

Smoothing and dispersive properties of evolution equations with potential perturbations

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Abstract. We survey a group of joint results with different authors concerning the decay properties of evolution equations with variable coefficients. The problems studied include the wave, Schrödinger and Dirac equation, perturbed with electromagnetic potentials, and the main focus of the paper is on global dispersive and Strichartz estimates when the coefficients are of low regularity and of critical decay.

Key words: hyperbolic equations, dispersive estimates, Strichartz estimates, dispersive equations, Schrödinger equation, Dirac equation, magnetic potential.

In recent years an intense activity has been devoted to the study of dispersive properties of evolution equations. After pioneering works of Segal, von Wahl, Pecher, Kato, Yajima, Strichartz, Ginibre and Velo, it became rapidly clear that decay properties of classical linear equations such as the Schrödinger, wave, Klein-Gordon, Dirac, Korteweg – de Vries equations were of central importance in the theory of nonlinear PDEs. This justifies the large number of papers that have appeared on the subject.

In this paper we would like to review a group of recent joint works with different collaborators ([8], [9], [10], [11], [12]) concerning dispersive, smoothing and Strichartz estimates for the main evolution equations of mathematical physics, in the case when potential (electromagnetic) perturbations are present. The focus of these results is on global estimates in time, on one hand, and on the lowest possible assumptions of decay at infinity and smoothness on the coefficients, on the other side. As we mentioned above, such estimates have become a standard tool in the study of linear and nonlinear evolution equations. In particular Strichartz estimates enjoy a wide popularity, thanks to their usefulness in many different situations. They can be proved for a large class of constant coefficients equations, using the methods of [16] and [23]. In a sense, they represent the modern energy estimates, and are especially effective in questions of low regularity solutions and global existence for nonlinear equations.

The term “dispersive estimate” is now in common use to denote a decay estimate in time of a suitable spatial norm of the solution to an evolution equation, typically the L^∞ norm. For the Schrödinger equation the dispersive estimate is

$$\|e^{it\Delta}f\|_{L^\infty} \lesssim t^{-n/2}\|f\|_{L^1},$$

while for the wave equation it can be written in the form

$$\|e^{it|D|}f\|_{L^\infty} \lesssim t^{-(n-1)/2}\|f\|_{\dot{B}_{1,1}^{(n+1)/2}}$$

and for the Klein-Gordon equation

$$\|e^{it\langle D \rangle}f\|_{L^\infty} \lesssim t^{-n/2}\|f\|_{B_{1,1}^{(n+1)/2}}$$

Here $B_{p,q}^s$ denotes a Besov space (a dot meaning the homogeneous version of the space), $A \lesssim B$ means $A \leq CB$ for a suitable constant C , and we use the operators $|D| = (-\Delta)^{1/2}$ and $\langle D \rangle = (1 - \Delta)^{1/2}$. We shall also be interested in the decay properties of the Dirac equation, which is a 4×4 constant coefficient system of the form

$$iu_t + \mathcal{D}u = 0$$

in the *massless* case, and

$$iu_t + \mathcal{D}u + \beta u = 0$$

in the *massive* case. Here $u(t, x)$ is a so-called *spinor*, i.e., $u: \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}^4$, the operator \mathcal{D} is defined as

$$\mathcal{D} = \frac{1}{i} \sum_{k=1}^3 \alpha_k \partial_k$$

and the 4×4 *Dirac matrices* can be written

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad k = 1, 2, 3$$

in terms of the *Pauli matrices*

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The dispersive estimate for the Dirac equation can be written, in the mass-

less case $u = e^{it\mathcal{D}}f$, as follows:

$$\|e^{it\mathcal{D}}f\|_{L^\infty} \lesssim t^{-1}\|f\|_{\dot{B}_{1,1}^2}.$$

We pass now to describe the Strichartz estimates, which can be deduced from the dispersive estimates and hence represent a weaker form of decay. Using the notations $L^p L^q = L^p(\mathbb{R}_t; L^q(\mathbb{R}_x^n))$, $\|f\| \lesssim \|g\|$ to mean $\|f\| \leq C\|g\|$, and H_q^s and \dot{H}_q^s to denote the spaces with norms

$$\|f\|_{H_q^s} = \|\langle D \rangle^s f\|_{L^q}, \quad \|f\|_{\dot{H}_q^s} = \| |D|^s f \|_{L^q},$$

the Strichartz estimates for the Schrödinger equation take the following form: for $n \geq 2$,

$$\|e^{it\Delta}f\|_{L^p L^q} \lesssim \|f\|_{L^2},$$

provided the couple (p, q) is *Schrödinger admissible*:

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2n}{n-2} \geq q \geq 2, \quad q \neq \infty. \quad (0.1)$$

The couple $(p, q) = (2, 2n/(n-2))$ is called the *endpoint* and is allowed when $n > 2$.

For the wave equation the estimates can be written as follows: for $n \geq 3$,

$$\|e^{it|D|}f\|_{L^p \dot{H}_q^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2},$$

provided the couple (p, q) is *wave admissible*:

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2(n-1)}{n-3} \geq q \geq 2, \quad q \neq \infty. \quad (0.2)$$

The wave equation endpoint is $(p, q) = (2, 2(n-1)/(n-3))$ and is allowed in dimension $n > 3$. Note that the wave endpoint in dimension n coincides with the Schrödinger endpoint in dimension $n-1$.

Finally for the Klein-Gordon equation we have: for $n \geq 2$,

$$\|e^{it\langle D \rangle}f\|_{L^p H_q^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2},$$

provided (p, q) is Schrödinger admissible.

As to the solution $u(t, x) = e^{it\mathcal{D}}f$ of the massless Dirac system with

initial value $u(0, x) = f(x)$, it satisfies the Strichartz estimate:

$$\|e^{it\mathcal{D}}f\|_{L^p\dot{H}_q^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2}, \quad n = 3,$$

for all wave admissible (p, q) , while in the massive case we have

$$\|e^{it(\mathcal{D}+\beta)}f\|_{L^pH_q^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2}, \quad n = 3,$$

for all Schrödinger admissible (p, q) .

In view of the applications, it is an important problem to extend Strichartz estimates to more general equations with variable coefficients, possibly of low regularity in order to retain the advantages over classical energy methods. Indeed, in recent years a large number of works have investigated this kind of problem. In the case of potential perturbations like

$$iu_t - \Delta u + V(x)u = 0, \quad \square u + V(x)u = 0,$$

Strichartz estimates are now fairly well understood. We mention among the many works [4], [18], [17], [25], [27] and the survey [26] for the Schrödinger equation, and [6], [14], [11] for the wave equation. We also mention the wave operator approach of Yajima ([32], [33], [34], [2]), which was recently optimized in dimension 1 in [8]. The question of the minimal assumptions on the potential for dispersion to hold is still largely open, although some major advances in this direction have been made recently in [19] and [20].

Results are much less complete in the case of first order perturbations i.e. *magnetic potentials*

$$iu_t + \Delta u + a \cdot \nabla u + bu = 0, \quad \square u + a \cdot \nabla u + bu = 0.$$

Concerning Strichartz estimates for the Schrödinger equation with small potentials a, b we recall at least the papers [29], [15]; in 3D the recent work [13] handles for the first time the case of large magnetic potentials. For the wave equation with small magnetic potentials, partial Strichartz estimates were obtained in 3D in [7] in the case of smooth, rapidly decaying coefficients. The dispersive estimate in 3D was proved in [9] for the magnetic wave equation with small singular potentials and for the massless Dirac system with a small singular matrix potential. We must also mention the papers [28], [24], [30] containig some local estimates in the fully variable coefficient case. Only in the one dimensional case the optimal dispersive

estimates for the case of fully variable singular coefficients have been proved in [8].

The plan of the paper is as follows. In Section 1 we recall our results concerning smoothing and Strichartz estimates for evolution equations with electromagnetic perturbations. In Section 2 we review the dispersive estimate for the 3D wave equation, in which case results are close to optimal. Section 3 is devoted to the dispersive estimates for the 3D wave and Dirac equations, in presence of magnetic potentials. In Section 4 we describe our results in the 1D case, which give a complete picture of the case of coefficients independent of time. Finally the last Section 5 covers the Schrödinger equation with point interactions in 3D, i.e. with a very singular potential given by a sum of delta functions; in this case we get an optimal weighted dispersive estimate comparable with the classical one.

1. Strichartz estimates

A method of proof which is very efficient in the case of electric potentials was introduced in [25] and further developed in [4]. The main idea is to combine Strichartz estimates for the free equation with Kato smoothing estimates for the perturbed equation. The same method is used in [13] for the 3D Schrödinger equation with a large magnetic potential.

Our first group of results ([10]) is based on a suitable modification of this method, applied in a systematic way to all of the above equations perturbed with magnetic potentials.

Thus consider a magnetic Schrödinger operator

$$H = -(\nabla + iA(x))^2 + B(x), \quad (1.1)$$

which is selfadjoint under the following assumptions: A_j and B are real valued, and

$$\|B\|_{L^{n/2,\infty}} < \infty, \quad \|B_-\|_{L^{n/2,\infty}} < \delta, \quad \|A\|_{L^{n,\infty}} < \delta \quad (1.2)$$

for some δ sufficiently small. Here and in the following, $B_-(x) = \max\{-B(x), 0\}$ denotes the negative part of the function B , and $L^{p,\infty} = L_w^p$ denotes the Lorentz or weak Lebesgue space. However, in order to state our results, it is more convenient to represent the operator in the form

$$H \equiv -\Delta + W(x, D) \equiv -\Delta + a(x) \cdot \nabla + b(x) \quad (1.3)$$

and to make the abstract assumption that H is selfadjoint. In view of (1.2), the following explicit conditions on a, b are sufficient (but not necessary) for the selfadjointness of H :

$$a(x) \text{ is pure imaginary, } \operatorname{Im} b = -i\nabla \cdot a \quad (1.4)$$

and

$$\begin{aligned} \|\nabla a\|_{L^{n/2,\infty}} + \|b\|_{L^{n/2,\infty}} &< \infty, \\ \|(\operatorname{Re} b)_-\|_{L^{n/2,\infty}} &< \delta, \quad \|a\|_{L^{n,\infty}} < \delta \end{aligned} \quad (1.5)$$

for a small enough δ .

Our first result concerns smoothing estimates of Kato-Yajima type for the scalar Schrödinger, wave and Klein-Gordon equations. Besides being a necessary tool to prove the Strichartz estimates, they have also an independent interest (see e.g. [3], [21], [22]). Notice in particular that we allow a singularity at 0 in the coefficient, and that the electric potential can be large, while the magnetic term must satisfy a smallness condition. We shall use the following weight functions:

$$\tau_\epsilon(x) = \begin{cases} |x|^{1/2-\epsilon} + |x| & \text{if } n \geq 3, \\ |x|^{1/2-\epsilon} + |x|^{1+\epsilon} & \text{if } n = 2 \end{cases}$$

and

$$w_\sigma(x) = |x|(1 + |\log |x||)^\sigma, \quad \sigma > 1.$$

Then we have:

Proposition 1.1 Smoothing estimates for scalar equations *Let $n \geq 2$. Assume the operator*

$$-\Delta + W(x, D) = -\Delta + a(x) \cdot \nabla + b_1(x) + b_2(x)$$

is selfadjoint with

$$|a(x)| \leq \frac{\delta}{\tau_\epsilon w_\sigma^{1/2}}, \quad |b_1(x)| \leq \frac{\delta}{\tau_\epsilon^2}, \quad 0 \leq b_2(x) \leq \frac{C}{\tau_\epsilon^2} \quad (1.6)$$

for some $\delta, \epsilon > 0$ sufficiently small and some $\sigma > 1$, $C > 0$. Moreover assume that 0 is not a resonance for $-\Delta + b_2$.

Then the following smoothing estimates hold: for the Schrödinger equa-

tion

$$\|\tau_\epsilon^{-1} e^{it(-\Delta+W)} f\|_{L^2 L^2} + \|\tau_\epsilon^{-1} |D|^{1/2} e^{it(-\Delta+W)} f\|_{L^2 L^2} \lesssim \|f\|_{L^2}$$

while for the wave and Klein-Gordon equations

$$\|\tau_\epsilon^{-1} e^{it\sqrt{-\Delta+W}} f\|_{L^2 L^2} + \|\tau_\epsilon^{-1} e^{it\sqrt{1-\Delta+W}} f\|_{L^2 L^2} \lesssim \|f\|_{L^2}.$$

The assumption that 0 is not a resonance for $-\Delta + b_2(x)$ here means: if $(-\Delta + b_2)f = 0$ and $\langle x \rangle^{-1} f \in L^2$ then $f \equiv 0$.

We can then prove Strichartz estimates for the perturbed scalar equations as a consequence of the above smoothing properties. The idea is essentially to rewrite the equation in the form

$$iu_t - \Delta u = -W(x, D)u, \quad u(0, x) = f(x)$$

and hence write the solution as

$$u = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta} W(x, D)u(s) ds.$$

To the first term clearly we can apply the classical Strichartz estimates. In order to handle the second one, we can use the Christ-Kiselev lemma [5] which states that: given two Banach spaces X, Y and a bounded integral operator

$$Tf = \int_{\mathbb{R}} K(t, s) f(s) ds$$

from $L^p(\mathbb{R}, X)$ to $L^{\tilde{p}}(\mathbb{R}, Y)$, then its truncated version

$$Sf = \int_0^t K(t, s) f(s) ds$$

is also bounded on the same spaces, provided $p < \tilde{p}$. Thus we see that in order to estimate u in an $L^p L^q$ space it is sufficient to estimate the untruncated integral as follows:

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} W(x, D)u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2}.$$

Notice however that the use of the Christ-Kiselev lemma prevents us from reaching the endpoint. The advantage of this method is that the untrun-

cated integral can be split as

$$e^{it\Delta} \int_{\mathbb{R}} e^{-is\Delta} W(x, D) u(s) ds;$$

then we can combine the classical Strichartz estimates for $e^{it\Delta}$ with (the dual of) the classical smoothing estimate for $e^{it\Delta}$

$$\left\| \int e^{-is\Delta} F(s) ds \right\|_{L^2} \lesssim \|\langle x \rangle^{1/2+\epsilon} |D|^{-1/2} F\|_{L^2 L^2}$$

(or more refined versions if necessary). In this way we obtain an inequality of the form

$$\|u\|_{L^p L^q} \lesssim \|f\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} |D|^{-1/2} W(x, D) u\|_{L^2 L^2}.$$

Since the operator acting on u at the right hand side has order at most $1/2$, after some delicate commutator estimates, which require additional regularity on the coefficients, we are reduced to the smoothing estimate of Proposition 1.1. By suitable refinements of these methods we can prove:

Theorem 1.2 Strichartz for Schrödinger *Let $n \geq 2$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that*

$$\langle x \rangle^{1+3\epsilon} \chi(x) a_j(x) \in C^{1/2+2\epsilon} \quad \text{for some function } \chi \gtrsim w_\sigma^{1/2}. \quad (1.7)$$

Then, for any non-endpoint Schrödinger admissible couple (p, q) , the following Strichartz estimate holds:

$$\|e^{it(-\Delta+W)} f\|_{L^p L^q} \lesssim \|f\|_{L^2}. \quad (1.8)$$

Theorem 1.3 Strichartz for wave *Let $n \geq 3$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that*

$$|a(x)| \leq \frac{C}{\tau_\epsilon^2}, \quad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{|x| \tau_\epsilon}. \quad (1.9)$$

Then, for any non-endpoint wave admissible couple (p, q) the following Strichartz estimate holds:

$$\|e^{it\sqrt{-\Delta+W}} f\|_{L^p \dot{H}_q^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2}. \quad (1.10)$$

Theorem 1.4 Strichartz for Klein-Gordon *Let $n \geq 2$, $-\Delta + W$ be as in Proposition 1.1 and assume in addition that*

$$|a(x)| \leq \frac{C}{\tau_\epsilon^2}, \quad |b_1 + b_2 - \nabla \cdot a| \leq \frac{C}{\langle x \rangle \tau_\epsilon}. \quad (1.11)$$

Then, for any non-endpoint Schrödinger admissible couple (p, q) , the following Strichartz estimate holds:

$$\|e^{it\sqrt{-\Delta+1+W}}\|_{L^p H_q^{1/q-1/p-1/2}} \leq C\|f\|_{L^2}. \quad (1.12)$$

Our final results concern the Dirac system:

Theorem 1.5 Massless Dirac *Let $n = 3$, and let $V(x) = V(x)^*$ be a 4×4 complex valued matrix such that*

$$|V(x)| \leq \frac{\delta}{w_\sigma(x)} \quad (1.13)$$

for some δ sufficiently small and some $\sigma > 1$. Then the following smoothing estimate holds:

$$\|w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f\|_{L^2 L^2} \lesssim \|f\|_{L^2} \quad (1.14)$$

and, for any non-endpoint wave admissible couple (p, q) , the following Strichartz estimate holds:

$$\|e^{it(\mathcal{D}+V)} f\|_{L^p \dot{H}^{1/q-1/p-1/2}} \lesssim \|f\|_{L^2}. \quad (1.15)$$

2. Dispersive estimates for the wave equation

Dispersive estimates are a much stronger result compared with Strichartz estimates, and indeed it is possible to deduce the latter from the former but not viceversa. Thus one expects in general stronger assumptions on the coefficients. However, for the 3D wave equation in [11] we proved an optimal dispersive estimate for a potential perturbation with a minimal smoothness assumption on the potential. We recall the relevant definitions.

A measurable function $V(x)$ on \mathbb{R}^n , $n \geq 3$, is said to belong to the *Kato class* if

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0. \quad (2.1)$$

Moreover, the *Kato norm* of $V(x)$ is defined as

$$\|V\|_K = \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy. \quad (2.2)$$

For $n = 2$ the kernel $|x-y|^{2-n}$ is replaced by $\log(|x-y|^{-1})$.

As it is well known, the presence of eigenvalues or resonances can influence the decay properties of the solutions. The standard way out of this difficulty is to assume that no resonances are present on the positive real axis, and in many cases this reduces to assuming that 0 is not a resonance. In our first result this assumption takes the following form. We denote as usual by $R_0(z) = (-z - \Delta)^{-1}$ the resolvent operator of $-\Delta$, and by $R_0(\lambda \pm i0)$ the limits $\lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon)$ at a point $\lambda \geq 0$. Then we assume that

The integral equation $f + R_0(\lambda + i0)Vf = 0$ has no nontrivial bounded solution for any $\lambda \geq 0$,

or, equivalently,

$$f + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} V(y)f(y)dy = 0, \quad f \in L^\infty, \quad \lambda \geq 0$$

$$\implies f \equiv 0. \quad (2.3)$$

In several cases this assumption can be drastically weakened (see below). Then we can state our result:

Theorem 2.1 *Let $V = V_1 + V_2$ be a real valued potential of Kato class on \mathbb{R}^3 . Assume that:*

- (i) V_1 is compactly supported and has a bounded Kato norm;
- (ii) V_2 has a small Kato norm and precisely

$$\|V_2\|_K \cdot \left(1 + \frac{1}{4\pi} \|V_1\|_K\right) < 4\pi; \quad (2.4)$$

- (iii) the negative part $V_- = \max\{-V, 0\}$ satisfies

$$\|V_-\|_K < 2\pi; \quad (2.5)$$

- (iv) the non resonant condition (2.3) holds for all $\lambda \geq 0$.

Then any solution $u(t, x)$ to the Cauchy problem

$$\square u + V(x)u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = f(x) \quad (2.6)$$

satisfies the dispersive estimate

$$\|u(t, \cdot)\|_{L^\infty} \leq Ct^{-1} \|f\|_{\dot{B}_{1,1}^1(\mathbb{R}^3)}. \quad (2.7)$$

In order to see a real word application of this result, we recall the following corollary:

Corollary 2.2 *Assume the real valued potential V belongs to $L^{3/2,1}$ with $\|V_-\|_K < 2\pi$ and satisfies the non resonant condition (2.3). Then the same conclusion of Theorem 2.1 holds.*

Here $L^{3/2,1}$ denotes a Lorentz space. In particular, this applies to potentials belonging to $L^{3/2-\delta}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3)$ for some $\delta > 0$, in view of the embedding

$$L^{3/2-\delta}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3) \subseteq L^{3/2,1}(\mathbb{R}^3).$$

Concerning the spectral assumption on the nonexistence of resonances or embedded eigenvalues, this can be replaced by a stronger decay assumption on the potential as follows:

Theorem 2.3 *Let V_1 be a nonnegative L^2 function such that $V_1(x) \leq C|x|^{-3-\delta}$ ($\delta > 0$) for large x . Then there exists a constant $\epsilon(V_1) > 0$ such that: for all real valued functions V_2 of Kato class with*

$$\|V_2\|_K < \epsilon(V_1) \quad (2.8)$$

and for $V = V_1 + V_2$, the solution $u(t, x)$ of problem (2.6) satisfies the dispersive estimate (2.7).

3. Dispersive estimates for magnetic potentials

For equations perturbed with magnetic potentials, to our knowledge our results in [9] are the only available. These include the case of the 3D wave equation, and the closely connected problem of the 3D Dirac equation.

Our first result in this group concerns the Cauchy problem for the wave equation perturbed with a small rough electromagnetic potential

$$u_{tt}(t, x) - (\nabla + iA(x))^2 u + B(x)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \quad (3.1)$$

$$u(0, x) = 0, \quad u_t(0, x) = g(x). \quad (3.2)$$

We shall assume that the operator $-(\nabla + iA)^2 + B$ is selfadjoint, and we

shall denote by ϕ_j a standard (nonhomogeneous) Paley-Littlewood partition of unity, $j \geq 0$. Then we can prove:

Theorem 3.1 *Assume the potentials $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $B: \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfy*

$$|A_j| \leq \frac{C_0}{|x|\langle x \rangle (|\log |x|| + 1)^\beta},$$

$$\sum_{j=1}^3 |\partial_j A_j| + |B| \leq \frac{C_0}{|x|^2 (|\log |x|| + 1)^\beta}, \quad (3.3)$$

for some constant $C_0 > 0$ sufficiently small and some $\beta > 1$. Then any solution of the Cauchy problem (3.1), (3.2) satisfies the decay estimate

$$|u(t, x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{2j} \|\langle x \rangle w_\beta^{1/2} \varphi_j(\sqrt{H}) g\|_{L^2}, \quad (3.4)$$

where $w_\beta(x) := |x|(|\log |x|| + 1)^\beta$. If in addition we assume that, for some $\epsilon > 0$,

$$\langle D \rangle^{1+\epsilon} A_j \in L^\infty, \quad \langle D \rangle^\epsilon B \in L^\infty \quad (3.5)$$

then u satisfies for any $\delta > 0$ the estimate

$$|u(t, x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} g\|_{H^{2+\epsilon}}. \quad (3.6)$$

Our second result concerns the perturbed Dirac system

$$iu_t - \mathcal{D}u + V(x)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (3.7)$$

$$u(0, x) = f(x). \quad (3.8)$$

By exploiting the above mentioned relation between the magnetic wave equation and the Dirac system, we can prove the following Theorem as a direct consequence of Theorem 3.1:

Theorem 3.2 *Assume the 4×4 complex valued matrix $V(x) = V^*(x)$ satisfies*

$$|V(x)| \leq \frac{C_0}{|x|\langle x \rangle (|\log |x|| + 1)^\beta},$$

$$|DV(x)| \leq \frac{C_0}{|x|^2 (|\log |x|| + 1)^\beta}, \quad (3.9)$$

for some $C_0 > 0$ small enough and some $\beta > 1$. Then the solution of the Cauchy problem (3.7), (3.8) satisfies the decay estimate

$$|u(t, x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{3j} \|\langle x \rangle w_\beta^{1/2} \varphi_j(\mathcal{D} + V)f\|_{L^2}, \quad (3.10)$$

where $w_\beta(x) = |x|(|\log|x|| + 1)^\beta$. If in addition we assume that, for some $\epsilon > 0$,

$$\langle D \rangle^{2+\epsilon} V \in L^\infty, \quad (3.11)$$

then u satisfies for any $\delta > 0$ the estimate

$$|u(t, x)| \leq \frac{C}{t} \|\langle x \rangle^{3/2+\delta} f\|_{H^{3+\epsilon}}. \quad (3.12)$$

We remark that for the unperturbed Dirac system, with vanishing mass, the loss of derivatives is exactly 2.

Since Theorem 3.2 is proved essentially by “squaring” the perturbed Dirac operator, a condition on the derivative DV is essential in order to apply Theorem 3.1 to the resulting wave equation. On the other hand, we can study the Cauchy problem (3.7), (3.8) by a direct application of the spectral calculus for the selfadjoint operator $\mathcal{D} + V(x)$; this alternative approach allows us to consider much rougher potentials $V(x)$. The price to pay is an additional loss of one derivative, so that the total loss is 4 derivatives in our last result:

Theorem 3.3 *Assume the 4×4 complex valued matrix $V(x) = V^*(x)$ satisfies*

$$|V(x)| \leq \frac{C_0}{|x|^{1/2} \langle x \rangle^{3/2} (|\log|x|| + 1)^{\beta/2}}, \quad (3.13)$$

for some $C_0 > 0$ small enough and some $\beta > 1$. Then the solution of the Cauchy problem (3.7), (3.8) satisfies for any $\epsilon > 0$ the decay estimate

$$|u(t, x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{4j} \|\langle x \rangle^{3/2+\epsilon} \varphi_j(\mathcal{D} + V)f\|_{L^2}. \quad (3.14)$$

4. The one dimensional case

A general and powerful method to prove dispersive estimates (and much more) was introduced by K. Yajima. To illustrate it we recall a few basic

notions of scattering theory; we specialize to the case of dimension 1 for the sake of simplicity, but analogous arguments hold also in higher dimensions. More details can be found in our paper [8].

Let $H_0 = -d^2/dx^2$ be the one-dimensional Laplace operator on the line, and consider the perturbed operator $H = H_0 + V(x)$. For a potential $V(x) \in L^1(\mathbb{R})$, the operator H can be realized uniquely as a selfadjoint operator on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$. The absolutely continuous spectrum of H is $[0, +\infty[$, the singular continuous spectrum is absent, and the possible eigenvalues are all strictly negative. Moreover, the *wave operators*

$$W_{\pm}f = L^2 - \lim_{s \rightarrow \pm\infty} e^{isH} e^{-isH_0} f \quad (4.1)$$

exist and are unitary from $L^2(\mathbb{R})$ to the absolutely continuous space $L^2_{ac}(\mathbb{R})$ of H . A very useful feature of W_{\pm} is the *intertwining property*. If we denote by P_{ac} the projection of L^2 onto $L^2_{ac}(\mathbb{R})$, the property can be stated as follows: for any Borel function f ,

$$W_{\pm}f(H_0)W_{\pm}^* = f(H)P_{ac}. \quad (4.2)$$

Thanks to (4.2), one can reduce the study of an operator $f(H)$, or more generally $f(t, H)$, to the study of $f(t, H_0)$ which has a much simpler structure. When applied to the operators

$$e^{itH}, \quad \sin(t\sqrt{H})/\sqrt{H}, \quad \sin(t\sqrt{H+1})/\sqrt{H+1},$$

this method can be used to prove decay estimates for the Schrödinger, wave and Klein-Gordon equations

$$\begin{aligned} iu_t - \Delta u + Vu &= 0, & u_{tt} - \Delta u + Vu &= 0, \\ u_{tt} - u_{xx} - \Delta u + u + Vu &= 0, \end{aligned}$$

provided one has some control on the L^p behaviour of W_{\pm} , W_{\pm}^* . Indeed, if the wave operators are bounded on L^p , the $L^q - L^{q'}$ estimates valid for the free operators extend immediately to the perturbed ones via the elementary argument

$$\begin{aligned} \|e^{itH} P_{ac} f\|_{L^q} &\equiv \|W_+ e^{itH_0} W_+^* f\|_{L^q} \\ &\leq C \|e^{itH_0} W_+^* f\|_{L^q} \leq C t^{-\alpha} \|W_+^* f\|_{L^{q'}} \leq C t^{-\alpha} \|f\|_{L^{q'}} \end{aligned}$$

Such a program was developed systematically by K. Yajima in a series of papers [33], [32], [34] where he obtained the L^p boundedness for all p

of W_{\pm} , under suitable assumptions on the potential V , for space dimension $n \geq 2$. The analysis was completed in the one dimensional case in Artbazar-Yajma [2] and by Ricardo Weder [31]. We remark that in high dimension $n \geq 4$ the decay estimates obtained by this method are the best available from the point of view of the assumptions on the potential; only in low dimension $n \leq 3$ more precise results have been proved, as mentioned in the preceding sections.

In order to explain our results in more detail we recall a few notions. The relevant potential classes are the spaces

$$L^1_{\gamma}(\mathbb{R}) \equiv \{f: (1 + |x|)^{\gamma} f \in L^1(\mathbb{R})\}. \quad (4.3)$$

Moreover, given a potential $V(x)$, the *Jost functions* are the solutions $f_{\pm}(\lambda, x)$ of the equation $-f'' + Vf = \lambda^2 f$ satisfying the asymptotic conditions $|f_{\pm}(\lambda, x) - e^{\pm i\lambda x}| \rightarrow 0$ as $x \rightarrow \pm\infty$. When $V(x) \in L^1_1$, the solutions f_{\pm} are uniquely defined. Now consider the Wronskian

$$W(\lambda) = f_+(\lambda, 0)\partial_x f_-(\lambda, 0) - \partial_x f_+(\lambda, 0)f_-(\lambda, 0). \quad (4.4)$$

The function $W(\lambda)$ is always different from zero for $\lambda \in \mathbb{R} \setminus 0$, and hence for real λ it can only vanish at $\lambda = 0$. Then we say that 0 is a *resonance for H* when $W(0) = 0$, and that it is not a resonance when $W(0) \neq 0$. The first one is also called the *exceptional case*.

In [31] Weder proved that the wave operators are bounded on L^p for all $1 < p < \infty$, provided $V \in L^1_{\gamma}$ for $\gamma > 5/2$. The assumption can be relaxed to $\gamma > 3/2$ provided 0 is not a resonance.

Our result in 1D is the following:

Theorem 4.1 *Assume $V \in L^1_1$ and 0 is not a resonance, or $V \in L^1_2$ in the general case. Then the wave operators W_{\pm} , W_{\pm}^* can be extended to bounded operators on L^p for all $1 < p < \infty$. Moreover, in the endpoint L^{∞} case we have the estimate*

$$\|W_{\pm}g\|_{L^{\infty}} \leq C\|g\|_{L^{\infty}} + C\|\mathcal{H}g\|_{L^{\infty}}, \quad (4.5)$$

for all $g \in L^{\infty} \cap L^p$ for some $p < \infty$ such that $\mathcal{H}g \in L^{\infty}$, where \mathcal{H} is the Hilbert transform on \mathbb{R} ; the conjugate operators W_{\pm}^* satisfy the same estimate.

As a first application, consider the initial value problem

$$iu_t - a(x)u_{xx} + b(x)u_x + V(x)u = 0, \quad u(0, x) = f(x). \quad (4.6)$$

A great advantage of the one dimensional case is that by a suitable change of variable equation (4.6) can be reduced to the case of an electric potential, which is covered by Theorem 4.1. Indeed, if we set $u(t, x) = \sigma(x)w(t, c(x))$ with

$$c(x) = \int_0^x a(s)^{-1/2} ds, \quad \sigma(x) = a(x)^{1/4} \exp\left(\int_0^x \frac{b(s)}{2a(s)} ds\right) \quad (4.7)$$

we see easily that $w(t, x)$ is a solution of

$$iw_t - \Delta w + \tilde{V}(x)w = 0,$$

where the potential $\tilde{V}(y)$ is given by

$$\begin{aligned} \tilde{V}(c(x)) = V(x) + \frac{1}{16a(x)}(2b(x) + a'(x))(2b(x) + 3a'(x)) \\ - \frac{1}{4}(2b(x) + a''(x)). \end{aligned} \quad (4.8)$$

Then applying the Theorem and coming back to $u(t, x)$ we obtain the following decay result, where the notation $f \in L_1^2$ means $(1 + |x|)f \in L^2$.

Proposition 4.2 *Assume $V \in L_2^1$, $a \in W^{2,1}(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R})$ with*

$$a(x) \geq c_0 > 0 \quad a', b \in L_1^2, \quad a'', b' \in L_2^1 \quad (4.9)$$

for some constant c_0 . Then the solution of the initial value problem (4.6) satisfies

$$\|P_{ac}u(t, \cdot)\|_{L^q} \leq Ct^{1/q-1/2} \|f\|_{L^{q'}}, \quad 2 \leq q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1. \quad (4.10)$$

The same result holds if $a = 1$, $b = 0$ and $V \in L_1^1$, provided 0 is not a resonance for H .

A similar result holds for the wave equation with fully variable coefficients (depending only on x).

5. The Schrödinger equation with point interactions

We conclude this review by mentioning the results of the paper [12], where we considered the case of a Schrödinger operator with “delta poten-

tials". The operator can be written formally as

$$H = -\Delta + \sum_{j=1}^N \mu_j \delta_{y_j} \quad (5.1)$$

where δ_{y_j} is the Dirac measure placed at $y_j \in \mathbb{R}^3$ and the parameters μ_j are coupling constants. The basic question here is whether dispersive properties are preserved even for potentials of measure type, even rougher than the ones considered in the preceding sections.

As a matter of fact the Dirac measure in \mathbb{R}^3 is not a small perturbation of the Laplacian, even in the sense of quadratic forms. In order to obtain a rigorous counterpart of (5.1) one considers the following restriction of the free Laplacian

$$\hat{H} = -\Delta, \quad D(\hat{H}) = C_0^\infty(\mathbb{R}^3 \setminus Y) \quad (5.2)$$

where $Y = (y_1, \dots, y_N)$. The operator (5.2) is symmetric but not self-adjoint in $L^2(\mathbb{R}^3)$ and, obviously, one possible self-adjoint extension is trivial, i.e. it coincides with the free Laplacian $H_0 = -\Delta$, $D(H_0) = H^2(\mathbb{R}^3)$. Using the theory of self-adjoint extensions of symmetric operators, developed by von Neumann and Krein, one can show that the operator (5.2) has N^2 (non trivial) self-adjoint extensions which, by definition, are all the possible Schrödinger operators with point interactions at Y (for a comprehensive treatment we refer to the monograph [1]). Any such extension can be considered as a Laplace operator with a singular boundary condition satisfied at each point $y_j \in Y$.

In the following we shall only consider the case of local boundary conditions which are more relevant from the physical point of view. More precisely, we shall restrict to the self-adjoint extensions $H_{\alpha,Y}$ parametrized by $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_j \in \mathbb{R}$, and corresponding to the singular boundary condition at Y

$$\lim_{r_j \rightarrow 0} \left[\frac{\partial(r_j u)}{\partial r_j} - 4\pi\alpha_j(r_j u) \right] = 0, \quad r_j = |x - y_j|, \quad j = 1, \dots, N \quad (5.3)$$

Due to the presence of the (unavoidable) singularity at the points where the interaction is placed, we are forced to introduce the following weight function

$$w(x) = \sum_{j=1}^N \left(1 + \frac{1}{|x - y_j|} \right). \quad (5.4)$$

Moreover, for $z \in \mathbb{C}$, we define the matrix

$$[\Gamma_{\alpha,Y}(z)]_{j,\ell} = \left[\left(\alpha_j - \frac{iz}{4\pi} \right) \delta_{j,\ell} - \tilde{G}_z(y_j - y_\ell) \right]_{j,\ell=1}^N, \quad (5.5)$$

with

$$\tilde{G}_z(x) = \begin{cases} \frac{e^{iz|x|}}{4\pi|x|} & x \neq 0, \\ 0 & x = 0. \end{cases} \quad (5.6)$$

The role of this matrix is that it allows to represent in an explicit form the resolvent of the perturbed Schrödinger operator, via Krein's theory. Indeed, $R_{\alpha,Y}(z) = (H_{\alpha,Y} - z)^{-1}$ can be written for $\text{Im } z > 0$ as

$$\begin{aligned} (R_{\alpha,Y}(z^2)f)(x) &= (R_0(z^2)f)(x) \\ &+ \sum_{j,\ell=1}^N [\Gamma_{\alpha,Y}(z)]_{j,\ell}^{-1} \frac{e^{-iz|x-y_j|}}{4\pi|x-y_j|} \int_{\mathbb{R}^3} \frac{e^{iz|y-y_\ell|}}{4\pi|y-y_\ell|} f(y) dy. \end{aligned}$$

Notice that as a consequence the limit absorption principle (i.e., the existence of the limit of the resolvent as z approaches the real axis) holds true for $R_{\alpha,Y}$.

Our main result is the following theorem (where as usual P_{ac} denotes the projection on the absolutely continuous subspace of L^2 relative to our operator):

Theorem 5.1 *Assume that the matrix $\Gamma_{\alpha,Y}(\mu)$ is invertible for $\mu \in [0, +\infty)$ with a locally bounded inverse. Then the following dispersive estimate holds*

$$\|w^{-1}e^{itH_{\alpha,Y}}P_{ac}f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{3/2}} \|w \cdot f\|_{L^1(\mathbb{R}^3)} \quad (5.7)$$

for any $f \in L^2(\mathbb{R}^3)$ such that $w \cdot f \in L^1(\mathbb{R}^3)$.

In the special case $N = 1$, estimate (5.7) holds for all $\alpha \neq 0$; moreover, when $\alpha > 0$ the projection P_{ac} can be replaced by the identity. Finally, in the resonant case $\alpha = 0$ we have the slower decay estimate

$$\|w^{-1}e^{itH_{0,Y}}f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{1/2}} \|w \cdot f\|_{L^1(\mathbb{R}^3)} \quad (5.8)$$

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