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Weighted sharing of three values by meromorphic functions

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Abstract. Using the notion of weighted sharing of values we prove some uniqueness theorems for meromorphic functions which improve some existing results.

Key words: uniqueness, weighted sharing, meromorphic function.

1. Introduction and Definitions

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$ f and g have the same set of a-points with the same multiplicity, we say that f, g share the value a CM (counting multiplicities) and if we do not take the multiplicities into account f, g are said to share the value a IM (ignoring multiplicities). We denote by E a set of nonnegative real numbers of finite Lebesgue measure which is not necessarily the same in each of its occurrences.

Though for the standard definitions and notations of the value distribution theory we refer to [3], in the following definition we explain some notations.

Definition 1.1 ([4, 8]) We denote by $N(r, a; f \mid\leq 1)$ and $N(r, a; f \mid\geq 2)$ the counting functions of simple and multiple a-points of f respectively. Also by $\overline{N}(r, a; f \mid\geq 2)$ we denote the reduced counting function of multiple a-points of f, where each a-point is counted only once.

A number of authors viz., R. Nevanlinna [10], M. Ozawa [11], G.G. Gundersen [2], H. Ueda [12, 13], G. Brosch [1], H.X. Yi et al. [14, 15, 16, 17], P.Li and C.C. Yang [9], Q.C. Zhang [18] worked on the problem of uniqueness of meromorphic functions sharing three values.

M. Ozawa [11] proved the following result.

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Theorem A ([11]) *Let f and g be two distinct nonconstant entire functions of finite order sharing* 0, 1 *CM. If $\delta(0; f) > 1/2$ then $fg \equiv 1$.*

Removing the order restriction on the functions H. Ueda [12] proved the following theorem.

Theorem B ([12]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\limsup_{r\to\infty} \frac{N(r,\,\infty; f) + N(r,\,0; f)}{T(r,\,f)} < \frac{1}{2}$$

then $fg \equiv 1$.

G. Brosch [1] further improved Theorem B and proved the following result.

Theorem C ([1]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,\,\infty; f) + \overline{N}(r,\,0; f) - (1/2)m(r,\,1; g)}{T(r,\,f)} < \frac{1}{2}$$

then $fg \equiv 1$.

G. Brosch [1] also proved the following two results.

Theorem D ([1]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\limsup_{r\to\infty,\, r\notin E} \frac{N(r)}{T(r,\,f)} > \frac{2}{3}$$

then f is a bilinear transformation of g, where $N(r)$ denotes the counting function of those zeros of $f - g$ which are not the zeros of $f(f-1)$ and $1/f$.

Theorem E ([1]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\overline{N}(r,\,0; f') + \overline{N}(r,\,0; g') = S(r,\,f)$$

then f is a bilinear transformation of g.

Improving Theorem D and Theorem E, Q.C. Zhang [18] proved the following two results.

Theorem F ([18]) *Let f and g be two distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If*

$$\limsup_{r\to\infty,\, r\notin E} \frac{N(r)}{T(r,\, f)} > \frac{1}{2}$$

then f and g satisfy one of the following relations: (i) $f + g \equiv 1$, (ii) $(f-1)(g-1) \equiv 1$ *and* (iii) $fg \equiv 1$.

Theorem G ([18]) *Let f and g be two distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If*

$$\overline{N}(r,\, 0; f') + \overline{N}(r,\, 0; g') < \lambda\{T(r,\, f) + T(r,\, g)\}$$

for a constant λ $(0 < \lambda < 1/2)$, then f is a bilinear transformation of g.

Q.C. Zhang [18] also proved the following result which is a sort of complement to Theorem F.

Theorem H ([18]) *Let f and g be two distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If*

$$0 < \limsup_{r\to\infty,\, r\notin E} \frac{N(r)}{T(r,\, f)} \le \frac{1}{2}$$

then $N(r) = (1/p)T(r,\, f) + S(r,\, f)$ and f is not a bilinear transformation of g and they assume one of the following forms:

(i) $f = \dfrac{e^{s\gamma} - 1}{e^{(p+1)\gamma} - 1}, \quad g = \dfrac{e^{-s\gamma} - 1}{e^{-(p+1)\gamma} - 1};$

(ii) $f = \dfrac{e^{(p+1)\gamma} - 1}{e^{(p+1-s)\gamma} - 1}, \quad g = \dfrac{e^{-(p+1)\gamma} - 1}{e^{-(p+1-s)\gamma} - 1};$

(iii) $f = \dfrac{e^{s\gamma} - 1}{e^{-(p+1-s)\gamma} - 1}, \quad g = \dfrac{e^{-s\gamma} - 1}{e^{(p+1-s)\gamma} - 1};$

where s and p are positive integers with $1 \le s \le p$ and s, $p+1$ are relatively prime and γ is a nonconstant entire function.

Using the results of Zhang [18] recently H.X. Yi and W.R. Lü [17] proved the following results, the first of which extends Theorem C and the second extends Theorem B.

Theorem I ([17]) *Let f and g be two distinct nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If there exists a set I of infinite linear mea-*

sure such that

$$\limsup_{r\to\infty,\, r\in I} \frac{N(r,\,\infty; f\mid\leq 1) + N(r,\,0; f\mid\leq 1) - m(r,\,1; g)}{T(r,\,f)} < 1$$

then f and g satisfy one one of the following relations:

(i) $f = \dfrac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}, \quad g = \dfrac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1};$

(ii) $f = \dfrac{e^{(p+1)\gamma}-1}{e^{s\gamma}-1}, \quad g = \dfrac{e^{-(p+1)\gamma}-1}{e^{-s\gamma}-1};$

(iii) $f = \dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}, \quad g = \dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1};$

where s and p are positive integers with $1 \leq s \leq p$ and $s,\ p+1$ are relatively prime and γ is a nonconstant entire function. Also

$$N(r,\,\infty; f\mid\leq 1) + N(r,\,0; f\mid\leq 1) - m(r,\,1; g) = \Big(1 - \frac{1}{p}\Big) T(r,\,f) + S(r,\,f).$$

Theorem J ([17]) *Let f and g be two distinct nonconstant meromorphic functions sharing* $0,\,1,\,\infty$ *CM. If there exists a set I of infinite linear measure such that*

$$\limsup_{r\to\infty,\, r\in I} \frac{N(r,\,\infty; f\mid\leq 1) + N(r,0; f\mid\leq 1)}{T(r,\,f)} < 1,$$

then f and g assume the following form:

$$f = \frac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}, \quad g = \frac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1},$$

where $s,\ p$ are positive integers with $1 \leq s \leq p$, and $s,\ p+1$ are relatively prime and γ is a nonconstant entire function. Also

$$N(r,\,\infty; f\mid\leq 1) + N(r,\,0; f\mid\leq 1) = \Big(1 - \frac{1}{p}\Big) T(r,\,f) + S(r,\,f).$$

The purpose of the paper is to show that the supposition of sharing the values $0,\,1,\,\infty$ CM is in fact redundant and it is possible to achieve the same conclusion under a remarkable relaxation on the nature of sharing the values. To this end we use the notion of weighted sharing of values as introduced in [4, 5], which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1.2 ([4, 5]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_o is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.3 ([8]) Let f and g share a value a IM. Let z be an a-point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively.

We put

$$\begin{aligned} \overline{\nu}_f(z) &= 1 \quad \text{if } p_f(z) > p_g(z) \\ &= 0 \quad \text{if } p_f(z) \leq p_g(z) \end{aligned}$$

and

$$\begin{aligned} \overline{\mu}_f(z) &= 1 \quad \text{if } p_f(z) < p_g(z) \\ &= 0 \quad \text{if } p_f(z) \geq p_g(z). \end{aligned}$$

Let $\overline{n}(r, a; f > g) = \sum_{|z| \leq r} \overline{\nu}_f(z)$ and $\overline{n}(r, a; f < g) = \sum_{|z| \leq r} \overline{\mu}_f(z)$. We now denote by $\overline{N}(r, a; f > g)$ and $\overline{N}(r, a; f < g)$ the integrated counting functions obtained from $\overline{n}(r, a; f > g)$ and $\overline{n}(r, a; f < g)$ respectively.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([18]) *Let f_1 and f_2 be nonconstant meromorphic functions satisfying $\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S_0(r)$ for $i = 1, 2$. Then either $\overline{N}_0(r, 1; f_1, f_2) = S_0(r)$ or there exist two integers s, t $(|s| + |t| > 0)$ such* *that* $f_1^s f_2^t \equiv 1$, *where* $\overline{N}_0(r,\, 1; f_1,\, f_2)$ *denotes the reduced counting function of* f_1 *and* f_2 *related to the common* 1*-points and* $T(r) = T(r,\, f_1) + T(r,\, f_2)$, $S_0(r) = o(T(r))$ *as* $r \to \infty$ $(r \notin E)$.

Lemma 2.2 ([2]) *If* $f,\, g$ *share* $(0,\, 0)$, $(1,\, 0)$, $(\infty,\, 0)$ *then* (i) $T(r,\, f) \le 3T(r,\, g) + S(r,\, f)$, (ii) $T(r,\, g) \le 3T(r,\, f) + S(r,\, g)$.

This shows that $S(r,\, f) = S(r,\, g)$ and we denote them by $S(r)$.

Lemma 2.3 ([6]) *Let* $f,\, g$ *share* $(0,\, 1)$, $(1,\, m)$, $(\infty,\, k)$ *and* $f \not\equiv g$, *where* $(m-1)(mk-1) > (1+m)^2$. *Then* $\overline{N}(r,\, a; f \mid\ge 2) = S(r)$ *and* $\overline{N}(r,\, a; g \mid\ge 2) = S(r)$ *for* $a = 0,\, 1,\, \infty$.

Lemma 2.4 ([7]) *Let* $f,\, g$ *share* $(0,\, 0)$, $(1,\, 0)$, $(\infty,\, 0)$ *and* $f \not\equiv g$. *If* $\alpha = f/g$ *and* $\beta = (f-1)/(g-1)$ *then*

(i) $\overline{N}(r,\, 0; \alpha) = \overline{N}(r,\, \infty; f < g) + \overline{N}(r,\, 0; f > g)$,

(ii) $\overline{N}(r,\, \infty; \alpha) = \overline{N}(r,\, \infty; f > g) + \overline{N}(r,\, 0; f < g)$,

(iii) $\overline{N}(r,\, 0; \beta) = \overline{N}(r,\, \infty; f < g) + \overline{N}(r,\, 1; f > g)$,

(iv) $\overline{N}(r,\, \infty; \beta) = \overline{N}(r,\, \infty; f > g) + \overline{N}(r,\, 1; f < g)$.

Lemma 2.5 *Let* $f,\, g$ *share* $(0,\, 1)$, $(1,\, m)$, $(\infty,\, k)$ *and* $f \not\equiv g$, *where* $(m-1)(mk-1) > (1+m)^2$. *If* α *and* β *are defined as in* Lemma 2.4 *then* $\overline{N}(r,\, a; \alpha) = S(r)$ *and* $\overline{N}(r,\, a; \beta) = S(r)$ *for* $a = 0,\, \infty$.

Proof. The lemma follows from Lemmas 2.3 and 2.4 because $\overline{N}(r,\, a; f > g) \le \overline{N}(r,\, a; f \mid\ge 2)$ and $\overline{N}(r,\, a; f < g) \le \overline{N}(r,\, a; g \mid\ge 2)$ for $a = 0,\, 1,\, \infty$. □

Lemma 2.6 *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(0,\, 0)$, $(1,\, 0)$ *and* $(\infty,\, 0)$. *If* f *is a bilinear transformation of* g *then* f *and* g *satisfy exactly one of the following*:

(i) $f \equiv g$,

(ii) $f + g \equiv 1$ *with* $N(r) = N(r,\, 1/2; f) = T(r,\, f) + S(r,\, f)$;

(iii) $(f-1)(g-1) \equiv 1$ *with* $N(r) = N(r,\, 2; f) = T(r,\, f) + S(r,\, f)$;

(iv) $fg \equiv 1$ *with* $N(r) = N(r,\, -1; f) = T(r,\, f) + S(r,\, f)$;

(v) $f \equiv Ag + 1 - A$ *with* $N(r) = 0$;

(vi) $f \equiv Ag$ *with* $N(r) = 0$;

(vii) $f(g + A - 1) \equiv Ag$ *with* $N(r) = 0$;

where A $(\ne 0,\, 1)$ *is a finite constant.*

We omit the proof as it can be carried out in the line of Lemma 6 [7] and Lemma 7 [18].

Lemma 2.7 *Let α and β be two meromorphic functions such that $\alpha^s\beta^t \equiv 1$, where s, t are relatively prime integers and $s \neq 0$. If h is a function defined by $h^s \equiv \beta$ then h is a single valued meromorphic function.*

Proof. Let $S = \{sx + ty\colon x,\, y \in \mathbb{Z}$ and $sx + ty > 0\}$, where $\mathbb{Z}$ is the set of all integers. Since $s \neq 0$, $|s| = sx + t.0$ is an element of S, where we put $x = 1$ if $s > 0$ and $x = -1$ if $s < 0$. Therefore S is a nonempty set of positive integers and so by well ordering principle S contains a least element d, say. Hence $d = su + tv$ for some integers u and v.

Now by division algorithm we get $s = dq + r$, where q, r are integers and $0 \leq r < d$. Since $r = s - dq = s(1 - uq) + t(-vq)$, it follows that if $r > 0$ then $r \in S$, which is impossible because d is the least element of S. Hence $r = 0$ and so s is divisible by d. Similarly we can show that t is also divisible by d. So d is a common divisor of s and t.

If a is a common divisor of s and t then $d = su + tv$ is divisible by a. Therefore $d = su + tv$ is the greatest common divisor of s and t. Since s and t are relatively prime, $d = 1$ and so we get $su + tv = 1$ for two integers u and v.

Since $\alpha^s\beta^t \equiv 1$ and $h^s \equiv \beta$, it follows that $\alpha h^t \equiv c$, where c is a constant satisfying $c^s = 1$. This shows that h^t is a single valued meromorphic function because α is meromorphic and c is a constant. Therefore

$$h = h^{su+tv} = (h^s)^u(h^t)^v = \beta^u(h^t)^v$$

is a single valued meromorphic function. This proves the lemma. $\square$

Lemma 2.8 ([8]) *Let f, g be two distinct nonconstant meromorphic functions sharing $(0,\, 1)$, $(1,\, m)$ and $(\infty,\, k)$, where $(m-1)(mk-1) > (1+m)^2$. If f is not a bilinear transformation of g then each of the following holds*:

(i) $T(r,\, f) + T(r,\, g) = N(r,\, 0; g \mid\leq 1) + N(r,\, 1; g \mid\leq 1) + N(r,\, \infty; g \mid\leq 1) + N_0(r) + S(r)$,

(ii) $T(r,\, f) + T(r,\, g) = N(r,\, 0; f \mid\leq 1) + N(r,\, 1; f \mid\leq 1) + N(r,\, \infty; f \mid\leq 1) + N_0(r) + S(r)$,

(iii) $N_1(r) = S(r)$,

(iv) $T(r,\, f) = N(r,\, 0; g' \mid\leq 1) + N_0(r) + S(r)$,

(v) $T(r,\, g) = N(r,\, 0; f' \mid\leq 1) + N_0(r) + S(r)$,

where $N_0(r)(N_1(r))$ denotes the counting function of those simple (multiple) zeros of $f - g$ which are not the zeros of $g(g-1)$, $1/g$ and so are not the *zeros of* $f(f-1)$, $1/f$.

Lemma 2.9 ([16]) *If* f, g *are two distinct meromorphic functions sharing* 0, 1, ∞ *CM then* $N(r, a; f \mid\geq 2) = S(r)$ *and* $N(r, a; g \mid\geq 2) = S(r)$ *for* $a = 0, 1, \infty$.

3. Main results

In this section we present the main results of the paper.

Theorem 3.1 *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $(0, 1)$, $(1, m)$ *and* (∞, k), *where* $(m-1)(mk-1) > (1+m)^2$. *If* $N(r) \neq S(r)$ *then one of the following holds*:

(i) f *is a bilinear transformation of* g *with* $N(r) = T(r, f) + S(r) = T(r, g) + S(r)$;

(ii) f *is not a bilinear transformation of* g *with* $T(r, f) = T(r, g) + S(r)$ *and* $N(r) \leq (1/2)T(r, f) + S(r)$.

Proof. If f is a bilinear transformation of g then by Lemma 2.6 we see that $N(r) = T(r, f) + S(r) = T(r, g) + S(r)$, which is (i).

Now we suppose that f is not a bilinear transformation of g. We note in view of Lemma 2.8(iii) that $N(r) = N_0(r) + N_1(r) = N_0(r) + S(r)$. Also α, β (as defined in Lemma 2.4) and $\alpha\beta$ are nonconstant. Since $N_0(r) \neq S(r)$ and $N_0(r) \leq \overline{N}_0(r, 1; \alpha, \beta)$, it follows that $\overline{N}_0(r, 1; \alpha, \beta) \neq S(r)$. Also in view of Lemma 2.2 we note that $S(r) = S_0(r)$. So by Lemma 2.1 there exist two integers s and t ($|s| + |t| > 0$) such that

$$\alpha^s \beta^t \equiv 1. \tag{3.1}$$

Without loss of generality we may suppose that $s > 0$ and s, t are relatively prime. From (3.1) it follows that

$$f^s(f-1)^t \equiv g^s(g-1)^t. \tag{3.2}$$

and so

$$T(r, f) = T(r, g) + S(r). \tag{3.3}$$

Since α, β are nonconstant, it follows from (3.1) that $st \neq 0$. We now consider the following cases.

Case I: Let $t > 0$. If $s + t = 2$ then $s = t = 1$ and so f becomes a bilinear transformation of g, which is not the case. Hence $s + t \geq 3$.

Let h be a function defined by $h^s \equiv \beta$. Then by Lemma 2.7 h is a single valued meromorphic function. We put $c = \alpha h^t$ so that $c^s = 1$ and $\alpha = ch^{-t}$.

Since $g = (1-\beta)/(\alpha-\beta)$ and $g - 1 = (1-\alpha)/(\alpha-\beta)$, it follows that

$$g = \frac{(h^s-1)h^t}{h^{s+t}-c} \tag{3.4}$$

and

$$g - 1 = -\frac{h^t - c}{h^{s+t}-c}. \tag{3.5}$$

By Lemma 2.5 we see that

$$\overline{N}(r,\, 0; h) = \overline{N}(r,\, 0; h^s) = \overline{N}(r,\, 0; \beta) = S(r)$$

and

$$\overline{N}(r,\, \infty; h) = \overline{N}(r,\, \infty; h^s) = \overline{N}(r,\, \infty; \beta) = S(r).$$

Let $L = \{\exp(2lt\pi i/s)\colon l = 0,\, 1,\, 2,\, \ldots,\, s-1\}$. If $c \notin L$ then the numerators and denominators (considered as polynomials in h) in (3.4) and (3.5) have no common factor. So noting that $S(r,\, h) = S(r)$ we get $T(r,\, g) = (s+t)T(r,\, h) + S(r)$, $\overline{N}(r,\, 0; g) = \overline{N}(r,\, 1; h^s) + S(r) = sT(r,\, h) + S(r)$, $\overline{N}(r,\, \infty; g) = \overline{N}(r,\, c; h^{s+t}) + S(r) = (s+t)T(r,\, h) + S(r)$ and $\overline{N}(r,\, 1; g) = \overline{N}(r,\, c; h^t) + S(r) = tT(r,\, h) + S(r)$.

So by Lemma 2.3, Lemma 2.8(i) and (3.3) we get $N_0(r) = S(r)$ and so $N(r) = S(r)$, which is not the case.

Therefore $c \in L$ and so $h - \exp(2l\pi i/s)$ is the only common factor of the numerators and denominators in (3.4) and (3.5). Hence $T(r,\, g) = (s + t - 1)T(r,\, h) + S(r)$, $\overline{N}(r,\, 0; g) = (s-1)T(r,\, h) + S(r)$, $\overline{N}(r,\, \infty; g) = (s + t - 1)T(r,\, h) + S(r)$ and $\overline{N}(r,\, 1; g) = (t-1)T(r,\, h) + S(r)$.

So by Lemma 2.3, Lemma 2.8 (i) and (3.3) we get

$$\begin{aligned} N(r) = N_0(r) + S(r) &= \frac{1}{s+t-1}T(r,\, g) + S(r) \\ &= \frac{1}{s+t-1}T(r,\, f) + S(r) \end{aligned} \tag{3.6}$$

and so $N(r) \le (1/2)T(r,\, f) + S(r)$.

Case II: Let $t < 0$. If $s + t = 0$ then from (3.2) we see that f becomes a bilinear transformation of g, which is not the case. We now consider the following subcases.

Subcase (i) Let $s + t \geq 1$. If $s = 2$ then $t = -1$ and so from (3.2) we see that f becomes a bilinear transformation of g, which is not the case. So $s \geq 3$. In this case we get

$$g = \frac{h^s - 1}{h^{-t}(h^{s+t} - c)} \tag{3.7}$$

and

$$g - 1 = \frac{c(h^{-t} - 1/c)}{h^{-t}(h^{s+t} - c)}. \tag{3.8}$$

Since $N(r) \neq S(r)$, proceeding as *Case I* we see that $c \in L$. Therefore $T(r, g) = (s-1)T(r, h) + S(r)$, $\overline{N}(r, 0; g) = (s-1)T(r, h) + S(r)$, $\overline{N}(r, 1; g) = (-t-1)T(r, h) + S(r)$ and $\overline{N}(r, \infty; g) = (s+t-1)T(r, h) + S(r)$. So by Lemma 2.3, Lemma 2.8 (i) and (3.3) we get

$$N(r) = N_0(r) + S(r) = \frac{1}{s-1}T(r, f) + S(r) \tag{3.9}$$

and so $N(r) \leq (1/2)T(r, f) + S(r)$.

Subcase (ii) Let $s + t \leq -1$. If $t = -2$ then $s = 1$ and so from (3.2) we see that f becomes a bilinear transformation of g, which is not the case. So $t \leq -3$. In this case we get

$$g = -\frac{h^s - 1}{ch^s(h^{-s-t} - 1/c)}, \tag{3.10}$$

and

$$g - 1 = -\frac{c(h^{-t} - 1/c)}{ch^s(h^{-s-t} - 1/c)}. \tag{3.11}$$

Since $N(r) \neq S(r)$, proceeding as *Case I* we see that $c \in L$. Therefore $T(r, g) = (-t-1)T(r, h) + S(r)$, $\overline{N}(r, 0; g) = (s-1)T(r, h) + S(r)$, $\overline{N}(r, 1; g) = (-t-1)T(r, h) + S(r)$ and $\overline{N}(r, \infty; g) = (-t-s-1)T(r, h) + S(r)$. So by Lemma 2.3, Lemma 2.8(i) and (3.3) we get

$$N(r) = N_0(r) + S(r) = \frac{1}{-t-1}T(r, f) + S(r) \tag{3.12}$$

and so $N(r) \leq (1/2)T(r, f) + S(r)$. This proves the theorem. $\square$

Remark 3.1 The condition $(m-1)(mk-1) > (1+m)^2$ of Theorem 3.1 is equivalent to $(m-1)(k-1) > 4$ and so it is symmetric in m and k. Also we see that Theorem 3.1 and the subsequent theorems are valid for the following pairs of least values of m and k: (i) $m = 2$, $k = 6$, (ii) $m = 6$, $k = 2$, (iii) $m = 3$, $k = 4$ and (iv) $m = 4$, $k = 3$.

Remark 3.2 Considering $f = (1-e^z)^3/(1-3e^z)$ and $g = 4(1-e^z)/(1-3e^z)$ we see that in Theorem 3.1 sharing $(0,1)$ cannot be relaxed to sharing (0, 0) because $N(r) = N(r, -1; e^z) = T(r, e^z) = (1/3)T(r, f)$ and $T(r, f) \sim 3T(r, g)$ as $r \to \infty$.

Remark 3.3 Replacing f, g by $1-f$ and $1-g$ we see that Theorem 3.1 holds good if f, g share $(0, m)$, $(1, 1)$ and (∞, k). Also replacing f, g by $1/f$ and $1/g$ we see that Theorem 3.1 holds good if f, g share $(0, k)$, $(1, m)$ and $(\infty, 1)$.

Following theorem improves Theorem F.

Theorem 3.2 *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0, 1)$, $(1, m)$ *and* (∞, k), *where* $(m-1)(mk-1) > (1+m)^2$. *If $N(r) \geq \lambda T(r, f) + S(r)$ for some $\lambda > 1/2$ then f is a bilinear transformation of g and $N(r) = T(r, f) + S(r) = T(r, g) + S(r)$. Further f and g satisfy one of the following*: (i) $f+g \equiv 1$, (ii) $(f-1)(g-1) \equiv 1$, (iii) $fg \equiv 1$.

Proof. Since $N(r) \geq \lambda T(r, f) + S(r)$ for some $\lambda > 1/2$, by Theorem 3.1 we see that f is a bilinear transformation of g and $N(r) = T(r, f) + S(r) = T(r, g) + S(r)$. Also by Lemma 2.6 we see that f and g satisfy one of the following: (i) $f + g \equiv 1$, (ii) $(f-1)(g-1) \equiv 1$, (iii) $fg \equiv 1$. This proves the theorem. □

Following theorem improves Theorem H.

Theorem 3.3 *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0, 1)$, $(1, m)$ *and* (∞, k), *where* $(m-1)(mk-1) > (1+m)^2$. *If $N(r) \leq \lambda T(r, f) + S(r)$ for some λ $(0 < \lambda < 1)$ and $N(r) \not\equiv S(r)$ then f is not a bilinear transformation of g and $N(r) = (1/p)T(r, f) + S(r)$, $T(r, f) = T(r, g) + S(r)$ and f, g satisfy one of the following*:

(i) $f = \dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}, \quad g = \dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1};$

(ii) $f = \dfrac{e^{(p+1)\gamma}-1}{e^{(p+1-s)\gamma}-1}, \quad g = \dfrac{e^{-(p+1)\gamma}-1}{e^{-(p+1-s)\gamma}-1};$

(iii) $f = \dfrac{e^{s\gamma} - 1}{e^{-(p+1-s)\gamma} - 1}, \quad g = \dfrac{e^{-s\gamma} - 1}{e^{(p+1-s)\gamma} - 1};$
where s and $p(\geq 2)$ are positive integers with $1 \leq s \leq p$ and s, $p+1$ are relatively prime and γ is a nonconstant entire function.

Proof. Since $N(r) \leq \lambda T(r, f) + S(r)$ and $0 < \lambda < 1$, by Theorem 3.1 we see that f is not a bilinear transformation of g and $T(r, f) = T(r, g) + S(r)$. Now we consider the following cases and subcases of the proof of Theorem 3.1.

Case I: Let $t > 0$ and so $s + t \geq 3$. Using (3.4) we see that

$$g = \frac{(h^s - 1)h^t}{h^{s+t} - c} \tag{3.13}$$

and

$$f = \alpha g = \frac{c(h^s - 1)}{h^{s+t} - c}. \tag{3.14}$$

From (3.13) and (3.14) we see that if z_0 is a zero of h then $g(z_0) = 0$ and $f(z_0) = 1$. Since f, g share $(0, 1)$ and $(1, m)$, it follows that h has no zero. Again if z_1 is a pole of h then $g(z_1) = 1$ and $f(z_1) = 0$. Since f, g share $(0, 1)$ and $(1, m)$, it follows that h has no pole.

Now we put $p = s + t - 1$ so that $1 \leq s \leq p$ and $p \geq 2$. Since $N(r) \neq S(r)$, it follows that $c = \exp(2tl\pi i/s)$ for some $l \in \{0, 1, 2, \ldots, s-1\}$. We now put $h = ae^{\gamma}$, where γ is a nonconstant entire function and $a = \exp(2l\pi i/s)$. Since $a^s = 1$ and $a^{p+1} = c$, it follows from (3.13) and (3.14) that

$$f = \frac{e^{s\gamma} - 1}{e^{(p+1)\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-s\gamma} - 1}{e^{-(p+1)\gamma} - 1}.$$

Also from (3.6) we get $N(r) = (1/p)T(r, f) + S(r)$.

Case II: Let $t < 0$. We now consider the following subcases.

Subcase (i) Let $s + t \geq 1$ so that $s \geq 3$. Using (3.7) we see that

$$g = \frac{h^s - 1}{h^s - ch^{-t}} \tag{3.15}$$

and

$$f = \alpha g = \frac{c(h^s - 1)}{h^{s+t} - c}. \tag{3.16}$$

Since a zero of h is an 1-point of f and a pole of g and f, g share $(1, m)$, (∞, k), it follows that h has no zero. Also since a pole of h is a pole of f and an 1-point of g, it follows that h has no pole.

We put $p = s - 1$ and $t = -u$ so that $p \geq 2$ and $1 \leq u \leq p$. Since $N(r) \neq S(r)$, it follows that $c = \exp(2tl\pi i/s)$ for some $l \in \{0, 1, 2, \ldots, s - 1\}$. We now put $h = ae^{\gamma}$, where γ is a nonconstant entire function and $a = \exp(2l\pi i/s)$.

Since $a^{p+1} = 1$ and $a^{p+1-u} = c$, it follows from (3.15) and (3.16) that

$$f = \frac{e^{(p+1)\gamma} - 1}{e^{(p+1-u)\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-(p+1)\gamma} - 1}{e^{-(p+1-u)\gamma} - 1}$$

and now we rename u as s.

Also from (3.9) we get $N(r) = (1/p)T(r, f) + S(r)$.

Subcase (ii) Let $s + t \leq -1$ so that $t \leq -3$. Using (3.10) we get

$$g = -\frac{h^s - 1}{ch^s(h^{-s-t} - 1/c)} \tag{3.17}$$

and

$$f = \alpha g = -\frac{h^{-t}(h^s - 1)}{h^s(h^{-s-t} - 1/c)}. \tag{3.18}$$

Since a zero of h is a zero of f and a pole of g and f, g share $(0, 1)$, (∞, k), it follows that h has no zero. Again since a pole of h is a pole of f and a zero of g, it follows that h has no pole.

We put $p = -t - 1$ so that $p \geq 2$ and $1 \leq s \leq p$. Since $N(r) \neq S(r)$, we see that $c = \exp(2tl\pi i/s)$ for some $l \in \{0, 1, 2, \ldots, s - 1\}$. Now we put $h = ae^{\gamma}$, where γ is a nonconstant entire function and $a = \exp(2l\pi i/s)$.

Since $a^s = 1$ and $a^{s+t} = c$, it follows from (3.17) and (3.18) that

$$f = \frac{e^{s\gamma} - 1}{e^{-(p+1-s)\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-s\gamma} - 1}{e^{(p+1-s)\gamma} - 1}.$$

Also from (3.12) we get $N(r) = (1/p)T(r, f) + S(r)$. This proves the theorem. □

Following theorem improves Theorem G.

Theorem 3.4 *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0,\,1)$*,* $(1,\,m)$ *and* $(\infty,\,k)$*, where* $(m-1)(mk-1) > (1+m)^2$*. If* $N(r,\,0;f' \mid\leq 1) + N(r,\,0;g' \mid\leq 1) \leq \lambda\{T(r,\,f)+T(r,\,g)\} + S(r)$ *for some* λ $(0<\lambda<1/2)$ *then f is a bilinear transformation of g.*

Proof. If possible suppose that f is not a bilinear transformation of g. Then by Lemma 2.8 (iv)–(v) we see that

$$\begin{aligned} &T(r,\,f)+T(r,\,g) \\ &\qquad = N(r,\,0;f' \mid\leq 1) + N(r,\,0;g' \mid\leq 1) + 2N_0(r) + S(r) \\ &\qquad \leq \lambda\{T(r,\,f)+T(r,\,g)\} + 2N_0(r) + S(r) \end{aligned}$$

and so

$$2N_0(r) \geq (1-\lambda)\{T(r,\,f)+T(r,\,g)\} + S(r). \tag{3.19}$$

This shows that $N_0(r) \neq S(r)$ and so $N(r) \neq S(r)$. Hence by Theorem 3.1 we get $T(r,\,f) = T(r,\,g) + S(r)$ and $N(r) \leq (1/2)T(r,\,f) + S(r)$. Since $N_0(r) \leq N(r)$, it follows from (3.19) that $(1-\lambda)T(r,\,f) \leq (1/2)T(r,\,f) + S(r)$, which is a contradiction. This proves the theorem. □

Remark 3.4 Considering $f = e^{2z} + e^{z} + 1$ and $g = e^{-2z} + e^{-z} + 1$ we see that the condition $\lambda < 1/2$ in Theorem 3.4 is essential.

Next theorem improves Theorem I.

Theorem 3.5 *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0,\,1)$*,* $(1,\,m)$ *and* $(\infty,\,k)$*, where* $(m-1)(mk-1) > (1+m)^2$*. If*

$$N(r,\,\infty;f \mid\leq 1) + N(r,\,0;f \mid\leq 1) - m(r,\,1;g) \leq \lambda T(r,\,f) + S(r)$$

for some λ $(0<\lambda<1)$ *then f and g assume one of the following forms*:

(I) $f = \dfrac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}, \quad g = \dfrac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1};$

(II) $f = \dfrac{e^{(p+1)\gamma}-1}{e^{s\gamma}-1}, \quad g = \dfrac{e^{-(p+1)\gamma}-1}{e^{-s\gamma}-1};$

(III) $f = \dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}, \quad g = \dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1};$

where s and p are positive integers with $1 \leq s \leq p$ and s, $p+1$ are relatively prime and γ is a nonconstant entire function. Also

$$N(r,\,\infty;f\mid\leq 1)+N(r,\,0;f\mid\leq 1)-m(r,\,1;g)$$
$$=\Big(1-\frac{1}{p}\Big)T(r,\,f)+S(r,\,f). \quad (3.20)$$

Proof. We consider the following two cases.

Case I: Let f be a bilinear transformation of g. Then one of the possibilities (ii)–(vii) of Lemma 2.6 will occur. If the possibility (v) occurs then 0 and $1-A$ are Picard exceptional values of f. So by the second fundamental theorem, Lemma 2.3 and the given condition we get

$$\begin{aligned}2T(r,\,f)&\leq N(r,\,\infty;f\mid\leq 1)+N(r,\,1;f\mid\leq 1)+S(r)\\&\leq \lambda T(r,\,f)+m(r,\,1;g)+N(r,\,1;g)+S(r)\\&=\lambda T(r,\,f)+T(r,\,g)+S(r)\\&=(\lambda+1)T(r,\,f)+S(r),\end{aligned}$$

which is a contradiction.

Similarly we can show that the possibilities (vi) and (vii) do not occur.

If $f+g\equiv 1$ then 0, 1 are Picard exceptional values of f and g. Hence there exists a nonconstant entire function γ such that $f=1/(1+e^{\gamma})$ and $g=1/(1+e^{-\gamma})$, which is the possibility (III) for $s=1$, $p=1$. Also $N(r,\,0;f\mid\leq 1)\equiv 0$, $N(r,\,\infty;f\mid\leq 1)=T(r,\,f)+O(1)$, $m(r,\,1;g)=T(r,\,f)+O(1)$ and so (3.20) is satisfied for $p=1$.

If $(f-1)(g-1)\equiv 1$ then 1, ∞ are Picard exceptional values of f and g. Hence there exists a nonconstant entire function γ such that $f=1+e^{\gamma}$ and $g=1+e^{-\gamma}$, which is the possibility (II) for $s=1$, $p=1$. Since $N(r,\,0;f\mid\leq 1)=T(r,\,f)+O(1)$, $N(r,\,\infty;f\mid\leq 1)\equiv 0$ and $m(r,\,1;g)=T(r,\,f)+O(1)$, we see that (3.20) is satisfied for $p=1$.

If $fg\equiv 1$ then 0, ∞ are Picard exceptional values of f and g. Hence there exists a nonconstant entire function γ such that $f=-e^{\gamma}$ and $g=-e^{-\gamma}$, which is the possibility (I) for $s=1$, $p=1$. Also $N(r,\,0;f\mid\leq 1)\equiv 0$, $N(r,\,\infty;f\mid\leq 1)\equiv 0$, $m(r,\,1;g)\equiv 0$ and so (3.20) is satisfied for $p=1$.

Case II: Let f be not a bilinear transformation of g. By Lemma 2.8(ii) and the given condition we get

$$\begin{aligned}&T(r,\,f)+T(r,\,g)\\&=N(r,\,0;f\mid\leq 1)+N(r,\,1;f\mid\leq 1)+N(r,\,\infty;f\mid\leq 1)+N_0(r)+S(r)\\&\leq \lambda T(r,\,f)+N(r,\,1;g)+m(r,\,1;g)+N_0(r)+S(r)\end{aligned}$$

i.e.,

$$(1-\lambda)T(r,\,f) \le N_0(r) + S(r) \le N(r) + S(r)$$

and so $N(r) \neq S(r)$. Hence by Theorem 3.1 we see that $N(r) \le (1/2)T(r,\,f)$ $+S(r)$. Therefore by Theorem 3.3 for a nonconstant entire function γ f and g assume one of the following forms:

$$f = \frac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}, \quad g = \frac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1}$$

which is the possibility (III);

$$f = \frac{e^{(p+1)\gamma}-1}{e^{(p+1-s)\gamma}-1}, \quad g = \frac{e^{-(p+1)\gamma}-1}{e^{-(p+1-s)\gamma}-1}$$

which reduces to the possibility (II) if we rename $p+1-s$ as s;

$$f = \frac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}, \quad g = \frac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1}$$

which is the possibility (I).

Now from above we see that f, g share $0, 1, \infty$ CM and so by Lemma 2.9 we get

$$N(r,\,1; f \mid\le 1) = N(r,\,1; g \mid\le 1) = N(r,\,1; g) + S(r).$$

Since $N(r) \le (1/2)T(r,\,f) + S(r)$, by Theorem 3.3 we get $N(r) = (1/p)T(r,\,f) + S(r)$. Since by Lemma 2.8(iii) $N(r) = N_0(r) + N_1(r) = N_0(r) + S(r)$, we get by Lemma 2.8(i)

$$\begin{aligned} &T(r,\,f) + T(r,\,g) \\ &= N(r,\,0; f \mid\le 1) + N(r,\,\infty; f \mid\le 1) + N(r,\,1; f \mid\le 1) + N_0(r) + S(r) \\ &= N(r,\,0; f \mid\le 1) + N(r,\,\infty; f \mid\le 1) + N(r,\,1; g) + \frac{1}{p}T(r,\,f) + S(r,\,f) \end{aligned}$$

and so

$$\begin{aligned} &N(r,\,0; f \mid\le 1) + N(r,\,\infty; f \mid\le 1) - m(r,\,1; g) \\ &\qquad\qquad = \Big(1 - \frac{1}{p}\Big)T(r,\,f) + S(r,\,f), \end{aligned}$$

which is (3.20). This proves the theorem. □

Following theorem improves Theorem J.

Theorem 3.6 *Let f and g be two distinct nonconstant meromorphic functions sharing $(0, 1)$, $(1, m)$ and (∞, k), where $(m-1)(mk-1) > (1+m)^2$. If*

$$N(r, 0; f \mid\leq 1) + N(r, \infty; f \mid\leq 1) \leq \lambda T(r, f) + S(r, f)$$

for some λ $(0 < \lambda < 1)$ then f and g assume the following form:

$$f = \frac{e^{s\gamma} - 1}{e^{-(p+1-s)\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{e^{(p+1-s)\gamma} - 1},$$

where s, p are positive integers with $1 \leq s \leq p$, and s, $p+1$ are relatively prime and γ is a nonconstant entire function. Also

$$N(r, \infty; f \mid\leq 1) + N(r, 0; f \mid\leq 1) = \Big(1 - \frac{1}{p}\Big) T(r, f) + S(r, f).$$

Proof. By Theorem 3.5 f and g satisfy one of the possibilities (I), (II) and (III).

If f and g satisfy the possibility (II) then $N(r, 0; f \mid\leq 1) = pT(r, e^{\gamma}) + S(r)$, $N(r, \infty; f \mid\leq 1) = (s-1)T(r, e^{\gamma}) + S(r)$ and $T(r, f) = pT(r, e^{\gamma}) + S(r)$, which contradicts the given condition.

Again if f and g satisfy the possibility (III) then $N(r, 0; f \mid\leq 1) = (s-1)T(r, e^{\gamma}) + S(r)$, $N(r, \infty; f \mid\leq 1) = pT(r, e^{\gamma}) + S(r)$ and $T(r, f) = pT(r, e^{\gamma}) + S(r)$, which also contradicts the given condition.

Therefore f and g satisfy the possibility (I) of Theorem 3.5. In this case we see that $N(r, 1; g) = pT(r, e^{\gamma}) + S(r) = T(r, g) + S(r)$ and so $m(r, 1; g) = S(r)$. Hence by (3.20) we get

$$N(r, 0; f \mid\leq 1) + N(r, \infty; f \mid\leq 1) = \Big(1 - \frac{1}{p}\Big) T(r, f) + S(r, f).$$

This proves the theorem. □

Remark 3.5 Considering $f = e^{2z} + e^{z} + 1$ and $g = e^{-2z} + e^{-z} + 1$ we see that the condition $\lambda < 1$ in Theorem 3.6 is essential.

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