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A short review of scattering for the Schrödinger-improved Boussinesq system

(Dedicated to Professor Rentaro Agemi on his seventieth birthday)

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Abstract. This article is a review of [6], which is concerned with the scattering theory for the Schrödinger-improved Boussinesq system in two space dimensions.

Key words: Schrödinger-improved boussinesq system, asymptotic behavior, scattering theory.

1. Introduction

We study the asymptotic behavior in time of solutions, (in particular, scattering theory), for the Schrödinger-improved Boussinesq system (hereafter referred to as the Schrödinger-IBq system) in two space dimensions:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = vu, \\ \partial_t^2 v - \Delta v - \Delta \partial_t^2 v = \Delta |u|^2, \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, $\partial_t = \partial/\partial t$, Δ is the Laplace operator for the space variable x, and u and v are complex-valued and real-valued unknown functions of (t, x), respectively. Following [6], we consider the existence of wave operators for the system (1.1).

There are several results on the local and global existence of solutions and the asymptotic behavior in time of solutions to the Schrödinger-IBq system (1.1). Ozawa and Tsutaya [4] proved the local well-posedness for the system (1.1) in the space $L^2 \times L^2 \times L^2 \ni (u, v, \partial_t v)$, when the space dimension $n \le 3$ by the Strichartz estimate for the Schrödinger equation. They also showed the global well-posedness in the energy class $H^1 \times L^2 \times (L^2 \cap \dot{H}^{-1})$ when $n \le 2$. Furthermore, in [4], when the space dimension $n = 4$, the local well-posedness for this system in the L^2-level for small initial

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data was shown. Cho and Ozawa [2] proved the existence of a unique global solution to the system (1.1) in the energy class for small initial data, when the space dimension $n = 3$ or 4. Furthermore, in [2], when $n = 4$, it was also shown that the small global solution has a free profile in $L^2 \times L^2 \times \dot{H}^{-1}$ by the Strichartz estimate for the Schrödinger equation on the time interval $[0, \infty)$. Recently, Akahori [1] proved the local well-posedness in $H^{s_1} \times H^{s_2} \times H^{s_2}$ for $-1/4 < s_1 < 0$ and $-1/2 < s_2 < 0$ when $n \leq 3$, and the global well-posedness in $L^2 \times L^2 \times L^2$ when $n \leq 2$. For investigating the large time behavior of solutions to the system (1.1) in relatively low dimensional case, (that is, $n \leq 3$), it seems that the method via the Strichartz estimate for the Schrödinger equation on the time interval $[0, \infty)$ in four-dimensional case in [2] does not work, because the Strichartz estimate for the solution u on the time interval $[0, \infty)$ does not derive the optimal time decay of the solution (u, v). In [6], the author proved that when the space dimension $n = 2$, for given asymptotic data, there exists an asymptotically free solution to the system (1.1) if the Schrödinger data is sufficiently small.

We compare the full dynamical system given by (1.1) and the free dynamics. We summarize properties of the free solutions. We introduce the following operators

$$
\begin{aligned}
&U(t) = e^{it\Delta/2}, \quad \mathcal{L} = i\partial_t + \frac{1}{2}\Delta, \\
&\Omega = (1-\Delta)^{1/2}, \quad A = (-\Delta)^{1/2}(1-\Delta)^{-1/2}, \\
&K(t) = A^{-1}\sin tA, \quad \dot{K}(t) = \cos tA.
\end{aligned}
$$

We note that the solution to the Cauchy problem of the free Schrödinger equation

$$
\begin{cases}
i\partial_t u + \dfrac{1}{2}\Delta u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0, x) = \phi(x), & x \in \mathbb{R}^n
\end{cases}
$$

is given by $u(t, \cdot\,) = U(t)\phi$, and that the solution to the Cauchy problem of the free IBq equation

$$
\begin{cases}
\partial_t^2 v - \Delta v - \Delta \partial_t^2 v = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
v(0, x) = \psi_0(x), \quad \partial_t v(0, x) = \psi_1(x), & x \in \mathbb{R}^n
\end{cases}
$$

is given by $v(t, \cdot\,) = \dot{K}(t)\psi_0 + K(t)\psi_1$. It is well-known that the solutions to the free Schrödinger equation satisfy

$$\|U(t)\phi\|_{L^\infty_x(\mathbb{R}^n)} \le C|t|^{-n/2}\|\phi\|_{L^1(\mathbb{R}^n)} \tag{1.2}$$

if $t \neq 0$. Cho and Ozawa [3] proved the estimate for the solution to the free IBq equation

$$\begin{aligned} &\|\dot{K}(t)\phi\|_{L^\infty_x(\mathbb{R}^n)} + \|K(t)\psi\|_{L^\infty_x(\mathbb{R}^n)} \\ &\le C|t|^{-n/2}(\|\Omega^{2n}A^{n/2-1}\phi\|_{\dot{B}^0_{1,1}} + \|\Omega^{2n}A^{n/2-2}\psi\|_{\dot{B}^0_{1,1}}) \end{aligned} \tag{1.3}$$

if $t \neq 0$, where $\dot{B}^0_{1,1}$ is the homogeneous Besov space defined later. Hence it seems that $\|v(t)u(t)\|_{L^2(\mathbb{R}^n)}$, where vu is the interaction of the Schrödinger component in the system (1.1), decays like $t^{-n/2}$ as $t \to \infty$, and that the scattering problem for the system (1.1) in relatively low dimension is rather difficult because $\|v(t)u(t)\|_{L^2(\mathbb{R}^n)}$ decays slowly in low dimension. Since the system (1.1) has quadratic nonlinearities, according to the general theory, the two-dimensional case is the borderline between the short range and the long range scattering. In this article, following [6], we consider the existence of wave operators for the system (1.1) in two space dimensions, when the Schrödinger asymptotic data is suitably small. The three-dimensional case is easier than the two-dimensional case, according to the decay rate of the free solutions (see the estimates (1.2) and (1.3)), since deriving integrability of the L^2-norms of the nonlinearities in the former case is easier than in the latter one.

We introduce several notations. For $\psi \in \mathcal{S}'$, we denote the Fourier transform of ψ by $\hat{\psi}$ or $\mathcal{F}\psi$. For m, $s \in \mathbb{R}$, we introduce the weighted Sobolev space:

$$H^{m,s} = \{\psi \in \mathcal{S}'; \|\psi\|_{H^{m,s}} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_{L^2} < \infty\}.$$

H^m denotes $H^{m,0}$. $\dot{H}^m$ is the homogeneous Sobolev space:

$$\dot{H}^m = \{\psi \in \mathcal{S}'; \|\psi\|_{\dot{H}^m} = \|(-\Delta)^{m/2}\psi\|_{L^2} < \infty\}.$$

For $m \in \mathbb{R}$ and $1 \le q$, $r \le \infty$, the homogeneous Besov space $\dot{B}^m_{q,r}$ is defined as follows:

$$\dot{B}^m_{q,r} = \{\psi \in \mathcal{S}'; \|\psi\|_{\dot{B}^m_{q,r}} = \|\{2^{jm}\|\varphi_j * \psi\|_{L^q}\}_j\|_{l^r_j(\mathbb{Z})} < \infty\},$$

where $\hat{\varphi}_j(\xi) = \hat{\phi}(2^{-j}\xi) - \hat{\phi}(2^{-(j+1)}\xi)$ and $\phi \in \mathcal{S}$ is a function such that $\hat{\phi} \in C^\infty_0$ with $0 \le \hat{\phi} \le 1$, $\hat{\phi}(\xi) = 1$ when $|\xi| \le 1$ and $\hat{\phi}(\xi) = 0$ when $|\xi| \ge 2$.

The following theorem is essentially obtained in [6].

Theorem 1.1 *Let the space dimension $n = 2$, and let (u_+, v_{+0}, v_{+1}) be asymptotic data such that u_+ is complex valued, v_{+0} and v_{+1} are real valued, $u_+ \in L^2$, $(1+|x|)u_+ \in L^1$, $v_{+0} \in H^{0,1} \cap \dot{H}^{-2} \cap \Omega^{-4}\dot{B}^0_{1,1}$, $xv_{+0} \in \dot{H}^{-1}$, $A^{-1}v_{+1} \in L^2 \cap \dot{H}^{-2} \cap \Omega^{-4}\dot{B}^0_{1,1}$ and $xv_{+1} \in \dot{H}^{-1} \cap \dot{H}^{-3}$. Assume that $\|u_+\|_{L^1_x}$ is sufficiently small. Then the system* (1.1) *has a unique solution (u, v) satisfying*

$$u \in C(\mathbb{R}; L^2_x), \quad v \in C^1(\mathbb{R}; L^2_x), \tag{1.4}$$

$$\sup_{t\geq 1} t^k \|u(t) - U(t)u_+\|_{L^2_x} < \infty, \tag{1.5}$$

$$\sup_{t\geq 1} t^k \|u - U(\,\cdot\,)u_+\|_{L^4((t,\infty);L^4_x)} < \infty, \tag{1.6}$$

$$\begin{aligned}&\sup_{t\geq 1} t^k (\|v(t) - (\dot{K}(t)v_{+0} + K(t)v_{+1})\|_{L^2_x} \\ &\qquad + \|\partial_t v(t) - \partial_t(\dot{K}(t)v_{+0} + K(t)v_{+1})\|_{L^2_x}) < \infty,\end{aligned} \tag{1.7}$$

where $1/2 < k \leq 3/4$. Furthermore the wave operator

$$W_+ : (u_+, v_{0+}, v_{1+}) \mapsto (u(0), v(0), (\partial_t v)(0))$$

is well-defined.

A similar result holds for the negative time.

Remark 1.1 In [6], the existence and uniqueness of an asymptotically free solution to the system (1.1) are proved by solving the local Cauchy problem at $t = +\infty$ on the interval $[T, \infty)$ for sufficiently large $T \geq 1$. According to the result on the global well-posedness for that system in $L^2 \times L^2 \times L^2$ by Akahori [1], we can extend the (local) asymptotically free solution obtained in [6] to the whole time interval $\mathbb{R}$ so that (1.4)–(1.7) hold.

2. Strategy of the proof of Theorem 1.1

We introduce the strategy of the proof of Theorem 1.1. For the detailed proof of it, see [6].

The proof consists of the following three steps.

(I) *Solving the Cauchy problem at infinite initial time.* We introduce the following function space:

$$\begin{aligned} X([T,\infty)) = \{(w,z) \in C([T,\infty); L_x^2) \oplus C^1([T,\infty); L_x^2);\\ \|(w,z)\|_{X([T,\infty))} < \infty\},\\ \|(w,z)\|_{X([T,\infty))} = \sup_{t\geq T}(t^k\|w(t)\|_{L_x^2} + t^k\|w\|_{L^4((t,\infty);L_x^4)}\\ + t^k(\|z(t)\|_{L_x^2} + \|\partial_t z(t)\|_{L_x^2})), \end{aligned}$$

where $k > 1/2$. By the contraction argument, for any functions (u_a, v_a) satisfying

$$\|u_a(t)\|_{L_x^\infty} \leq at^{-1}, \tag{2.1}$$

$$\|v_a(t)\|_{L_x^\infty} \leq bt^{-1}, \tag{2.2}$$

$$\|R_1(u_a, v_a)\|_{L^1((t,\infty)L_x^2)} \leq r_1 t^{-\alpha}, \tag{2.3}$$

$$\|A^{-1}R_2(u_a, v_a)\|_{L^1((t,\infty);H_x^{-2})} \leq r_2 t^{-\beta}, \tag{2.4}$$

where

$$\begin{aligned} R_1(u_a, v_a) &= \mathcal{L}u_a - v_a u_a, \\ R_2(u_a, v_a) &= (\partial_t^2 - \Delta - \Delta\partial_t^2)v_a - \Delta|u_a|^2, \end{aligned} \tag{2.5}$$

$\alpha > k$ and $\beta > k$, we prove the existence and uniqueness of a solution (u, v) to the system (1.1) such that $(u - u_a, v - v_a) \in X([T,\infty))$ for sufficiently large $T \geq 1$ and sufficiently small $a > 0$. (For the proof, see Section 3 in [6].) The condition $k > 1/2$ comes from the following reason. We put $(w, z) = (u - u_a, v - v_a)$ and intend to solve the integral equation

$$\begin{cases} w(t) = i\displaystyle\int_t^\infty U(t-s)\{z(s)w(s) + v_a(s)w(s) \\ \qquad\qquad + z(s)u_a(s) + R_1(u_a, v_a)(s)\}ds, \\ z(t) = \displaystyle\int_t^\infty K(t-s)\{A^2(|w(s)|^2 + 2\,\mathrm{Re}(w(s)\overline{u_a(s)})) \\ \qquad\qquad + (1-\Delta)^{-1}R_2(u_a, v_a)(s)\}ds \end{cases}$$

in $X([T,\infty))$. To show the existence and uniqueness of a solution $(w, z) \in X([T,\infty))$ to the above system, by the Strichartz estimate for the Schrödinger equation, we estimate $\|zw\|_{L^{4/3}((t,\infty);L^{4/3})}$ for $(w, z) \in X([T,\infty))$. For $t \geq T$,

$$\|zw\|_{L^{4/3}((t,\infty);L_x^{4/3})} \leq \|z\|_{L^2((t,\infty);L_x^2)}\|w\|_{L^4((t,\infty);L_x^4)},$$

so $z \in L^2((t,\infty); L_x^2)$ is required. To realize $z \in L^2((t,\infty); L_x^2)$, we assume $k > 1/2$. Furthermore, roughly speaking, we assume the smallness condition only on a and do not need the smallness of b, because the nonlinearities vu and $\Delta|u|^2$ of the system (1.1) involve the Schrödinger component u of the solution, but the nonlinearity $\Delta|u|^2$ in the IBq component does not contain the IBq component v.

(II) *Constructing an approximate solution for large time.* Let (u_+, v_{+0}, v_{+1}) be given asymptotic data. We construct an approximate solution (u_a, v_a) satisfying the estimates (2.1)–(2.4) explicitly. It is natural to define the principal terms u_1 and v_1 by

$$u_1(t, \cdot) = U(t)u_+ \tag{2.6}$$

and

$$v_1(t, \cdot) = \dot{K}(t)v_{+0} + K(t)v_{+1}, \tag{2.7}$$

which are free solutions to the Schrödinger and the IBq equations, respectively. But $\|v_1(t)u_1(t)\|_{L^2_x} \sim t^{-1}$, and hence $\|v_1(t)u_1(t)\|_{L^2_x}$ is not integrable over the interval $[1, \infty)$. Therefore we see that if we choose an asymptotics as $(u_a, v_a) = (u_1, v_1)$, then the estimate (2.3) is not satisfied. So we determine our asymptotics of the form $(u_a, v_a) = (u_1 + u_2, v_1)$ and choose a second correcting term u_2 such that (u_a, v_a) satisfies the estimates (2.1)–(2.4) with $a = \|u_+\|_{L^1}$ and $\alpha = \beta = 1$. For the explicit representation of the correcting term u_2 and how to find it, see Section 3. Combining with Part (I), we have the unique existence of a solution (u, v) to the system (1.1) such that $(u - u_a, v - v_a) = (u - u_1 - u_2, v - v_1) \in X([T, \infty))$ with $1/2 < k < 1$, when $\|u_+\|_{L^1}$ is sufficiently small and T is sufficiently large. The condition $k \leq 3/4$ in Theorem 1.1 comes from the following reason. Since $\|u_2\|_{L^4((t,\infty);L^4_x)}$ only decays as $t^{-3/4}$ (see the estimate (3.17) below), we see that $(u - u_1, v - v_1) \in X([T, \infty))$ under the condition $1/2 < k \leq 3/4$.

(III) *Extending the solution globally.* As in Remark 1.1, by Akahori's result [1] on the global well-posedness in $L^2 \times L^2 \times L^2$, we extend the local solution of the system (1.1) on $[T, \infty)$ obtained in the above two steps to whole time interval.

These three steps imply Theorem 1.1.

3. Construction of an approximate solution

For convenience for readers, we illustrate construction of an asymptotics $(u_a,\, v_a)$ mentioned in Part (II) in Section 2, though the construction of it in this section is the same as in [6].

Let $(u_+,\, v_{+0},\, v_{+1})$ be asymptotic data satisfying the assumptions of the theorem. We construct an approximate solution $(u_a,\, v_a)$ satisfying the estimates (2.1)–(2.4) explicitly. We determine the asymptotics $(u_a,\, v_a)$ of the form

$$(u_a,\, v_a) = (u_1 + u_2,\, v_1),$$

where u_1 and v_1 are the free solutions defined by (2.6) and (2.7), respectively. As we mentioned in Part (II) in Section 2, $\|v_1(t)u_1(t)\|_{L^2_x} \sim t^{-1}$, and hence $\|v_1(t)u_1(t)\|_{L^2_x}$ is not integrable over the interval $[1,\, \infty)$. So we find a second correcting term u_2 such that $(u_a,\, v_a)$ satisfies the estimates (2.1)–(2.4). We consider the error term $R_1(u_a,\, v_a)$ defined by (2.5). Since $\mathcal{L}u_1 = 0$, we obtain

$$R_1(u_a,\, v_a) = (\mathcal{L}u_2 - v_1u_1) - v_1u_2.$$

Therefore, roughly speaking, it is sufficient to find u_2 such that $\mathcal{L}u_2 - v_1u_1$ and u_2 decay faster than v_1u_1 and u_1, respectively. We remark that the nonlinearity $\Delta|u|^2$ in the IBq equation does not cause such difficulty, because derivatives of $|u_1(t)|^2$ decay faster than $|u_1(t)|^2$. (See Lemma 2.3 in [6].) Hence we do not need a correction term for the IBq component. Hereafter we concentrate on the Schrödinger component.

Construction of u_2 is as follows. We find u_2 of the form

$$u_2 = Vu_1, \tag{3.1}$$

where u_1 is the free solution for the Schrödinger equation defined by (2.6),

$$V(t,\, \cdot\,) = \dot{K}(t)\phi_0 + K(t)\phi_1 \tag{3.2}$$

is a solution to the free IBq equation, and ϕ_0 and ϕ_1 are complex-valued functions of $x \in \mathbb{R}^2$ which will be determined later. We determine V (namely ϕ_0 and ϕ_1) so that $\mathcal{L}u_2 - v_1u_1$ and u_2 decay faster than v_1u_1 and u_1, respectively. We compute $\mathcal{L}u_2$. Since $\mathcal{L}u_1 = 0$, we have

$$\begin{aligned}\mathcal{L}u_2 &= \frac{1}{2}u_1\Delta V + \frac{i}{t}u_1 PV - \frac{i}{t}(Ju_1)\cdot\nabla V + V\mathcal{L}u_1 \\ &= \frac{1}{2}u_1\Delta V + \frac{i}{t}u_1 PV - \frac{i}{t}(Ju_1)\cdot\nabla V,\end{aligned} \tag{3.3}$$

where J and P are the operators defined by

$$J = x + it\nabla, \quad P = t\partial_t + x\cdot\nabla.$$

Noting that u_1 and ∇V are solutions to the free Schrödinger and the free IBq equations, respectively, we see that the third term in the right hand side of (3.3) is a remainder term. We consider the first and the second terms in the right hand side of (3.3). We calculate PV in the second term as follows. Since $(\partial_t^2 - \Delta - \Delta\partial_t^2)P = (P+2)(\partial_t^2 - \Delta - \Delta\partial_t^2) - 2\Delta\partial_t^2$ and $(\partial_t^2 - \Delta - \Delta\partial_t^2)V = 0$, PV satisfies

$$(\partial_t^2 - \Delta - \Delta\partial_t^2)PV = -2\Delta\partial_t^2 V. \tag{3.4}$$

It is easy to check that

$$(PV)(0, x) = x\cdot\nabla\phi_0(x), \tag{3.5}$$

$$(\partial_t(PV))(0, x) = \phi_1(x) + x\cdot\nabla\phi_1(x). \tag{3.6}$$

By the equalities (3.4)–(3.6), we obtain

$$\begin{aligned}PV(t) =& \dot{K}(t)(x\cdot\nabla\phi_0) + K(t)(\phi_1 + x\cdot\nabla\phi_1) \\ &+ 2\int_0^t K(t-s)A^2\partial_s^2 V(s)ds \\ =& \dot{K}(t)(x\cdot\nabla\phi_0) + K(t)(\phi_1 + x\cdot\nabla\phi_1) \\ &- 2A^4\int_0^t K(t-s)V(s)ds.\end{aligned} \tag{3.7}$$

By a direct calculation, we have

$$\int_0^t K(t-s)(\dot{K}(s)\phi_0)ds = \frac{t}{2}K(t)\phi_0,$$

$$\int_0^t K(t-s)(K(s)\phi_1)ds = -\frac{t}{2}A^{-2}\dot{K}(t)\phi_1 + \frac{1}{2}A^{-2}K(t)\phi_1.$$

The above identities and the equality (3.7) imply

$$
\begin{aligned}
PV(t) =& t\dot{K}(t)(A^2\phi_1) - tK(t)(A^4\phi_0) \\
&+ \dot{K}(t)(x\cdot\nabla\phi_0) + K(t)(\phi_1 + x\cdot\nabla\phi_1 - A^2\phi_1) \\
=& t\dot{K}(t)(A^2\phi_1) - tK(t)(A^4\phi_0) \\
&+ \dot{K}(t)(x\cdot\nabla\phi_0) + K(t)(x\cdot\nabla\phi_1 + (1-\Delta)^{-1}\phi_1).
\end{aligned} \tag{3.8}
$$

By the equalities (3.3) and (3.8), and the representation (3.2) of V, we have

$$
\begin{aligned}
\mathcal{L}u_2 =& \frac{1}{2}u_1\{\dot{K}(t)(\Delta\phi_0 + 2iA^2\phi_1) + K(t)(-2iA^4\phi_0 + \Delta\phi_1)\} \\
&+ \frac{i}{t}u_1\{\dot{K}(t)(x\cdot\nabla\phi_0) + K(t)(x\cdot\nabla\phi_1 + (1-\Delta)^{-1}\phi_1)\} \\
&- \frac{i}{t}(Ju_1)\cdot\nabla V,
\end{aligned} \tag{3.9}
$$

Note that the most slowly decaying term in the right hand side of (3.9) is the first one. Now we define functions $\phi_0 = \mathcal{F}^{-1}\hat{\phi}_0$ and $\phi_1 = \mathcal{F}^{-1}\hat{\phi}_1$ by

$$
\begin{aligned}
\hat{\phi}_0(\xi) =& -\frac{2(1+|\xi|^2)^3}{|\xi|^6 + 3|\xi|^4 - |\xi|^2 + 1} \\
&\times \Big(\frac{1}{|\xi|^2}\hat{v}_{+0}(\xi) + \frac{2i}{|\xi|^2(1+|\xi|^2)}\hat{v}_{+1}(\xi)\Big),
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\hat{\phi}_1(\xi) =& \frac{2(1+|\xi|^2)^3}{|\xi|^6 + 3|\xi|^4 - |\xi|^2 + 1} \\
&\times \Big(\frac{2i}{(1+|\xi|^2)^2}\hat{v}_{+0}(\xi) - \frac{1}{|\xi|^2}\hat{v}_{+1}(\xi)\Big).
\end{aligned} \tag{3.11}
$$

Namely, we determine the second correcting term u_2 by (3.1), (3.2), (3.10) and (3.11). Then we see that

$$
\begin{cases}
-\frac{1}{2}|\xi|^2\hat{\phi}_0 + i\frac{|\xi|^2}{1+|\xi|^2}\hat{\phi}_1 = \hat{v}_{+0}, \\
-i\frac{|\xi|^4}{(1+|\xi|^2)^2}\hat{\phi}_0 - \frac{1}{2}|\xi|^2\hat{\phi}_1 = \hat{v}_{+1}.
\end{cases} \tag{3.12}
$$

By the definition (2.7) of v_1, we have

$$
\frac{1}{2}u_1\{\dot{K}(t)(\Delta\phi_0 - 2iA^2\phi_1) + K(t)(-2iA^4\phi_0 + \Delta\phi_1)\} = v_1u_1. \tag{3.13}
$$

In fact, we have constructed the functions $\hat{\phi}_0$ and $\hat{\phi}_1$ by solving the system (3.12). We note that the factor $2(1+|\xi|^2)^3/(|\xi|^6 + 3|\xi|^4 - |\xi|^2 + 1)$ in (3.10) and (3.11) is a smooth and bounded function on $\mathbb{R}^2$, since $|\xi|^6 + 3|\xi|^4 - |\xi|^2 + 1 = |\xi|^6 + 3(|\xi|^2 - 1/6)^2 + 11/12 \geq 11/12$ for any $\xi \in \mathbb{R}^2$. Therefore it follows from the equalities (3.9) and (3.13) that

$$\begin{aligned}\mathcal{L}u_2 - v_1u_1 =&\frac{i}{t}u_1\{\dot{K}(t)(x\cdot\nabla\phi_0)\\&+K(t)(x\cdot\nabla\phi_1+(1-\Delta)^{-1}\phi_1)\}-\frac{i}{t}(Ju_1)\cdot\nabla V.\end{aligned} \tag{3.14}$$

The equality (3.14) yields that $\mathcal{L}u_2 - v_1u_1$ decays faster than v_1u_1. In fact, we have the following estimates related to u_2. There exists a constant $C > 0$ such that for $t \geq 1$,

$$\|u_2(t)\|_{L^2_x} \leq Ct^{-1}\|u_+\|_{L^1_x}(\|v_{+0}\|_{\dot{H}^{-1}\cap\dot{H}^{-2}} + \|A^{-1}v_{+1}\|_{\dot{H}^{-2}}), \tag{3.15}$$

$$\begin{aligned}\|u_2(t)\|_{L^\infty_x} \leq& Ct^{-1}\|u_+\|_{L^1_x}\\&\times(\|v_{+0}\|_{L^2\cap\dot{H}^{-2}}+\|A^{-1}v_{+1}\|_{L^2\cap\dot{H}^{-2}}),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|u_2\|_{L^4((t,\infty);L^4_x)} \leq& Ct^{-3/4}\|u_+\|_{L^1_x}\\&\times(\|v_{+0}\|_{L^2\cap\dot{H}^{-2}}+\|A^{-1}v_{+1}\|_{L^2\cap\dot{H}^{-2}}),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\|v_1(t)u_2(t)\|_{L^2_x} \leq& Ct^{-2}\|u_+\|_{L^1_x}\\&\times(\|v_{+0}\|_{L^2\cap\dot{H}^{-2}}+\|A^{-1}v_{+1}\|_{L^2\cap\dot{H}^{-2}})\\&\times(\|\Omega^4v_{+0}\|_{\dot{B}^0_{1,1}}+\|\Omega^4A^{-1}v_{+1}\|_{\dot{B}^0_{1,1}}),\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\|\mathcal{L}u_2(t) - v_1(t)u_1(t)\|_{L^2_x}\\&\leq Ct^{-2}(\|u_+\|_{L^1_x}+\|xu_+\|_{L^1_x})(\|v_{+0}\|_{H^{0,1}\cap\dot{H}^{-2}}+\|xv_{+0}\|_{\dot{H}^{-1}}\\&\quad+\|A^{-1}v_{+1}\|_{\dot{H}^{-1}\cap\dot{H}^{-2}}+\|xv_{+1}\|_{\dot{H}^{-1}\cap\dot{H}^{-3}}).\end{aligned} \tag{3.19}$$

Remark 3.1 The estimates (3.15)–(3.19) were obtained in Lemma 4.2 in [6]. The author found a misprint in the statement of that lemma in [6], and he would like to revise it. In the estimate (4.18) for $\|u_2\|_{L^4((t,\infty);L^4_x)}$ in Lemma 4.2 of [6], there is the factor t^{-1}. But the factor t^{-1} is a misprint, and it should be substituted by $t^{-3/4}$ as the above estimate (3.17). (The above estimate (3.17) is correct.) The proof of (4.18) in [6] is described correctly, and this misprint does not bring any trouble for the rest of [6].

As we mentioned above, we put $(u_a, v_a) = (u_1 + u_2, v_1)$. Then by the estimates (1.2), (1.3) and (3.15)–(3.19), we see that the estimates (2.1)–(2.4) are satisfied with $a = \|u_+\|_{L^1}$ and $\alpha = \beta = 1$. (For the proof, see Proposition 4.1 in [6].) We complete the construction of (u_a, v_a).

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