each ε and for each $x \in \partial X^\varepsilon$ we have

$$f(x) = \frac{\varepsilon}{\varepsilon_0} f(h_\varepsilon(x)).$$

Our main results are the following.

Theorem 2.5 (Finiteness Theorem) *Let $P^k(n, 2)$ be the set of all polynomial map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, $f = (f_1, f_2)$, where the degrees of f_1 and f_2 are less than or equal to $k \in \mathbb{N}$. Then the set of the equivalence classes with respect to topological contact equivalence is finite.*

Theorem 2.6 (Conical Structure) *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be a semialgebraic continuous map-germ. Then there exists a topologically conical map-germ $\tilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ such that f and $\tilde{f}$ are topologically contact equivalent and $f^{-1}(0) = \tilde{f}^{-1}(0)$.*

3. Proofs

Proposition 3.1 *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be a semialgebraic continuous map-germ. Then there exists a semialgebraic topologically pseudo-conical map-germ $\tilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ such that f and $\tilde{f}$ are topologically contact* *equivalent and* $f^{-1}(0) = \tilde{f}^{-1}(0)$.

Proof. Let us construct the map $\tilde{f}$ in the following way: for each $x \in \mathbb{R}^n$ we define

$$\tilde{f}(x) = \begin{cases} d(x, f^{-1}(0))\frac{f(x)}{\|f(x)\|} & \text{if } x \notin f^{-1}(0) \\ 0 & \text{if } x \in f^{-1}(0) \end{cases}$$

By the Lemma of Nishimura [8], f and $\tilde{f}$ are topologically contact equivalent. □

Proposition 3.2 *Let* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ *be semialgebraic continuous map-germs such that* $f^{-1}(0) = g^{-1}(0) = X$ *and suppose that the restrictions*

$$f, g \colon \mathbb{R}^n \setminus X \to \mathbb{R}^2 \setminus \{0\}$$

are homotopic. Then the map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ *are topologically contact equivalent.*

Proof. From Proposition 3.1 we can consider $\tilde{f}$ and $\tilde{g}$ be two topologically pseudo-conical map germs such that f is topologically contact equivalent to $\tilde{f}$, $f^{-1}(0) = \tilde{f}^{-1}(0)$ and g is topologically contact equivalent to $\tilde{g}$, $g^{-1}(0) = \tilde{g}^{-1}(0)$. Identify $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$. We can consider the map $\Theta : \mathbb{R}^n \setminus X \to \mathbb{C} \setminus \{0\}$ defined as follows:

$$\Theta(x) = \frac{\tilde{f}(x)}{\tilde{g}(x)} \quad \text{(a complex divison).}$$

Lemma 3.3 *If* Θ *is nullhomotopic then* $\tilde{f}$ *and* $\tilde{g}$ *are topologically contact equivalent.*

Observe that Θ is nullhomotopic if, and only if, $\tilde{f}, \tilde{g} \colon \mathbb{R}^n \setminus X \to \mathbb{R}^2 \setminus \{0\}$ are homotopic. Since $\tilde{f}$ is topologically contact equivalent to f and $\tilde{g}$ is topologically contact equivalent to g, then f and g are topologically contact equivalent (Lemma 3.3). This concludes the proof of the proposition. □

Proof of the Lemma 3.3. Since $\tilde{f}$ and $\tilde{g}$ are topologically pseudo-conical map germs with respect to $X = f^{-1}(0) = g^{-1}(0)$, we can consider $\|\tilde{f}(x)\| = \|\tilde{g}(x)\| = \|x\|$ and the mapping $\Theta = \frac{\tilde{f}}{\tilde{g}}$ defined from $\mathbb{R}^n \setminus X$ to S^1.

Let $e^{i\theta}$ be the universal covering $\mathbb{R} \to S^1$. Since Θ is nullhomotopic, there exists a continuous map $m\colon \mathbb{R}^n \setminus X \to \mathbb{R}$ such that

$$\tilde{f}(x) = e^{im(x)}\tilde{g}(x),\ \forall\, x \in \mathbb{R}^n \setminus X.$$

Let $\lambda_x\colon \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of the norm $\|y\|$ given by

$$\lambda_x(\|y\|) = \begin{cases} 1 & \text{if } 0 \le \|y\| \le d(x,X) \\ \frac{-\|y\|+2d(x,X)}{d(x,X)} & \text{if } d(x,X) \le \|y\| \le 2d(x,X) \\ 0 & \text{if } \|y\| \ge 2d(x,X) \end{cases}$$

For all $x \in \mathbb{R}^n \setminus X$ and $y \in \mathbb{C}$ we define the mapping $L_x(y) = \mathrm{e}^{im(x)\lambda_x(\|y\|)}\, y$.

For all $x \in X = f^{-1}(0)$, we define L_x as identity mapping. So the map L_x is defined for all $\mathbb{R}^2 \equiv \mathbb{C}$ and it satisfies $L_x(0) = 0$ and $L_x(\tilde{g}(x)) = \tilde{f}(x)$.

Then we define the mapping $H(x,y) = (x, L_x(y))$, $(x,y) \in \mathbb{R}^n \times \mathbb{C}$. This mapping H is a homeomorphism. From our construction, the pair of mappings (id, H) satisfies the condition of (h,H) in the Definition 2.1. Therefore, we can conclude that $\tilde{f}$ and $\tilde{g}$ are topologically contact equivalent and the Lemma 3.3 is proved. $\square$

Remark 3.4 The presentation of the map L_x corresponds to the Mather's description of the contact equivalence in the smooth case (Proposition 2.3 of [4]). In the general case $(p > 2)$, we do not know if this proposition holds.

Proof of the Conical Structure. By Proposition 3.1, we can suppose that f has a pseudo-conical structure. Let us construct a map $\bar{f}$ as follows. Let $\epsilon_0 > 0$ be a small number. Let $0 < \epsilon < \epsilon_0$ and let x belongs to ϵ-shell of $X = f^{-1}(0)$. Let h_ϵ be a trivialization family as defined in page 02. Set $\bar{f}(x) = \frac{\epsilon}{\epsilon_0} f(h_\epsilon(x))$. Since $\bar{f}$ and f are equal on the ϵ_0-shell of X, we conclude that their restrictions $U \setminus X \to \mathbb{R}^2 \setminus \{0\}$, where U is a neighborhood of X in $\mathbb{R}^n$. By Proposition 3.2, $\bar{f}$ and f are topologically contact equivalent. $\square$

In order to show the Finiteness Theorem, we need the following result.

Proposition 3.5 *Let Z and Y be two semialgebraic sets. Let $\mathcal{L}^k(Z,Y)$ be the set of all semialgebraic maps with complexity less than or equal to $k \in \mathbb{N}$. Then the set of homotopy types of maps in $\mathcal{L}^k(Z,Y)$ is finite.*

Proof. Clearly $\mathcal{L}^k(Z, Y)$ is semialgebraic and thus it has a finite number of connected components. If the two maps F and G belong to the same connected component of $\mathcal{L}^k(Z, Y)$ they are homotopic. □

Proof of the Finiteness Theorem. Consider the set $P^m(n, 2)$ of the all polynomial maps of degree less than or equal to $m \in \mathbb{N}$. By the Hardt's Theorem (cf. [3]), there exists a number $k \in \mathbb{N}$ (depending only on m and n) and there exists a finite set of maps $F_1, \ldots, F_r$ in $P^m(n, 2)$ such that for any $F \in P^m(n, 2)$ there exist a map $F_i \in \{F_1, \ldots, F_r\}$ and a semialgebraic homeomorphism $H\colon \mathbb{R}^n \to \mathbb{R}^n; H(F_i^{-1}(0)) = F^{-1}(0)$ and, moreover, the complexity of $F \circ H$ is less than or equal to k.

Since the number of the homotopy types of the maps in $\mathcal{L}^k(\mathbb{R}^n \setminus F_i^{-1}(0), \mathbb{R}^2 \setminus \{0\})$ is finite (by Proposition 3.5), the theorem is proved. □

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