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The Tate conjecture over finite fields for projective schemes related to Coxeter orbits

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Abstract. Let G be a simple algebraic group, defined over a finite field $\mathbb{F}_q$, with Frobenius map F. Let $X_f^\bullet$ be the Hansen-Demazure-Deligne-Lusztig compactification of the Deligne-Lusztig variety X_f of G associated with a Coxeter element in the Weyl group W_G of G, and let $X_{f,0}^\bullet$ be the $\mathbb{F}_q\delta$-structure on $X_f^\bullet$ over the finite extension $\mathbb{F}_q\delta$ of $\mathbb{F}_q$ determined by $F^\delta : X_f^\bullet \to X_f^\bullet$, where δ is the smallest positive integer such that F^δ is the identity map on W_G. We shall give an affirmative answer to the Tate conjecture over finite fields for algebraic cycles on $X_{f,0}^\bullet$ and related projective schemes.

Key words: The Tate conjecture over finite fields, Coxeter orbits.

Introduction

Let k_0 be a finite field, k an algebraic closure of k_0 and $\Pi = \mathrm{Gal}(k/k_0)$. Let X_0 be an equidimensional smooth projective scheme of finite type over k_0, purely of dimension d. For an integer s, $0 \leq s \leq d$, let $Z^s(X_0)$ be the free abelian group generated by the closed integral subschemes of X_0 of codimension s. Let ℓ be a prime number different from the characteristic of k_0. Let $X = X \times_{k_0} k$. Let

$$cl^s_{X_0} : Z^s(X_0) \longrightarrow H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi$$

be the cycle map, where (s) is the Tate twist and $H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi$ is the Π-invariant part of $H^{2s}(X, \mathbb{Q}_\ell(s))$. Let

$$A^s = A^s(X_0) = \mathbb{Q} \cdot \mathrm{Im}\, cl^s_{X_0} \quad (\subset H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi)$$

and

$$N^s = N^s(X_0) = \big\{ a \in A^s \mid \langle a, a' \rangle_X = 0 \quad \text{for all } a' \in A^{d-s} \big\},$$

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where $\langle\ ,\ \rangle_X$ is the Poincaré duality pairing via cup product:

$$\langle\ ,\ \rangle_X : H^{2s}(X, \mathbb{Q}_\ell(s)) \times H^{2(d-s)}(X, \mathbb{Q}_\ell(d-s))$$
$$\xrightarrow{\cup} H^{2d}(X, \mathbb{Q}_\ell(d)) \xrightarrow{\mathrm{Tr}_X} \mathbb{Q}_\ell.$$

The Tate conjecture over finite fields consists of the following two statements:

$$T^s : \mathbb{Q}_\ell \cdot A^s = H^{2s}(X, \mathbb{Q}_\ell(s))^{\Pi}$$

and

$$E^s : N_s = 0.$$

(See Tate [Ta II]). Since k_0 is finite, the Tate conjecture over finite fields is equivalent to the following statement:

The order of the pole of the zeta function $Z(X_0, t)$ at $t = q^{-s}$ is equal to $\dim_{\mathbb{Q}}(A^s/N^s)$. (See [Ta II, Theorem (2.9)]). Here $q = |k_0|$.

The Tate conjecture over finite fields is the base of Grothendieck-Milne's theory of motives over finite fields (Milne [Mi II]).

In this paper, we give an affirmative answer to the Tate conjecture over finite fields for very special projective schemes $X^{\bullet}_{f(I),0} = \bar{X}^{\bullet}_{f}(I)_0$ related to the Deligne-Lusztig's theory of representations of finite reductive groups G^F over algebraically closed fields of characteristic 0 (Deligne and Lusztig [DL]). Our main result is stated in the last paragraph of Section 3 (Theorem 1), and is proved in Sections 4, 5.

The motivation of our study is the "fact" that the rationality of a cuspidal unipotent representation of G^F has a "motivic explanation" ([Oh]).

Our result relies on Lusztig's calculation of the eigenvalues of Frobenius on the étale cohomology groups $H^i_c(X_f, \bar{\mathbb{Q}}_\ell)$ with compact supports of the Deligne-Lusztig variety X_f of G associated with a Coxeter element in the Weyl group of G (Lusztig [Lu]). Here $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$.

I wish to dedicate this paper to may daughter Chieko.

Preliminaries and convensions

Let K be an algebraically closed field. Let (X, O_X) be a separated, reduced scheme of finite type over K with structural sheaf O_X. Let $X(K)$

be the set of K-rational points of X, and let $O_{X(K)} = O_X \mid X(K)$. Then $(X(K), O_{X(K)})$ is a variety in the sense of Borel's book [Bo, Ch. AG]. The correspondence $(X, O_X) \mapsto (X(K), O_{X(K)})$ gives an equivalence of the category of separated, reduced schemes of finite type over K with morphisms over K and the category of varieties.

Throughout the paper, p is a fixed prime number and k is an algebaric closure of the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. By a variety, we mean a separated, reduced scheme X of finite type over k, and we shall identify X with $X(k)$. An algebraic group is the one in the sense of [Bo].

For an integral power p^a of p, $\mathbb{F}_{p^a}$ is the subfield of k with p^a elements. k_0 is a finite subfield of k and $\Pi = \mathrm{Gal}(k/k_0)$. φ is the arithmetic Frobenius automorphism of k over k_0, i.e., $\varphi(x) = x^{|k_0|}$, $x \in k$.

A sheaf is an abelian étale sheaf on a scheme.

ℓ is a fixed prime number different from p. $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$.

For a variety X, we write $H^i(X)$ and $H^i_c(X)$ instead of $H^i(X, \bar{\mathbb{Q}}_\ell)$ and $H^i_c(X, \bar{\mathbb{Q}}_\ell)$ respectively.

For a set S and a map $f : S \to S$, $S^f = \{x \in S \mid f(x) = x\}$, and if T is a set of maps $f : S \to S$, then $S^T = \{x \in S \mid f(x) = x \text{ for all } f \in T\}$.

If V is a finite dimensional vector space over a field E and $f : V \to V$ is a linear map, then, for $a \in E^*$, we set

$$V_a = \big\{v \in V \mid (f - aI_V)^n v = 0 \ \text{ for some integer } n \geq 1\big\}.$$

1. The Poincaré duality theorem

Let X be a smooth equidimensional variety, purely of dimension d. Then X is the disjoint union of its irreducible components $X_1, \ldots, X_m$. For an inteber u, $1 \leq u \leq m$, let $i_u : X_u \hookrightarrow X$ be the inclusion morphism. Let G be a sheaf on X. For $1 \leq u \leq m$, let $G_u = i_{u*} {i_u}^* G = i_{u!} {i_u}^* G$. Then we have $G = \bigoplus_{u=1}^m G_u$ and

$$H^i(X, G) = \bigoplus_{u=1}^{m} H^i(X, G_u) = \bigoplus_{u=1}^{m} H^i(X, i_{u*} {i_u}^* G).$$

Let H be another sheaf on X. Then there are cup product homomorphisms

$$\cup : H^i(X,G) \times H^j(X,H) \longrightarrow H^{i+j}(X, G\otimes H) \quad (i,j \geq 0).$$

For $1 \leq u \neq v \leq m$, we have

$$x \cup y = 0, \quad x \in H^i(X_u, {i_u}^*G), \quad y \in H^j(X_v, {i_v}^*H). \tag{1.1}$$

Assume that X is a projective variety. Let n be a positive integer coprime to p. Then, for $1 \leq u \leq m$, there is a canonical isomorphism $\mathrm{Tr}_{X_u} : H^{2d}(X_u, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$, where (d) denotes the Tate twist. Let

$$\begin{aligned} \mathrm{Tr}_X = \sum_{u=1}^{m} \mathrm{Tr}_{X_u} &: H^{2d}(X, \mathbb{Z}/n\mathbb{Z}(d)) \\ &= \bigoplus_{u=1}^{m} H^{2d}(X_u, \mathbb{Z}/n\mathbb{Z}(d)) \longrightarrow \mathbb{Z}/n\mathbb{Z}. \end{aligned}$$

Then, by the Poincaré duality theorem ([SGA 4, Ch. XVIII]), the pairing

$$\begin{aligned} \langle\ ,\ \rangle_{X,n} : H^i(X, \mathbb{Z}/n\mathbb{Z}(a)) \times H^{2d-i}(X, \mathbb{Z}/n\mathbb{Z}(d-a)) \\ \xrightarrow{\cup} H^{2d}(X, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{\mathrm{Tr}_X} \mathbb{Z}/n\mathbb{Z} \end{aligned} \tag{1.2}$$

is non-degenerate ($a \in \mathbb{Z}$). By replacing n by ℓ^n, on passing to the projective limit on n and by tensoring with $\mathbb{Q}_\ell$, we get a non-degenerate pairing

$$\langle\ ,\ \rangle_X : H^i(X, \mathbb{Q}_\ell(a)) \times H^{2d-i}(X, \mathbb{Q}_\ell(d-a)) \longrightarrow \mathbb{Q}_\ell.$$

Remark Deligne's proof of non-degenerateness of the pairing in [SGA 4, Ch. XVIII] is dificult to follow for the author. But, fortunately, we can see its proof in Milne's book [Mi I, Section 11] when X is irreducible. The general case follows from this special case by using (1.1).

Assume that X is obtained by the extension of scalars from a scheme X_0 over k_0 : $X = X_0 \times_{k_0} k$. Then the pairing $\langle\ ,\ \rangle_X$ is Π-equivarinat, where Π acts on $\mathbb{Q}_\ell$ trivially.

Let $Y = Y_0 \times_{k_0} k$ be another smooth equidimensional projective variety, purely of dimension e, let $f_0 : Y_0 \to X_0$ be a morphism over k_0 and let $f = f_0 \times_{k_0} k : Y \to X$. Then the inverse image homomorphism f^* :

$H^i(X, \mathbb{Q}_\ell(a)) \to H^i(Y, \mathbb{Q}_\ell(a))$ and the direct image homomorphism $f_* : H^{2e-i}(Y, \mathbb{Q}_\ell(e-a)) \to H^{2d-i}(X, \mathbb{Q}_\ell(d-a))$ (the dual of f^* via the Poincaré duality theorem) are Π-equivariant. Then f^* and f_* induce homomorphisms

$$f_1^* : H^i(X, \mathbb{Q}_\ell(a))_1 \longrightarrow H^i(Y, \mathbb{Q}_\ell(a))_1$$

and

$$f_{*1} : H^{2e-i}(Y, \mathbb{Q}_\ell(e-a))_1 \longrightarrow H^{2d-i}(X, \mathbb{Q}_\ell(d-a))_1.$$

The pairings $\langle\ ,\ \rangle_X$ and $\langle\ ,\ \rangle_Y$ induce non-degenerate pairings

$$\langle\ ,\ \rangle_{X,1} : H^i(X, \mathbb{Q}_\ell(a))_1 \times H^{2d-i}(X, \mathbb{Q}_\ell(d-a))_1 \longrightarrow \mathbb{Q}_\ell$$

and

$$\langle\ ,\ \rangle_{Y,1} : H^i(Y, \mathbb{Q}_\ell(a))_1 \times H^{2e-i}(Y, \mathbb{Q}_\ell(e-a))_1 \longrightarrow \mathbb{Q}_\ell.$$

Thererore $f_1^* : H^i(X, \mathbb{Q}_\ell(a))_1 \to H^i(Y, \mathbb{Q}_\ell(a))_1$ induces its dual homomorphism

$$(f_1^*)^\vee : H^{2e-i}(Y, \mathbb{Q}_\ell(e-a))_1 \longrightarrow H^{2d-i}(X, \mathbb{Q}_\ell(d-a))_1.$$

We see that

$$(f_1^*)^\vee = f_{*1}.$$

Therefore f_{*1} is surjective if f_1^* is injective.

Let V be a finite dimensional vector space over $\mathbb{Q}_\ell$ on which Π acts continuously. We note that

$$\overline{\langle \varphi \rangle} = \Pi.$$

Thus $V^\varphi = V^\Pi$. Thus, in particular, when φ acts semisimplly on V_1, we have $V_1 = V^\varphi = V^\pi$.

2. Tate conjecture

Let X_0 be a separated reduced smooth scheme of finite type over k_0, let $X = X_0 \times_{k_0} k$ and let $\pi_X : X \to X_0$ be the natural projection. Then X is a smooth variety over k. Assume that X_0 is purely of dimension d.

Let s be an integer, $0 \leq s \leq d$. Let $Z^s(X_0)$ (resp. $Z^s(X)$) be the free abelian group which is generated by the integral closed subschemes of X_0 (resp. X) of codimension s. For a prime cycle $Z_0 \in Z^s(X_0)$, let $Z_1, \ldots, Z_t$ be all the irreducible component of $Z_0 \times_{k_0} k$, and we put

$$\pi_X^* Z_0 = Z_1 + \cdots + Z_t \in Z^s(X)$$

(note that k_0 is perfect). Extending by additivity, we obtain a homomorphism

$$\pi_X^* : Z^s(X_0) \longrightarrow Z^s(X).$$

Let

$$\widetilde{\mathrm{cl}}^s_{X_0} : Z^s(X_0) \longrightarrow H^{2s}(X_0, \mathbb{Q}_\ell(s))$$

and

$$\widetilde{\mathrm{cl}}^s_X : Z^s(X) \longrightarrow H^{2s}(X, \mathbb{Q}_\ell(s))$$

be cycle maps (Grothendieck-Deligne [SGA 4 1/2]). Then we have the following commutative diagram:

$$\begin{tikzcd}
Z^s(X_0) \arrow[r, "\widetilde{\mathrm{cl}}^s_{X_0}"] \arrow[d, "\pi_X^*"'] & H^{2s}(X_0, \mathbb{Q}_\ell(s)) \arrow[d, "\pi_X^*"] \\
Z^s(X) \arrow[r, "\widetilde{\mathrm{cl}}^s_X"'] & H^{2s}(X, \mathbb{Q}_\ell(s)).
\end{tikzcd}$$

We see that the cycle map $\widetilde{\mathrm{cl}}^s_X$ coincides with the cycle map which is defined in [Mi I, Ch. VI, Section 9] and the map $\pi_X^* : H^{2s}(X_0, \mathbb{Q}_\ell(s)) \to H^{2s}(X, \mathbb{Q}_\ell(s))$ coincides with the edge homomorphism at position $(0, 2s)$ in the spectral sequence

$$H^i\bigl(\Pi, H^j(X, \mathbb{Q}_\ell(s))\bigr) \Longrightarrow H^{i+j}(X_0, \mathbb{Q}_\ell(s)).$$

Thus $\pi_X^* \circ \widetilde{\mathrm{cl}^s_{X_0}} = \widetilde{\mathrm{cl}^s_X} \circ \pi_X^*$ factors through $H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi$. We denote this map by

$$\mathrm{cl}^s_{X_0} : Z^s(X_0) \longrightarrow H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi.$$

Assume that X is projective. For an integer s, $0 \le s \le d$, let

$$A^s = A^s(X_0) = \mathbb{Q} \cdot \mathrm{Im}\bigl(\mathrm{cl}^s_{X_0}\bigr) \subset H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi$$

and

$$N^s = N^s(X_0) = \bigl\{a \in A^s \mid \langle a, a' \rangle_X = 0 \ \text{ for all } a' \in A^{d-s}\bigr\},$$

where $\langle\ ,\ \rangle_X$ is the Poincaré duality pairing via cup product.

The Tate conjecture over finite fields consists of the following two statements (see Tate [Ta I, II]):

$$\begin{aligned} &T^s : \mathbb{Q}_\ell \cdot A^s = H^{2s}(X, \mathbb{Q}_\ell(s))^\Pi, \\ &E^s : N^s = 0. \end{aligned}$$

Remark (1) In [Ta I], Tate defines his cycle map as follows:

Let $Z \in Z^s(X)$ be a prime cycle. We define $c(Z)$ to be an element of $H^{2s}(X, \mathbb{Q}_\ell(s))$ characterized by the property

$$\mathrm{Tr}_X(y \cup c(Z)) = \mathrm{Tr}_Z(y \mid Z)$$

for all $y \in H^{2(d-s)}(X, \mathbb{Q}_\ell(d-s))$. We see that

$$c(Z) = \widetilde{\mathrm{cl}^s_X}(Z)$$

if Z is smooth (see [Mi I, Ch. VI, Section 11, Remark 11.6(e), p. 284]). However I do not know whether this equality holds for singular Z.

(2) In [Ta II], Tate states his conjectures by using "the" cycle map whose definition is unknown to the author. Here we adopt Grothendieck-Deligne-Milne's definition of cycle maps.

Let Y_0 be a smooth equidimensional projective scheme over k_0, purely of dimension e, Let $Y = Y_0 \times_{k_0} k$ and let $\pi_Y : Y \to Y_0$ be the natural projection. Let $g_0 : Y_0 \to X_0$ be a morphism over k_0 and let $g = g_0 \times_{k_0} k : Y \to X$. Let s be an integer, $0 \leq s \leq e$. Let $W_0 \in Z^s(Y_0)$ be a prime cycle. Then the image $Z_0 = g_0(W_0)$ has a structure of closed integral subscheme of X_0. The function field $k_0(Z_0)$ of Z_0 can be regarded as a subfield of the function field $k_0(W_0)$ of W_0. Let $m = [k_0(W_0) : k_0(Z_0)]$. Then we define $g_{0*}W_0$ to be mZ_0 if m is finite and 0 otherwise. Extending by additivity, we obtain a homomorphism

$$g_{0*} : Z^s(Y_0) \longrightarrow Z^{d-e+s}(X_0).$$

Similarly, we can define a homomorphism

$$g_* : Z^s(Y) \longrightarrow Z^{d-e+s}(X).$$

The diagram

$$\begin{tikzcd} Y \arrow[r, "\pi_Y"] \arrow[d, "g"'] & Y_0 \arrow[d, "g_0"] \\ X \arrow[r, "\pi_X"'] & X_0 \end{tikzcd}$$

is cartesian and π_X is flat. Therefore

$$g_*\pi_Y^* = \pi_X^* g_{0*}$$

(see Fulton [Fu, Ch. I, Section 1.7, Proposition 1.7, p. 18]). Thus, if Y_0 is the disjoint union $\coprod_{j=1}^t Y_{0j}$ of closed subschemes Y_{0j} of X_0 and g_0 is the sum of the inclusion morphisms $Y_{0j} \hookrightarrow X_0$, then the following diagram is commutative:

$$\begin{tikzcd} Z^s(Y_0) \arrow[r, "\mathrm{cl}^s_{Y_0}"] \arrow[d, "g_{0*}"'] & H^{2s}(Y, \mathbb{Q}_\ell(s)) \arrow[d, "g_*"] \\ Z^{d-e+s}(X_0) \arrow[r, "\mathrm{cl}^{d-e+s}_{X_0}"'] & H^{2(d-e+s)}(X, \mathbb{Q}_\ell(d-e+s)). \end{tikzcd} \qquad (0 \leq s \leq e).$$

(Cf. [Mi I, Ch. VI, Section 9, Proposition 9.3, p. 269]).

3. Reductive groups

In the rest of this paper, we shall use the following notations. Almost all of them are extracted from Deligne and Lusztig's paper [DL] and Lusztig's paper [Lu].

G is a connected, reductive linear algebraic group over k. $F : G \to G$ is a surjective endomorphism of G such that some integral power F^d of F is the Frobenius endomorphism of G relative to a rational structure on G over a finite subfield k' of k and q is the positive real number such that $q^d = |k'|$ (uniquely determined by F). We assume that $d = 1$ or that $d = 2$ and q is an odd power of $\sqrt{2}$ or $\sqrt{3}$.

X_G is the set of Borel subgroups of G. G acts transitively on X_G by conjugation: $(g, B) \mapsto gBg^{-1}$, $g \in G$, $B \in X_G$. For each $B \in X_G$, the stabilizer $N_G(B)$ of B is just B, so the mapping $gB \mapsto gBg^{-1}$ defines a bijection $G/B \xrightarrow{\sim} X_G$. Therefore X_G has a structure of a projective variety. $F : X_G \to X_G$ is the map $B \mapsto F(B)$. This map is an endomorphism of X_G with respect to the structure of the projective variety of X_G:

By Lang-Steinberg theorem, there is an F-stable Borel subgroup B of G; for such B, the diagram

$$\begin{tikzcd} G/B \arrow[r, "\sim"] \arrow[d, "F"'] & X_G \arrow[d, "F"] \\ G/B \arrow[r, "\sim"] & X_G \end{tikzcd}$$

is commutative.

We let G act on $X_G \times X_G$ by $(g, (B, B')) \mapsto (gBg^{-1}, gB'g^{-1})$. Then the Weyl group W_G of G can be identified with the set $G \setminus (X_G \times X_G)$ of orbits of G on $X_G \times X_G$ as follows:

Let (T, B) be a pair of a maximal torus T of G and a Borel subgroup B of G containing T. Then the composite $\sigma(T, B)$ of the following bijectioins is an isomorphism of groups:

$$\begin{aligned} W_G(T) &= N_G(T)/T \xrightarrow{\sim} B \setminus G/B \xrightarrow{\sim} G \setminus (G/B \times G/B) \xrightarrow{\sim} G \setminus (X_G \times X_G) \\ &= W_G \end{aligned}$$

$$nT \longmapsto BnB \longmapsto G \cdot (B, nB) \longmapsto G \cdot (B, nBn^{-1}).$$

The law of compositioin in W_G will be written as $O \circ O'$ for O, $O' \in W_G$. The unit element is the diagonal $\Delta = \{(B, B) \mid B \in X_G\}$. The set

$$S = S_G = \{O \in W_G \mid \dim O = \dim X_G + 1\}$$

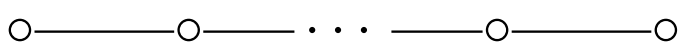
is the set of simple reflections in W_G. We denote by $\ell(\ )$ the length function on W_G with respect to S_G. $F : W_G \to W_G$ is the map $O \mapsto F(O)$. If (T, B) is an F-stable pair, then the diagram

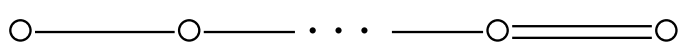
$$\begin{tikzcd} W_G(T) \arrow[r, "\sigma(T,B)", "\sim"'] \arrow[d, "F"'] & W_G \arrow[d, "F"] \\ W_G(T) \arrow[r, "\sim", "\sigma(T,B)"'] & W_G \end{tikzcd}$$

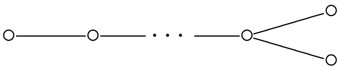
is commutative. We have $F(S_G) = S_G$. $S_F = (S_G)_F$ is the set of orbits of F on S_G. $\pi : S_G \to S_F$ is the natural map. $r = |S_F|$ is the rank of G. δ is the minimal positive integer such that F^δ is the identity map on W_G. q^δ is a power of p; we put $k_0 = \mathbb{F}_{q^\delta}$, and $\Pi = \mathrm{Gal}(k/k_0)$.

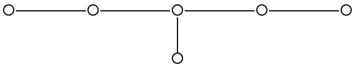
Let $B \in X_G^F$ (the F-invariant part of X_G). Then, in view of the construction of the structure of the projective variety on $G/B \xrightarrow{\sim} X_G$ (cf. Borel [Bo, Ch. II, Section 6, (6.8), pp. 181–2; Ch. IV, Section 11, (11.1), pp. 261–2]), we see that there is a projective space P^N over k with the "standard" k_0-structure with Frobenius map F^δ such that X_G is an F^δ-stable closed subvariety of P^N and that $F^\delta : X_G \to X_G$ is the restriction to X_G of $F^\delta : P^N \to P^N$.

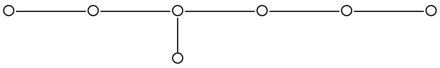
The Coxeter graph Γ of G is the graph with one vertex for each element of S_G and such that the vertices corresponding to O, $O' \in S_G$ $(O \neq O')$ are joined by 0, 1, 2 or 3 bonds according as $O \circ O'$ has order 2, 3, 4 or 6 respectively. $F : S_G \to S_G$ determines an automorphism F of Γ. When Γ is connected the possible (Γ, F) is as follows (cf. Bourbaki [Bour, Ch. 6, Section 4, $n^0 1$, Théoremè 1], Steinberg [St, Section 11]; also see Carter [Ca, pp. 37–8]);

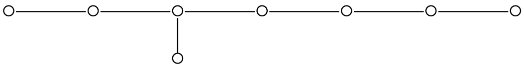
A_n $(n \geq 1)$  (n vertices, $\delta = 1$),

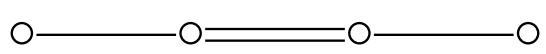
B_n $(n \geq 2)$  (n vertices, $\delta = 1$),

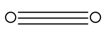
D_n $(n \geq 4)$  (n vertices, $\delta = 1$),

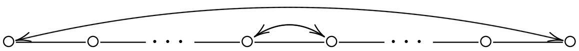
E_6  ($\delta = 1$),

E_7  ($\delta = 1$),

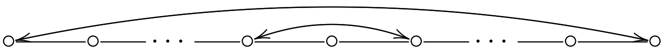
E_8  ($\delta = 1$),

F_4  ($\delta = 1$),

G_2  ($\delta = 1$),

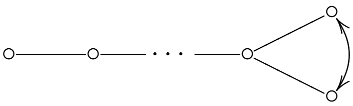
${}^2A_{2n}$ $(n \geq 1)$ 

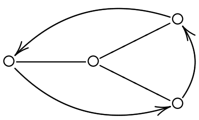
($2n$ vertices, $\delta = 2$),

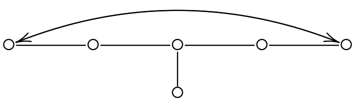
${}^2A_{2n+1}$ $(n \geq 1)$ 

($2n+1$ vertices, $\delta = 2$),

${}^2B_2$  ($\delta = 2$, $q = \sqrt{2}^{2m+1}$),

${}^2D_n$ $(n \geq 4)$  (n vertices, $\delta = 2$),

${}^3D_4$  ($\delta = 3$),

${}^2E_6$  ($\delta = 2$),

$^2F_4$  $(\delta = 2,\ q = \sqrt{2}^{2m+1})$,

$^2G_2$  $(\delta = 2,\ q = \sqrt{3}^{2m+1})$.

We continue to eatablish notations.

Let $O \in W_G$. We let

$$X(O) = X_G(O) = \big\{B \in X_G \mid (B, F(B)) \in O\big\}.$$

$X(O)$ is a smooth locally closed subvariety of X_G, purely of dimension $\ell(O)$. We call $X(O)$ the Deligne-Lusztig variety of G associated with O.

Let $O = O_1 \circ \cdots \circ O_n$ $(O_1, \ldots, O_n \in S_G)$ be a minimal expression for O. We let

$$\begin{aligned} X(O)^\bullet &= \bar{X}(O_1, \ldots, O_n) \\ &= \big\{(B_0, B_1, \ldots, B_n) \in X_G^{n+1} \mid (B_{i-1}, B_i) \in O_i \cup \Delta \\ &\qquad \text{for } 1 \le i \le n \text{ and } F(B_0) = B_n\big\} \end{aligned}$$

and

$$X(O_1, \ldots, O_n) = \big\{(B_0, B_1, \ldots, B_n) \in X(O)^\bullet \mid B_{i-1} \neq B_i, 1 \le i \le n\big\}.$$

Then $X(O)^\bullet$ is a smooth projective subvariety of X_G^{n+1}, purely of dimension $\ell(O)$, $X(O_1, \ldots, O_n)$ is an open dense subvariety of $X(O)^\bullet$ and the mapping $(B_0, B_1, \ldots, B_n) \mapsto B_0$ gives an isomorphism from $X(O_1, \ldots, O_n)$ onto $X(O)$. We call $X(O)^\bullet$ the Hansen-Demazure-Deligne-Lusztig campactification of $X(O)$ (with respect to a reduced expression $O = O_1 \circ \cdots \circ O_n$).

$X(O)^\bullet$ is an F^δ-stable subvariety of X_G^{n+1}. Therefore $F^\delta : X(O)^\bullet \to X(O)^\bullet$ determines a k_0-structure $X(O)_0^\bullet$ on $X(O)^\bullet$. $X(O)_0^\bullet$ is a smooth projective scheme of finite type over k_0, purely of dimension $\ell(O)$.

In the following, if P is a parabolic subgroup of G, then U_P is its unipotent radical, $L_P = P/U_P$ and $\pi_P : P \to L_P$ is the natural morphism. L_P is a connected, reductive linear algebraic group over k.

$f = (O_1, \ldots, O_r)$ is a sequence of elements of S_G such that $\{\pi(O_1), \ldots, \pi(O_r)\} = S_F$. We put

$$\begin{aligned} O_f &= O_1 \circ \cdots \circ O_r \in W_G, \\ X_f &= X(O_f), \\ X_f^\bullet &= X(O_f)^\bullet = \bar{X}(O_1, \ldots, O_r) \end{aligned}$$

and

$$X_{f,0}^\bullet = X(O_f)_0^\bullet.$$

O_f is called a Coxeter orbit of G on $X_G \times X_G$. X_f is a smooth irreducible affine variety of dimension r, $X_f^\bullet$ is a smooth irreducible projective variety of dimension r, and $X_{f,0}^\bullet$ is a smooth absolutely irreducible projective scheme of finite type over k_0.

Let I be any subset of S_F, and let $n = |I|$. $f(I) = (O_{i_1}, \ldots, O_{i_n})$ $(1 \le i_1 < \cdots < i_n \le r)$ is the subsequence of $f = (O_1, \ldots, O_r)$ such that $\{\pi(O_{i_1}), \ldots, \pi(O_{i_n})\} = I$. We put

$$\begin{aligned} O_{f(I)} &= O_{i_1} \circ \cdots \circ O_{i_n} \in W_G, \\ X_{f(I)} &= X(O_{f(I)}), \\ X_{f(I)}^\bullet &= X(O_{f(I)})^\bullet. \end{aligned}$$

We put

$$\bar{X}_f^\bullet(I) = \big\{(B_0, B_1, \ldots, B_r) \in X_f^\bullet \mid B_{i-1} = B_i \ \text{ if } \pi(O_i) \notin I\big\}$$

and

$$X_f^\bullet(I) = \big\{(B_0, B_1, \ldots, B_r) \in \bar{X}_f^\bullet(I) \mid B_{i-1} \neq B_i \ \text{ if } \pi(O_i) \in I\big\}.$$

$\bar{X}_f^\bullet(I)$ is isomorphic to $X(O_{f(I)})^\bullet = \bar{X}(O_{i_1}, \ldots, O_{i_n})$, so it is a smooth projective variety, purely of dimension $n = |I|$. The mapping $(B_0, B_1, \ldots, B_r) \mapsto B_0$ gives an isomorphism from $X_f^\bullet(I)$ onto $X_{f(I)}$. $X_f^\bullet(I)$ is an open dense subvariety of $\bar{X}_f^\bullet(I)$. $\bar{X}_f^\bullet(I)_0$ denotes the k_0-structure on $\bar{X}_f^\bullet(I)$ determined by $F^\delta : \bar{X}_f^\bullet(I) \to \bar{X}_f^\bullet(I)$. $\bar{X}_f^\bullet(I)_0$ is a smooth projective scheme of finite type over k_0, purely of dimension n.

We give the irreducible decompositions of $X_f^\bullet(I)$ $(\simeq X_{f(I)})$ and $\bar{X}_f^\bullet(I)$ $(\simeq X_{f(I)}^\bullet = X(O_{f(I)})^\bullet)$.

Let $\mathcal{P}_I$ be the (F-stable) conjugacy class of parabolic subgoups of G corresponding to $\pi^{-1}(I)$. More precisely, $\mathcal{P}_I$ is constructed as follows:

We fix an F-stable Borel subgroup B^* of G and an F-stable maximal torus T^* of G contained in B^*. Let $W_I = \langle \pi^{-1}(I) \rangle \subset W_G$, and let $W_I^* = \sigma(T^*, B^*)^{-1}(W_I) \subset W_G(T^*)$. Let $P_I^* = B^* W_I^* B^*$. Then

$$\mathcal{P}_I = \left\{ g P_I^* g^{-1} \mid g \in G \right\}.$$

Let $P \in \mathcal{P}_I^F$. Then the mapping $\bar{B} \mapsto \pi_P^{-1}(\bar{B})$ defines an isomorphism i_P from $X_P = X_{L_P}$ onto the closed subvariety $X_{G,P} = \{B \in X_G \mid B \subset P\}$ of X_G. $i_P \times i_P : X_P \times X_P \to X_G \times X_G$ induces an isomorphism i_P from $W_{L_P} = L_P \setminus (X_P \times X_P)$ onto the subgroup

$$\begin{aligned} W_I &= W(\mathcal{P}_I) \\ &= \left\{ O \in W_G \mid \text{ for } (B, B') \in O, \text{ there is } P' \in \mathcal{P}_I \text{ such that } B,\, B' \subset P' \right\} \end{aligned}$$

of W_G. We have

$$i_P(S_{L_P}) = S_G \cap W_I =: S_I = S(\mathcal{P}_I).$$

The bijection $i_P : S_{L_P} \xrightarrow{\sim} S_I$ determines a sequence $f(P) = f(L_P) = (\bar{O}_{i_1}, \ldots, \bar{O}_{i_n})$ of elements of S_{L_P} such that

$$\bar{O}_{f(P)} = \bar{O}_{i_1} \circ \cdots \circ \bar{O}_{i_n} \in W_{L_P}$$

is a Coxeter orbit of L_P on $X_P \times X_P$. $i_P : X_P \xrightarrow{\sim} X_{G,P}$ induces an isomorphism from $X_{f(P)} = X_{f(L_P)} = X_{L_P}(\bar{O}_{f(P)})$ onto the closed subvariety

$$X_{f(I),P} = \left\{ B \in X_{f(I)} \mid B \subset P \right\}$$

of $X_{f(I)}$. And

$$X_{f(I)} = \coprod_{P \in \mathcal{P}_I^F} X_{f(I),P}.$$

Thus

$$X_f^\bullet(I) \xrightarrow{\sim} X_{f(I)} \xleftarrow{\sim} \coprod_{P\in\mathcal{P}_I^F} X_{f(P)}, \tag{3.1}$$

which is the irreducible decomposition of $X_f^\bullet(I)$. These isomorphisms are F^δ-equivariant.

Similarly, the i_P, $P \in \mathcal{P}_I^F$, induce an F^δ-equivariant isomorphisms

$$\bar{X}_f^\bullet(I) \xrightarrow{\sim} X_{f(I)}^\bullet \xleftarrow{\sim} \coprod_{P\in\mathcal{P}_I^F} X_{f(P)}^\bullet. \tag{3.2}$$

(3.1) is proved in [Lu]. We give here a proof of (3.2).

Let R^*, R^{*+} and D^* be respectively the root system of G with respect to T^*, the set of positive roots determined by B^* and the set of corresponding simple roots. Put $J = \pi^{-1}(I)$, and $J^* = \sigma(T^*, B^*)^{-1}(J) \subset W_G(T^*)$. J^* is a subset of $S^* = \sigma(T^*, B^*)^{-1}(S_G)$ of simple reflections in $W_G(T^*)$ determined by B^*. Each $\alpha \in D^*$ determines a simple reflection $s_\alpha \in S^*$ and the mapping $\alpha \mapsto s_\alpha$ gives a bijection $a : D^* \xrightarrow{\sim} S^*$. Let $D_I^* = a^{-1}(J^*)$. For a root $\alpha \in R^*$, let U_α^* be the root subgroup of G associated with α. Then

$$P_I^* = \langle U_{-\alpha}^*, B^* \mid \alpha \in D_I^* \rangle.$$

Let

$$M_I^* = \langle U_\alpha^*, U_{-\alpha}^*, T^* \mid \alpha \in D_I^* \rangle.$$

Then M_I^* is an F-stable Levi subgroup of P_I^* ($P_I^* = M_I^* \ltimes U_{P_I^*}$). The composite $M_I^* \hookrightarrow P_I^* \to L_{P_I^*}$ induces and isomorphism $b : M_I^* \xrightarrow{\sim} L_{P_I^*}$.

Let $P \in \mathcal{P}_I^F$. Then $P = g_0 P_I^* g_0^{-1}$ for some $g_0 \in G^F$ (the F-invariant part of G). Let $M_P = g_0 M_I^* g_0^{-1}$. Then the composite $M_P \hookrightarrow P \to L_P$ induces an isomorphism $b_P : M_P \xrightarrow{\sim} L_P$. The morphism $i_P : X_P \to X_G$ is given by

$$i_P(\bar{B}) = b_P^{-1}(\bar{B}) \cdot U_P \quad (\bar{B} \in X_P).$$

Recall that $1 \le i_1 < \cdots < i_n \le r$. Put $i_0 = 0$. Then

$$X_{f(P)}^\bullet = \{(\bar{B}_{i_0}, \bar{B}_{i_1}, \dots, \bar{B}_{i_n}) \in X_P^{n+1} \mid (\bar{B}_{i_{j-1}}, \bar{B}_{i_j}) \in \bar{O}_{i_j} \cup \Delta_P$$
$$\text{for } 1 \le j \le n \text{ and } F(\bar{B}_{i_0}) = \bar{B}_{i_n}\},$$

where $\Delta_P = \{(\bar{B}, \bar{B}) \mid \bar{B} \in X_P\}$. For $(\bar{B}_{i_0}, \bar{B}_{i_1}, \ldots, \bar{B}_{i_n}) \in X_P^{n+1}$, put

$$i_P\big(\bar{B}_{i_0}, \bar{B}_{i_1}, \ldots, \bar{B}_{i_n}\big) = \big(i_P\big(\bar{B}_{i_0}\big), i_P\big(\bar{B}_{i_1}\big), \ldots, i_P\big(\bar{B}_{i_n}\big)\big) \in X_G^{n+1}.$$

Let

$$X^{\bullet}_{f(I),P} = \big\{\big(B_{i_0}, B_{i_1}, \ldots, B_{i_n}\big) \in X^{\bullet}_{f(I)} \mid B_{i_0}, B_{i_1}, \ldots, B_{i_n} \subset P\big\},$$

and we define $\pi_P : X^{\bullet}_{f(I),P} \to X^{\bullet}_{f(P)}$ by

$$\pi_P\big(B_{i_0}, B_{i_1}, \ldots, B_{i_n}\big) = \big(\pi_P\big(B_{i_0}\big), \pi_P\big(B_{i_1}\big), \ldots, \pi_P\big(B_{i_n}\big)\big).$$

Then $i_P : X_P^{n+1} \to X_G^{n+1}$ induces an isomorphism from $X^{\bullet}_{f(P)}$ onto $X^{\bullet}_{f(I),P}$ whose inverse is π_P.

Thus

$$(i_P)_{P \in \mathcal{P}_I^F} : \coprod_{P \in \mathcal{P}_I^F} X^{\bullet}_{f(P)} \xrightarrow{\sim} \coprod_{P \in \mathcal{P}_I^F} X^{\bullet}_{f(I),P} \subset X^{\bullet}_{f(I)}.$$

Let $(B_{i_0}, B_{i_1}, \ldots, B_{i_n}) \in X^{\bullet}_{f(I)}$. We show that there is some $P \in \mathcal{P}_I^F$ such that $B_{i_0}, B_{i_1}, \ldots, B_{i_n} \subset P$.

If $B_{i_0} = B_{i_1} = \cdots = B_{i_n}$, let P be a parabolic subgroup in $\mathcal{P}_I$ containing B_{i_0}. Then $F(B_{i_0}) = B_{i_n} = B_{i_0}$. So $B_{i_0} \subset F(P)$. Since $F(\mathcal{P}_I) = \mathcal{P}_I$, $F(P) \in \mathcal{P}_I$ and P and $F(P)$ are conjegate. So we must have $P = F(P)$. Thus $P \in \mathcal{P}_I^F$.

Otherwise, there is an integer $j, 1 \le j \le n$, such that $B_{i_{j-1}} \ne B_{i_j}$. Let j be minimal having this property. Then, by the definition of $X^{\bullet}_{f(I)}$, we must have $(B_{i_{j-1}}, B_{i_j}) \in O_{i_j}$. So there is an element $g \in G$ such that $B_{i_{j-1}} = gB^*g^{-1}$ and $B_{i_j} = gs_{i_j}B^*s_{i_j}g^{-1}$, where s_{i_j} is an element of $N_G(T^*)$ such that $\sigma(T^*, B^*)(s_{i_j}T^*) = O_{i_j}$. Put $P = gP_I^*g^{-1}$. Then $B_{i_{j-1}} \subset P$. As $g^{-1}B_{i_j}g = s_{i_j}B^*s_{i_j} \subset s_{i_j}P_I^*s_{i_j} = P_I^*$, $B_{i_j} \subset gP_I^*g^{-1} = P$.

If j is a unique integer such that $B_{i_{j-1}} \ne B_{i_j}$, then $(B_{i_0}, \ldots, B_{i_n}) = (B_{i_{j-1}}, \ldots, B_{i_{j-1}}, B_{i_j}, \ldots, B_{i_j})$. Therefore, in this case, as $B_{i_j} = B_{i_n} = F(B_{i_0}) = F(B_{i_{j-1}})$, $B_{i_j} \subset F(P)$. But P and $F(P)$ are conjugate, so we must have $F(P) = P$. Thus $P \in \mathcal{P}_I^F$.

Otherwise, let $j' > j$ be the minimal integer such that $B_{i_{j'-1}} \ne B_{i_{j'}}$. Then $(B_{i_{j'-1}}, B_{i_{j'}}) \in O_{i_{j'}}$. So there is an element $g' \in G$ such that $B_{i_{j'-1}} =$

$g'B^*g'^{-1}$ and $B_{i_{j'}} = g's_{i_{j'}}B^*s_{i_{j'}}g'^{-1}$. We have $B_{i_j} = B_{i_{j'-1}} \subset P$ and $B_{i_{j'-1}} \subset g'P_I^*g'^{-1}$. But P and $g'P_I^*g'^{-1}$ are conjugate, so we must have $P = g'P_I^*g'^{-1}$. As $g'^{-1}B_{i_{j'}}g' = s_{i_{j'}}B^*s_{i_{j'}} \subset s_{i_{j'}}P_I^*s_{i_{j'}} = P_I^*$, $B_{i_{j'}} \subset g'P_I^*g'^{-1} = P$.

By continuing the similar considerations, we see that there is some $P \in \mathcal{P}_I$ such that $B_{i_0}, B_{i_1}, \ldots, B_{i_n} \subset P$. We have $B_{i_n} = F(B_{i_0}) \subset F(P)$. So $B_{i_0} \subset P$, $F(P)$. But P and $F(P)$ are conjugate, we must have $F(P) = P$. Thus $P \in \mathcal{P}_I^F$.

Thus

$$X^{\bullet}_{f(I)} = \coprod_{P \in \mathcal{P}_I^F} X^{\bullet}_{f(I),P},$$

and

$$(i_P)_{P \in \mathcal{P}_I^F} : \coprod_{P \in \mathcal{P}_I^F} X^{\bullet}_{f(P)} \xrightarrow{\sim} X^{\bullet}_{f(I)}.$$

This isomorphism is F^δ-equivariant.

For $a \in Z$, $0 \leq a \leq n = |I|$, we put

$$D_a(I) = \bigcup_{\substack{J \subset I \\ |J| \leq a}} X^{\bullet}_f(J) \subset \bar{X}^{\bullet}_f(I);$$

we put $D_a(I) = \emptyset$ for $a < 0$. Then $D_0(I) \subset D_1(I) \subset \cdots \subset D_{n-1}(I)$ are closed subvarieties of $D_n(I) = \bar{X}^{\bullet}_f(I)$ and

$$D_a(I) - D_{a-1}(I) = \coprod_{\substack{J \subset I \\ |J| = a}} X^{\bullet}_f(I).$$

Our main result is

Theorem 1 *Assume that G is a simple algebraic group. Then, for any $I \subset S_F$, we have*

$$\mathbb{Q}_\ell \cdot A^s(X^{\bullet}_{f(I),0}) = H^{2s}(X^{\bullet}_{f(I)}, \mathbb{Q}_\ell(s))^{\Pi}$$

and

$$N^s(X^{\bullet}_{f(I),0}) = 0$$

for $0 \le s \le |I|$.

Corollary *Let* G *be a connected, reductive linear algebraic group, defined and split over* $\mathbb{F}_q$ $(\delta = 1)$. *Then*

$$\begin{aligned} Q_\ell \cdot A^1(X^{\bullet}_{f,0}) &= H^2\big(X^{\bullet}_f, \mathbb{Q}_\ell(1)\big)^{\Pi}, \\ Q_\ell \cdot A^{r-1}(X^{\bullet}_{f,0}) &= H^{2(r-1)}\big(X^{\bullet}_f, \mathbb{Q}_\ell(r-1)\big)^{\Pi}, \\ N^{r-1}(X^{\bullet}_{f,0}) &= 0. \end{aligned}$$

4. Start of the proof

Lemma 1 (Lusztig [Lu, Section 6, Theorem 6.1(i), p. 135]) *If* G *is a simple algebraic group, then* $(F^\delta)^*$ *acts semisimplly on* $H^i_c(X_f)$, $i \ge 0$.

As X_f is an irreducible affine variety of dimension r, we have $H^i_c(X_f) = 0$ unless $r \le i \le 2r$. Let $i \in Z$, $r \le i \le 2r$. Let $\lambda_1, \dots, \lambda_{n_i}$ be all the eigenvalues of $(F^\delta)^*$ on $H^i_c(X_f)$, and for each $j \in Z$, $1 \le j \le n_i$, let $H^i_c(X_f)_{\lambda_j}$ be the generalized λ_j-eigenspace of $(F^\delta)^*$ on $H^i_c(X_f)$. Then Lusztig proves that the $H^i_c(X_f)_{\lambda_j}$ are mutually non-isomorphic irreducible representations of G^F. For $1 \le j \le n_i$, let $v_j \in H^i_c(X_f)_{\lambda_j}$ be an eigenvector of $(F^\delta)^*$ associated with λ_j. Then $\bar{Q}_\ell[G^F]v_j$ is a G^F-submodule of $H^i_c(X_f)_{\lambda_j}$. But, as $H^i_c(X_f)_{\lambda_j}$ is irreducible, we must have $\bar{Q}_\ell[G^F]v_j = H^i_c(X_f)_{\lambda_j}$. Therefore there are elements $g_1, \dots, g_t$ of G^F such that the vectors $g_1 v_j, \dots, g_t v_j$ form a basis of the vector space $H^i_c(X_f)_{\lambda_j}$ over $\bar{Q}_\ell$. Since the action of $(F^\delta)^*$ and that of G^F commute, we see that $g_1 v_j, \dots, g_t v_j$ are eigenvectors of $(F^\delta)^*$. Therefore $(F^\delta)^*$ acts semisimplly on $H^i_c(X_f)_{\lambda_j}$. This holds for all j. Therefore $(F^\delta)^*$ acts semisimplly on $H^i_c(X_f) = \bigoplus_{j=1}^{n_i} H^i_c(X_f)_{\lambda_j}$.

Proposition 1 *Let* $s \in Z$ *and let* $I \subset S_F$. *Then* $(F^\delta)^*$ *acts semisimplly on* $H^i_c(X^{\bullet}_f(I))(s) = H^i_c(X^{\bullet}_f(I)) \otimes \bar{Q}(s)$, $i \ge 0$.

Let $(F^\delta)^*_0$ be the action of $(F^\delta)^*$ on $H^i_c(X^{\bullet}_f(I))$. Then $(F^\delta)^*$ acts on $H^i_c(X^{\bullet}_f(I))(s)$ by $(F^\delta)^*_0 \otimes (q^\delta)^{-s}$ $(q^\delta = |k_0|)$. Therefore we may assume that $s = 0$.

We recall that there is an F^δ-equivariant isomorphism

$$X_f^\bullet(I) \xleftarrow{\sim} \coprod_{P\in\mathcal{P}_I^F} X_{f(P)}.$$

Therefore there are $(F^\delta)^*$-equivariant isomorphisms:

$$\begin{aligned} H_c^i(X_f^\bullet(I)) &\xleftarrow{\sim} H_c^i\Big(\coprod_{P\in\mathcal{P}_I^F} X_{f(P)}\Big) \\ &\xrightarrow{\sim} \bigoplus_{P\in\mathcal{P}_I^F} H_c^i(X_{f(P)}) \\ &= \bigoplus_{P\in\mathcal{P}_I^F} H_c^i\big(X_{f(L_P)}\big). \end{aligned}$$

Let $P \in \mathcal{P}_I^F$. Let δ_P be the minimal positive integer such that F^{δ_P} is the identity map on W_{L_P}. Then, as F^δ is the identity map on W_{L_P}, we have $\delta_P \le \delta$. Let $\delta = \delta_P t + \delta'$ with $t,\ \delta' \in Z$, $t,\ \delta' \ge 0$, $0 \le \delta' < \delta_P$. Let $w \in W_{L_P}$. Then $w = F^\delta(w) = F^{\delta_P t+\delta'}(w) = F^{\delta'}(w)$. Since $0 \le \delta' < \delta_P$, by the minimality of δ_P, we must have $\delta' = 0$. Therefore δ_P divides δ. Thus to prove the assertion, it suffices to show that, for each $P \in \mathcal{P}_I^F$, $(F^{\delta_P})^*$ acts on $H_c^i(X_{f(L_P)})$ semisimplly. Thus we are reduced to the case where $I = S_F$.

But, by the argument in (1.18) of [Lu], we are reduced to the case where G is a simple algebraic group of adjoint type. Thus the assertion follows from Lemma 1.

In the rest of this paper, we shall assume that G is a simple algebraic group.

We quote from [Lu, (7.3)] the following table on the eigenvalues of $(F^\delta)^*$ on $H_c^i(X_f)$, $r \le i \le 2r$. Each table consists of $r+1$ colums. In the first column (from the left) we record the eigenvalues of $(F^\delta)^*$ occuring in $H_c^r(X_f)$, in the second column we record the eigenvalues of $(F^\delta)^*$ occuring in $H_c^{r+1}(X_f)$ and so on. θ, i, ζ will denote a primitive root of 1 in $\bar{Q}_\ell^*$ of order 3, 4, 5 respectively.

A_n $(n \ge 1)$:	$1,$	$q,$	$q^2, \dots, q^n,$		
B_n $(n \ge 2)$:	$1,$	$q,$	$q^2, \dots, q^{n-2},$	$q^{n-1},$	$q^n,$
	$-q,$	$-q^2,$	$-q^3, \dots, -q^{n-1},$		

$$
\begin{array}{llllllll}
D_n\ (n \geq 4): & 1, & q, & q^2, \ldots, q^{n-4}, & q^{n-3}, & q^{n-2}, & q^{n-1}, & q^n, \\
& -q^2, & -q^3, & -q^4, \ldots, -q^{n-2}, & & & &
\end{array}
$$

$$
\begin{array}{llllllll}
E_6: & 1, & q, & q^2, & q^3, & q^4, & q^5, & q^6, \\
& -q^2, & -q^3, & -q^4, & & & & \\
& \theta q^3, & & & & & & \\
& \theta^2 q^3, & & & & & &
\end{array}
$$

$$
\begin{array}{lllllllll}
E_7: & 1, & q, & q^2, & q^3, & q^4, & q^5, & q^6, & q^7, \\
& -q^2, & -q^3, & -q^4, & -q^5, & & & & \\
& \theta q^3, & \theta q^4, & & & & & & \\
& \theta^2 q^3, & \theta^2 q^4, & & & & & & \\
& i q^{7/2}, & & & & & & & \\
& -i q^{7/2}, & & & & & & &
\end{array}
$$

$$
\begin{array}{llllllllll}
E_8: & 1, & q, & q^2, & q^3, & q^4, & q^5, & q^6, & q^7, & q^8, \\
& -q^2, & -q^3, & -q^4, & -q^5, & -q^6, & & & & \\
& \theta q^3, & \theta q^4, & \theta q^5, & & & & & & \\
& \theta^2 q^3, & \theta^2 q^4, & \theta^2 q^5, & & & & & & \\
& i q^{7/2}, & i q^{9/2}, & & & & & & & \\
& -i q^{7/2}, & -i q^{9/2}, & & & & & & & \\
& \zeta^j q^4, & & & & & & & & \\
& -\theta q^4, & & & & & & & & \\
& -\theta^2 q^4, & (j = 1, 2, 3, 4), & & & & & & &
\end{array}
$$

$$
\begin{array}{llllll}
F_4: & 1, & q, & q^2, & q^3, & q^4, \\
& -q, & -q^2, & -q^3, & & \\
& i q^2, & & & & \\
& -i q^2, & & & & \\
& \theta q^2, & & & & \\
& \theta^2 q^2, & & & &
\end{array}
$$

$$
\begin{array}{llll}
G_5: & 1, & q, & q^2, \\
& -q, & & \\
& \theta q, & & \\
& \theta^2 q^2, & &
\end{array}
$$

$^2A_{2n}$ $(n \geq 1)$: $\quad 1, \quad q^2, \ldots, q^{2n-2}, \quad q^{2n},$
$\quad -q, \quad -q^3, \ldots, -q^{2n-1},$

$^2A_{2n+1}$ $(n \geq 2)$: $\quad 1, \quad q^2, \ldots, q^{2n-4}, \quad q^{2n-2}, \quad q^{2n}, \quad q^{2n+2},$
$\quad -q^3, \quad -q^5, \ldots, -q^{2n-1},$

^2D_n $(n \geq 3)$: $\quad 1, \quad q^2, \quad q^4, \ldots, q^{2n-2},$

$^3D_4$: $\quad 1, \quad q^3, \quad q^6,$
$\quad -q^3,$

$^2E_6$: $\quad 1, \quad q^2, \quad q^4, \quad q^6, \quad q^8,$
$\quad -q^3, \quad -q^5,$
$\quad \theta q^4,$
$\quad \theta^2 q^4,$

$^2B_2$: $\quad 1, \quad q^2,$
$\quad \dfrac{i-1}{\sqrt{2}}q,$
$\quad \dfrac{-i-1}{\sqrt{2}}q,$

$^2F_4$: $\quad 1, \quad q^2, \quad q^4,$
$\quad \dfrac{i-1}{\sqrt{2}}q, \quad \dfrac{i-1}{\sqrt{2}}q^3,$
$\quad \dfrac{-i-1}{\sqrt{2}}q, \quad \dfrac{-i-1}{\sqrt{2}}q^3,$
$\quad -q^2,$
$\quad iq^2,$
$\quad -iq^2,$
$\quad -\theta q^2,$
$\quad -\theta^2 q^2,$

$^2G_2$: $1,\quad q^2,$

$iq,$

$-iq,$

$\dfrac{i-\sqrt{3}}{2}q,$

$\dfrac{-i-\sqrt{3}}{2}q,$

Tee following is the key lemma.

Lemma 2 *Recall that G is a simple algebraic group. Let $J \subset S_F$ be such that $1 \leq |J| \leq r$. Then, for $i \in Z$, $0 \leq i \leq |J|-1$, we have*

$$H_c^{2i}(X_f^\bullet(J))_{(q^\delta)^i} = 0$$

(*the subspace of $H_c^{2i}(X_f^\bullet(J))$ on which $(F^\delta)^*$ acts by multipication by $(q^\delta)^i$*).

We first treat the cases $^2B_2$, $^2G_2$, $^2F_4$.

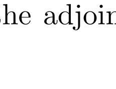
The case $^2B_2$ or $^2G_2$. We have $r=1$ and $X_f^\bullet(J) = X_f^\bullet(S_F) \simeq X_f$. X_f is an irreducible affine variety of dimension 1. Therefore $H_c^0(X_f^\bullet) = 0$.

The case $^2F_4$. We have $r=2$. Let $J = S_F$. Then $X_f^\bullet(J) \simeq X_f$ and X_f is an irreducible affine variety of dimension 2. Therefore $H_c^0(X_f) = 0$. Let $i=1$. Then the eigenvalues of $(F^2)^*$ ($\delta = 2$) on $H_c^2(X_f)$ are $\neq q^2$. Therefore $H_c^2(X_f)_{q^2} = 0$.

Let $|J| = 1$. We have an $(F^2)^*$-equivariant isomorphism

$$H_c^0(X_f^\bullet(J)) \overset{\sim}{\longleftarrow} \bigoplus_{P \in \mathcal{P}_J^F} H_c^0(X_{f(P)}).$$

Let $P \in \mathcal{P}_J^F$. Then, the Coxeter graph of the adjoint group L_P^{ad} of L_P is either

 or .

Therefore $X_{f(P)} \simeq X_{f(L_P^{ad})}$ is an irreducible affine variety of dimension 1. Therefore $H_c^0(X_{f(P)}) = 0$. Therefore $H_c^0(X_f^\bullet(J)) = 0$.

Next we treat the case where G is defined and split over F_q ($\delta = 1$).

The case A_n ($n \geq 1$). Let $0 \leq i \leq |J|-1$. We have an F^*-equivariant isomorphism

$$H_c^{2i}(X_f^\bullet(J)) \simeq \bigoplus_{P \in \mathcal{P}_J^F} H_c^{2i}(X_{f(P)}).$$

Let $P \in \mathcal{P}_J^F$. Then L_P^{ad} is of the form $G_1 \times \cdots \times G_m$, where, for $1 \leq j \leq m$, G_j is a simple algebraic group of type A_j with $r_j \geq 1$ and $r_1 + \cdots + r_m = |J|$. Therefore there is F-equivariant isomorphisms

$$X_{f(P)} \simeq X_{f(L_P^{ad})} \simeq X_{f_1} \times \cdots \times X_{f_m},$$

where, for $1 \leq j \leq m$, X_{f_j} is a variety for G_j similar to X_f for G. Then, by the Künneth formula, we have F^*-equivariant isomorphisms

$$\begin{aligned} H_c^{2i}(X_{f(P)}) &\simeq H_c^{2i}\big(X_{f_1} \times \cdots \times X_{f_m}\big) \\ &\bigoplus_{i_1 + \cdots + i_m = 2i} H_c^{i_1}\big(X_{f_1}\big) \otimes \cdots \otimes H_c^{i_m}\big(X_{f_m}\big). \end{aligned} \tag{$*$}$$

On each direct summand in the last term of $(*)$, F^* acts by the multiplication by

$$\begin{aligned} q^{i_1 - r_1} q^{i_2 - r_2} \dots q^{i_m - r_m} &= q^{(i_1 + \cdots + i_m) - (r_1 + \cdots + r_m)} \\ &= q^{2i - |J|} \neq q^i \qquad (\text{cf. } 0 \leq i \leq |J| - 1). \end{aligned}$$

Therefore

$$H_c^{2i}(X_{f(P)})_{q^i} = 0.$$

Therefore

$$H_c^{2i}(X_f^\bullet(J))_{q^i} \simeq \bigoplus_{P \in \mathcal{P}_J^F} H_c^{2i}(X_{f(P)})_{q^i} = 0.$$

The case B_n ($n \geq 2$). Let $P \in \mathcal{P}_J^F$. Then L_P^{ad} is of the form $G_1 \times \cdots \times G_m$, where either

(i) for $1 \leq j \leq m-1$, G_i is a simple algebraic group of type A_{r_j} with $r_j \geq 1$ and G_m is a simple algebraic group of type B_{r_m} with $r_m \geq 2$, and

$r_1 + \cdots + r_{m-1} + r_m = |J|$,

or

(ii) for $1 \leq j \leq m$, G_j is a simple algebraic group of type A_{r_j} with $r_j \geq 1$, and $r_1 + \cdots + r_m = |J|$.

We have a similar decomposition as $(*)$. In case (ii), F^* acts each direct summand by the multiplication by

$$q^{i_1 - r_1} \ldots q^{i_m - r_m} = q^{2i - |J|} \neq q^i.$$

In case (i), F^* acts by the multiplication by

$$q^{i_1 - r_1} \ldots q^{i_m - r_m} = q^{2i - |J|} \neq q^i$$

or

$$q^{i_1 - r_1} \ldots q^{i_{m-1} - r_{m-1}} (-q^{i_m - r_m + 1}) = -q^{2i - |J| + 1} \neq q^i.$$

Therefore $H_c^{2i}(X_{f(P)})_{q^i} = 0$. Thus

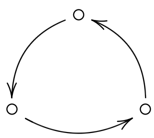
$$H_c^{2i}(X_f^{\bullet}(J))_{q^i} \simeq \bigoplus_{P \in \mathcal{P}_J^F} H_c^{2i}(X_{f(P)})_{q^i} = 0.$$

The remaining cases D_n, E_6, E_7, E_8, F_4 and G_2 can be treated similarly.

Thirdly we treat the non-split case.

The case ${}^3D_4$. We have $r = 2$. Let $J = S_F$. Then $X_f^{\bullet}(J) \simeq X_f$ and X_f is an irreducible affine variety of dimension 2. Therefore $H_c^0(X_f) = 0$. Let $i = 1$. Then the eigenvalues of $(F^3)^*$ $(\delta = 3)$ on $H_c^2(X_f)$ are $\neq q^3$. Therefore $H_c^2(X_f)_{q^3} = 0$.

Let $|J| = 1$. Let $P \in \mathcal{P}_J^F$. Then the Coxeter graph of L_P^{ad} is either

 or .

Therefore $X_{f(P)} \simeq X_{f(L_P^{ad})}$ is an irreducible affine variety of dimension 1. Therefore $H_c^0(X_{f(P)}) = 0$. Therefore

$$H_c^0(X_f^\bullet(J)) \simeq \bigoplus_{P\in\mathcal{P}_J^F} H_c^0(X_{f(P)}) = 0.$$

The case ${}^2A_{2n}$ $(n \geq 1)$. Let $P \in \mathcal{P}_J^F$. Then L_P^{ad} is of the form $G_1 \times \cdots \times G_m$, where either

(i) for $1 \leq j \leq m-1$, (G_i, F) is "isomorphic" to (A_{r_j}, F^2) for $r_j \geq 1$ and (G_m, F) is $({}^2A_{2r_m}, F)$ with $r_j \geq 1$, and $r_1 + \cdots + r_{m-1} + r_m = |J|$,

or

(ii) for $1 \leq j \leq m$, (G_j, F) is "isomorphic" to (A_{r_j}, F^2) for $r_j \geq 1$ and $r_1 + \cdots + r_m = |J|$.

We have a similar decomposition as $(*)$. In case (ii), on each direct summand, $(F^2)^*$ acts by the multiplication by

$$(q^2)^{i_1-r_1} \ldots (q^2)^{i_m-r_m} = (q^2)^{2i-|J|} \neq (q^2)^i.$$

In case (i), $(F^2)^*$ acts by the multiplication by

$$(q^2)^{i_1-r_1} \ldots (q^2)^{i_{m-1}-r_{m-1}} (q^2)^{i_m-r_m} = (q^2)^{2i-|J|} \neq (q^2)^i.$$

or

$$(q^2)^{i_1-r_1} \ldots (q^2)^{i_{m-1}-r_{m-1}} \big(-q^{2(i_m-r_m)+1}\big) \neq (q^2)^i.$$

Therefore $H_c^{2i}(X_{f(P)})_{(q^2)^i} = 0$. Therefore

$$H_c^{2i}(X_f^\bullet(J))_{(q^2)^i} \simeq \bigoplus_{P\in\mathcal{P}_J^F} H_c^{2i}(X_{f(P)})_{(q^2)^i} = 0.$$

The case ${}^2A_3$. This is the same as the case ${}^2D_3$. We have $r = 2$ and the eigenvalues of $(F^2)^*$ on $H_c^s(X_f)$ for $2 \leq s \leq 4$ are $1, q^2, q^4$, respectively. Thus, if $J = S_F$, then $X_f^\bullet(J) \simeq X_f$ and $H_c^0(X_f) = 0$ and $H_c^2(X_f)_{q^2} = 0$. Let $|J| = 1$. Let $P \in \mathcal{P}_J^F$. Then (L_P^{ad}, F) is "isomorphic" to (A_1, F^2). Therefore $X_{f(P)}$ is an irreducible affine variety of dimension 1. Therefore $H_c^0(X_{f(P)}) = 0$, and

$$H_c^0(X_f^\bullet(J)) \simeq \bigoplus_{P\in\mathcal{P}_J^F} H_c^0(X_{f(P)}) = 0.$$

The case ${}^2A_{2n+1}$ ($n \geq 2$). The first row in the table of the eigenvalues of $(F^2)^*$ on $H_c^s(X_f)$ is the same as that of (A_{n+1}, F^2) and any eigenvalue of $(F^2)^*$ in the second row is empty or of the form $(-1) \times$ (power of q). Therefore

$$H_c^{2i}(X_f^\bullet(J))_{(q^2)^i} \simeq \bigoplus_{P \in \mathcal{P}_J^F} H_c^{2i}(X_{f(P)})_{(q^2)^i} = 0.$$

The remaining cases ${}^2D_n$, ${}^2E_6$ can be treated similarly.

This completes the proof of Lemma 2.

Proposition 2 *Recall that G is a simple algebraic group. Let J be a subset of S_F such that $1 \leq |J| \leq r$. Then, for an integer a, $0 \leq a \leq |J|$, and for any integer i, $0 \leq i \leq a$, $i \leq |J| - 1$, $(F^\delta)^*$ acts semisimplly on $H^{2i}(D_a(J), \mathbb{Q}_\ell(i))_1$.*

Proposition 3 *Let J be any subset of S_F. Then, for any integer i, $0 \leq i \leq |J|$, $(F^\delta)^*$ acts semisimplly on $H^{2i}(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(i))_1$. Thus $H^{2i}(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(i))_1 = H^{2i}(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(i))^\varphi = H^{2i}(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(i))^\Pi$.*

Let $1 \leq |J| \leq r$ and let $0 \leq a \leq |J|$. Then the inclusions

$$D_a(J) - D_{a-1}(J) = \coprod_{\substack{J' \subset J \\ |J'| = a}} X_f^\bullet(J') \underset{\text{open}}{\hookrightarrow} D_a(J) \underset{\text{closed}}{\hookleftarrow} D_{a-1}(J)$$

give $(F^\delta)^*$-equivariant exact sequences:

$$\begin{aligned}
&H_c^{2i}(X_f^\bullet(J), \mathbb{Q}_\ell(i))_1 \\
&\qquad \longrightarrow H^{2i}(D_{|J|}(J), \mathbb{Q}_\ell(i))_1 \longrightarrow H^{2i}(D_{|J|-1}(J), \mathbb{Q}_\ell(i))_1, \\
&\bigoplus_{\substack{J' \subset J \\ |J'| = |J|-1}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1 \\
&\qquad \longrightarrow H^{2i}(D_{|J|-1}(J), \mathbb{Q}_\ell(i))_1 \longrightarrow H^{2i}(D_{|J|-2}(J), \mathbb{Q}_\ell(i))_1, \\
&\qquad\qquad \vdots \\
&\bigoplus_{\substack{J' \subset J \\ |J'| = i+1}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1 \\
&\qquad \longrightarrow H^{2i}(D_{i+1}(J), \mathbb{Q}_\ell(i))_1 \longrightarrow H^{2i}(D_i(J), \mathbb{Q}_\ell(i))_1,
\end{aligned}$$

$$\bigoplus_{\substack{J' \subset J \\ |J'|=i}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1 \longrightarrow H^{2i}(D_i(J), \mathbb{Q}_\ell(i))_1 \longrightarrow H^{2i}(D_{i-1}(J), \mathbb{Q}_\ell(i))_1 = 0.$$

By Proposition 1, we see that $(F^\delta)^*$ acts semisimplly on $\bigoplus_{\substack{J' \subset J \\ |J'|=i}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1$. Therefore we see from the last exact sequence that $(F^\delta)^*$ acts semisimplly on $H^{2i}(D_i(J), \mathbb{Q}_\ell(i))_1$. Since $i \leq |J|-1$, by Lemma 2, we have $H^{2i}(X_f^\bullet(J), \mathbb{Q}_\ell(i))_1 = 0$, $\bigoplus_{\substack{J' \subset J \\ |J'|=|J|-1}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1 = 0, \ldots, \bigoplus_{\substack{J' \subset J \\ |J'|=i+1}} H_c^{2i}(X_f^\bullet(J'), \mathbb{Q}_\ell(i))_1 = 0$. Therefore, by the second exact sequence from the bottom, we see that $(F^\delta)^*$ acts semisimplly on $H^{2i}(D_{i+1}(J), \mathbb{Q}_\ell(i))_1$. By the third exact sequence from the bottom, we see that $(F^\delta)^*$ acts semisimplly on $H^{2i}(D_{i+2}(J), \mathbb{Q}_\ell(i))_1$. ... By the last exact sequence from the bottom, we see that $(F^\delta)^*$ acts semisimplly on $H^{2i}(D_{|J|}(J), \mathbb{Q}_\ell(i))_1$. We note that $H^{2i}(D_{i'}(J), \mathbb{Q}_\ell(i))_1 = 0$ for $i' < i$.

This proves Proposition 2.

Next we prove Proposition 3. Since $D_{|J|}(J) = \bar{X}_f^\bullet(J)$, for $1 \leq |J| \leq r$ and for $1 \leq i \leq |J|-1$, the assertion follows from Proposition 2.

Let $i = |J|$. Then

$$H^{2i}(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(i)) \xrightarrow{\sim} \bigoplus_{P \in \mathcal{P}_J^F} H^{2i}(X_{f(P)}^\bullet, \mathbb{Q}_\ell(i)) \xrightarrow{\sim} \bigoplus_{P \in \mathcal{P}_J^F} \mathbb{Q}_\ell$$

($(F^\delta)^*$-equivariant). Thus the assertion holds for $1 \leq |J| \leq r$ and for $0 \leq i \leq |J|$.

Finally, let $|J| = 0$. Then $\bar{X}_f^\bullet(J) = \bar{X}_f^\bullet(\emptyset) = X_f^\bullet(\emptyset) = X_G^F$, and

$$H^0\big(\bar{X}_f^\bullet(\emptyset), \mathbb{Q}_\ell\big) = \bigoplus^{|X_f^\bullet(\emptyset)|} \mathbb{Q}_\ell,$$

on which $(F^\delta)^*$ acts trivially.

The final assertion follows from the fact that $(F^\delta)^* = \varphi^{-1}$ on the ℓ-adic cohomologies.

This proves Proposition 3.

5. End of the proof

Recall that G is a simple algebraic group. For an integer t, $0 \le t \le r$, I_t denotes a subset of S_F such that $|I_t| = r - t$.

There is a natural closed immersion $\bar{X}^\bullet_f(I_1)_0 \hookrightarrow \bar{X}^\bullet_f(I_0)_0 = \bar{X}^\bullet_f(S_F)_0 = X^\bullet_{f,0}$. Therefore there is a natural morphism

$$g_{1,0} : Z_{1,0} = \coprod_{I_1} \bar{X}^\bullet_f(I_1)_0 \longrightarrow Z_{0,0} = \bar{X}^\bullet_f(I_0)_0.$$

For $I_2 \subset I_1$, there is a natural closed immersion $\bar{X}^\bullet_f(I_2)_0 \hookrightarrow \bar{X}^\bullet_f(I_1)_0$. Therefore there is a natural morphism

$$g_{2,0} : Z_{2,0} = \coprod_{I_1} \coprod_{I_2 \subset I_1} \bar{X}^\bullet_f(I_2)_0 \longrightarrow Z_{1,0} = \coprod_{I_1} \bar{X}^\bullet_f(I_1)_0.$$

Similarly we obtain natural morphisms

$$g_{3,0} : Z_{3,0} = \coprod_{I_1} \coprod_{I_2 \subset I_1} \coprod_{I_3 \subset I_2} \bar{X}^\bullet_f(I_3)_0 \longrightarrow Z_{2,0},$$

$$g_{4,0} : Z_{4,0} = \coprod_{I_1} \coprod_{I_2 \subset I_1} \coprod_{I_3 \subset I_2} \coprod_{I_4 \subset I_3} \bar{X}^\bullet_f(I_4)_0 \longrightarrow Z_{3,0},$$

$$\vdots\ .$$

For an integer j, $j \ge 0$, let

$$Z_j = Z_{j,0} \times_{k_0} k = \coprod_{I_1} \coprod_{I_2 \subset I_1} \cdots \coprod_{I_j \subset I_{j-1}} \bar{X}^\bullet_f(I_j)$$

and, for $j \ge 1$, let

$$g_j = g_{j,0} \times_{k_0} k : Z_j \longrightarrow Z_{j-1}.$$

Then, for an integer s, $0 \le s \le r$, we obtain the following commutative diagram:

$$\begin{tikzcd}
Z^0(Z_{s,0}) \arrow[r, "\mathrm{cl}^0_{Z_{s,0}}"] \arrow[d, "(g_{s,0})_*"'] & H^0(Z_s, \mathbb{Q}_\ell)^\Pi \arrow[d, "(g_s)_*"] \\
Z^1(Z_{s-1,0}) \arrow[r, "\mathrm{cl}^1_{Z_{s-1,0}}"] \arrow[d, "(g_{s-1,0})_*"'] & H^2(Z_{s-1}, \mathbb{Q}_\ell(1))^\Pi \arrow[d, "(g_{s-1})_*"] \\
Z^2(Z_{s-2,0}) \arrow[r, "\mathrm{cl}^2_{Z_{s-2,0}}"] \arrow[d, "(g_{s-2,0})_*"'] & H^4(Z_{s-2}, \mathbb{Q}_\ell(2))^\Pi \arrow[d, "(g_{s-2})_*"] \\
\vdots \arrow[d, "(g_{2,0})_*"'] & \vdots \arrow[d, "(g_2)_*"] \\
Z^{s-1}(Z_{1,0}) \arrow[r, "\mathrm{cl}^{s-1}_{Z_{1,0}}"] \arrow[d, "(g_{1,0})_*"'] & H^{2(s-1)}(Z_1, \mathbb{Q}_\ell(s-1))^\Pi \arrow[d, "(g_1)_*"] \\
Z^s(Z_{0,0}) \arrow[r, "\mathrm{cl}^s_{Z_{0,0}}"] & H^{2s}(Z_0, \mathbb{Q}_\ell(s))^\Pi .
\end{tikzcd}$$

Firstly, since

$$\coprod_{P \in \mathcal{P}^F_{I_s}} X^\bullet_{f(P)} \xrightarrow{\sim} \bar{X}^\bullet_f(I_s)$$

is an F^δ-equivariant isomorphism, we have an isomorphism

$$\coprod_{P \in \mathcal{P}^F_{I_s}} X^\bullet_{f(P),0} \xrightarrow{\sim} \bar{X}^\bullet_f(I_s)_0,$$

so we have isomorphisms

$$\begin{aligned}
Z^0(Z_{s,0}) &= Z^0\Big(\coprod_{I_1} \coprod_{I_2 \subset I_1} \cdots \coprod_{I_s \subset I_{s-1}} \bar{X}^\bullet_f(I_s)_0\Big) \\
&= \bigoplus_{I_1} \bigoplus_{I_2 \subset I_1} \cdots \bigoplus_{I_s \subset I_{s-1}} Z^0\big(\bar{X}^\bullet_f(I_s)_0\big) \\
&\cong \bigoplus_{I_1} \bigoplus_{I_2 \subset I_1} \cdots \bigoplus_{I_s \subset I_{s-1}} \bigoplus_{P \in \mathcal{P}^F_{I_s}} Z^0(X^\bullet_{f(P)}) \cong \bigoplus_{I_1} \bigoplus_{I_2 \subset I_1} \cdots \bigoplus_{I_s \subset I_{s-1}} \bigoplus_{P \in \mathcal{P}^F_{I_s}} \mathbb{Z}
\end{aligned}$$

and

$$H^0(Z_s, Q_\ell)^\Pi \cong \bigoplus_{I_1} \bigoplus_{I_2 \subset I_1} \cdots \bigoplus_{I_s \subset I_{s-1}} \bigoplus_{P \in \mathcal{P}^F_{I_s}} H^0(X^\bullet_{f(P)}, \mathbb{Q}_\ell)^\Pi$$
$$\cong \bigoplus_{I_1} \bigoplus_{I_2 \subset I_1} \cdots \bigoplus_{I_s \subset I_{s-1}} \bigoplus_{P \in \mathcal{P}^F_{I_s}} \mathbb{Q}_\ell.$$

Therefore, as $\mathrm{cl}^0_{X^\bullet_{f(P),0}}$ is the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell$, we see that $\mathrm{cl}^0_{Z_{s,0}} \otimes \mathbb{Q}_\ell$ is an isomorphism.

We show that $(g_s)_*$, $(g_{s-1})_*, \ldots, (g_1)_*$ are surjective, which will imply that $\mathrm{cl}^s_{Z_{0,0}} \otimes \mathbb{Q}_\ell = \mathrm{cl}^s_{X^\bullet_{f,0}} \otimes \mathbb{Q}_\ell$ is surjective.

Let $1 \le s \le r$ and $1 \le j \le s$. Then the homomorphism

$$(g_j)_* : H^{2(s-j)}(Z_j, \mathbb{Q}_\ell(s-j))^\Pi \longrightarrow H^{2(s-j+1)}(Z_{j-1}, \mathbb{Q}_\ell(s-j+1))^\Pi$$

is the dual map of the homomorphism

$$(g_j)^* : H^{2(r-s)}(Z_{j-1}, \mathbb{Q}_\ell(r-s))^\Pi \longrightarrow H^{2(r-s)}(Z_j, \mathbb{Q}_\ell(r-s))^\Pi$$

(cf. Proposition 3). Therefore, to see that $(g_j)_*$ is surjective, it suffices to show that $(g_j)^*$ is injective. To see it, it suffices to show that, for any I_{j-1}, the homomorphism

$$(g_j)^* : H^{2(r-s)}(\bar{X}^\bullet_f(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \longrightarrow H^{2(r-s)}\Big(\coprod_{I_j \subset I_{j-1}} \bar{X}^\bullet_f(I_j), \mathbb{Q}_\ell(r-s)\Big)^\Pi$$

is injective $(0 \le j-1 \le s-1)$.

We have the following commutative diagram

$$\begin{tikzcd}
\coprod_{I_j \subset I_{j-1}} \bar{X}^\bullet_f(I_j) \arrow[rr, "g_j"] \arrow[dr, "h_j"'] & & \bar{X}^\bullet_f(I_{j-1}), \\
& \mathrm{Im}\, g_j \arrow[ur, hook, "i_j"'] &
\end{tikzcd}$$

where i_j is the closed immersion of $\operatorname{Im} g_j$ into $\bar{X}_f^\bullet(I_{j-1})$ and h_j is the restriction of g_j ($h_j = g_j$ but the image is restricted). Therefore we obtain the following commutative diagram:

$$\begin{tikzcd}
H^{2(r-s)}\Big(\coprod_{I_j \subset I_{j-1}} \bar{X}_f^\bullet(I_j), \mathbb{Q}_\ell(r-s)\Big)^\Pi & & H^{2(r-s)}(\bar{X}_f^\bullet(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \arrow[ll, "(g_j)^*"'] \arrow[dl, "(i_j)^*"] \\
& H^{2(r-s)}(\operatorname{Im} g_j, \mathbb{Q}_\ell(r-s))^\Pi \arrow[ul, "(h_j)^*"] &
\end{tikzcd}$$

Therefore it suffices to show that $(i_j)^*$ and $(h_j)^*$ are injective. We note that

$$\operatorname{Im} g_j = D_{r-j}(I_{j-1}).$$

In fact, let $(B_0, \dots, B_r) \in \operatorname{Im} g_j$. Then $(B_0, \dots, B_r) \in \bar{X}_f^\bullet(I_j)$ for some $I_j \subset I_{j-1}$. Let $J = \{O_i \mid 1 \leq i \leq r, B_{i-1} \neq B_i\}$. Then $a = |J| \leq r - j$, $J \subset I_{j-1}$ and $(B_0, \dots, B_r) \in X_f^\bullet(J) \subset D_a(J) \subset D_{r-j}(I_{j-1})$. Conversely, let $(B_0, \dots, B_r) \in D_{r-j}(I_{j-1})$. Then $(B_0, \dots, B_r) \in X_f^\bullet(J)$ for some $J \subset I_{j-1}$ with $|J| \leq r - j$. We have $X_f^\bullet(J) \subset \bar{X}_f^\bullet(J) \subset \bar{X}_f^\bullet(I_j)$ for some $I_j \subset I_{j-1}$. Therefore $(B_0, \dots, B_r) \in \operatorname{Im} g_j$.

Thus the map $(i_j)^*$ is the map

$$H^{2(r-s)}(\bar{X}_f^\bullet(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \longrightarrow H^{2(r-s)}(D_{r-j}(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi.$$

Since $\bar{X}_f^\bullet(I_{j-1}) = D_{r-(j-1)}(I_{j-1})$, $(i_j)^*$ is a part of the exact sequence

$$\begin{aligned} H_c^{2(r-s)}(X_f^\bullet(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi &\longrightarrow H^{2(r-s)}(D_{r-(j-1)}(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \\ &\longrightarrow H^{2(r-s)}(D_{r-j}(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \end{aligned}$$

which is obtained from the inclusions

$$\begin{aligned} X_f^\bullet(I_{j-1}) &= D_{r-(j-1)}(I_{j-1}) - D_{r-j}(I_{j-1}) \\ &\hookrightarrow D_{r-(j-1)}(I_{j-1}) \hookleftarrow D_{r-j}(I_{j-1}). \end{aligned}$$

But, as $r - s < r - (j-1)$ (cf. $j - 1 < s$), we have $H_c^{2(r-s)}(X_f^\bullet(I_{j-1}))_{(q^\delta)^{r-s}}$

$= 0$ by lemma 2. Therefore $H_c^{2(r-s)}(X_f^\bullet(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi = H_c^{2(r-s)}(X_f^\bullet(I_{j-1}), \mathbb{Q}_\ell(r-s))_1 = 0$. Therefore $(i_j)^*$ is injective.

Therefore it remains to show that the map

$$(h_j)^* : H^{2(r-s)}(D_{r-j}(I_{j-1}), \mathbb{Q}_\ell(r-s))^\Pi \longrightarrow H^{2(r-s)}\Big(\coprod_{I_j \subset I_{j-1}} \bar{X}_f^\bullet(I_j), \mathbb{Q}_\ell(r-s)\Big)^\Pi$$

is injective.

Suppose that $r = 1$. Then $s = 1$ and $j = 1$ (recall that $1 \le s \le r$ and $1 \le j \le s$). The map $(h_j)^* = (h_1)^*$ is

$$H^0(D_0(I_0), \mathbb{Q}_\ell)^\Pi \longrightarrow H^0\Big(\coprod_{I_1 \subset I_0} \bar{X}_f^\bullet(I_1), \mathbb{Q}_\ell\Big)^\Pi.$$

We have $D_0(I_0) = X_f^\bullet(\emptyset)$ and $\coprod_{I_1 \subset I_0} \bar{X}_f^\bullet(I_1) = X_f^\bullet(\emptyset)$. Therefore $(h_1)^*$ is the identity map.

Suppose that $r \ge 2$. First, let $j = s$:

$$h_s : \coprod_{I_s \subset I_{s-1}} \bar{X}_f^\bullet(I_s) \longrightarrow D_{r-s}(I_{s-1}).$$

Put:

$$Y_s = \coprod_{I_s \subset I_{s-1}} \bar{X}_f^\bullet(I_s),$$

$$U_s = \coprod_{I_s \subset I_{s-1}} X_f^\bullet(I_s) \quad (\underset{\text{open}}{\subset} Y_s),$$

$$W_s = Y_s - U_s = \coprod_{I_s \subset I_{s-1}} \big(\bar{X}_f^\bullet(I_s) - X_f^\bullet(I_s)\big) = \coprod_{I_s \subset I_{s-1}} D_{r-s-1}(I_s).$$

Then U_s is open in $D_{r-s}(I_{s-1})$ and $D_{r-s}(I_{s-1}) - U_s = D_{r-s-1}(I_{s-1})$. There is a commutative diagram

$$\begin{tikzcd}
U_s \arrow[r, hook, "\text{open}"'] \arrow[d, equal] & Y_s \arrow[d, "h_s"'] & W_s \arrow[l, hook', "\text{closed}"'] \arrow[d, "h_s|W_s"] \\
U_s \arrow[r, hook, "\text{open}"'] & D_{r-s}(I_{s-1}) & D_{r-s-1}(I_{s-1}). \arrow[l, hook', "\text{closed}"']
\end{tikzcd} \tag{5.1}$$

We note that $\dim W_s = \dim D_{r-s-1}(I_{s-1}) = r-s-1$ and $2(r-s-1) < 2(r-s)-1 < 2(r-s)$. Therefore $H^{2(r-s)-1}(W_s) = H^{2(r-s)}(W_s) = H^{2(r-s)-1}(D_{r-s-1}(I_{s-1})) = H^{2(r-s)}(D_{r-s-1}(I_{s-1})) = 0$. Put $D = D_{r-s}(I_{s-1})$ and $D' = D_{r-s-1}(I_{s-1})$. Then we obtain from (5.1) the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 = H^{2(r-s)-1}(W_s, \mathbb{Q}_\ell(r-s)) \arrow[r] & H_c^{2(r-s)}(U_s, \mathbb{Q}_\ell(r-s)) \\
0 = H^{2(r-s)-1}(D', \mathbb{Q}_\ell(r-s)) \arrow[r] \arrow[u, "(h_s|W_s)^*"] & H_c^{2(r-s)}(U_s, \mathbb{Q}_\ell(r-s)) \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & H^{2(r-s)}(Y_s, \mathbb{Q}_\ell(r-s)) \arrow[r] & H^{2(r-s)}(W_s, \mathbb{Q}_\ell(r-s)) = 0 \\
\arrow[r] & H^{2(r-s)}(D, \mathbb{Q}_\ell(r-s)) \arrow[r] \arrow[u, "h_s^*"] & H^{2(r-s)}(D', \mathbb{Q}_\ell(r-s)) = 0. \arrow[u, "(h_s|W_s)^*"']
\end{tikzcd}$$

Therefore

$$(h_s)^* : H^{2(r-s)}(D, \mathbb{Q}_\ell(r-s)) \longrightarrow H^{2(r-s)}(Y_s, \mathbb{Q}_\ell(r-s))$$

is an isomorphism. Therefore

$$(h_s)^* : H^{2(r-s)}(D, \mathbb{Q}_\ell(r-s))^\Pi \longrightarrow H^{2(r-s)}(Y_s, \mathbb{Q}_\ell(r-s))^\Pi$$

is injective.

Let $1 \le j \le s-1$. Put:

$$Z^{(0)} = \coprod_{I_j \subset I_{j-1}} \bar{X}_f^\bullet(I_j) = \coprod_{I_j \subset I_{j-1}} D_{r-j}(I_j),$$

$$Z^{(t)} = \coprod_{I_j \subset I_{j-1}} D_{r-j-t}(I_j) \qquad (t \ge 1),$$

$$\begin{aligned}
U^{(t)} &= Z^{(t)} - Z^{(t+1)} = \coprod_{I_j \subset I_{j-1}} (D_{r-j-t}(I_j) - D_{r-j-t-1}(I_j)) \\
&= \coprod_{I_j \subset I_{j-1}} \coprod_{\substack{J \subset I_j \\ |J| = r-j-t}} X_f^{\bullet}(J) \qquad (t \geq 0) \qquad\qquad (\text{open in } Z^{(t)}), \\
D^{(t)} &= D_{r-j-t}(I_{j-1}) \qquad (t \geq 0), \\
V^{(t)} &= D^{(t)} - D^{(t+1)} = \coprod_{\substack{J \subset I_{j-1} \\ |J| = r-j-t}} X_f^{\bullet}(J) \qquad (t \geq 0) \qquad (\text{open in } D^{(t)}).
\end{aligned}$$

For $t \geq 0$, let $h^{(t)} : Z^{(t)} \to D^{(t)}$ be the natural morphism, and let $u^{(t)} = h^{(t)} \mid U^{(t)} : U^{(t)} \to V^{(t)}$. Then we have the following commutative diagram ($t \geq 0$):

$$\begin{tikzcd}
U^{(t)} \arrow[r, hook, "\text{open}"'] \arrow[d, "u^{(t)}"'] & Z^{(t)} \arrow[d, "h^{(t)}"'] & Z^{(t+1)} \arrow[l, hook', "\text{closed}"'] \arrow[d, "h^{(t+1)}"] \\
V^{(t)} \arrow[r, hook, "\text{open}"'] & D^{(t)} & D^{(t+1)}. \arrow[l, hook', "\text{closed}"']
\end{tikzcd}$$

Therefore we obtain the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
H^{2(r-s)-1}(Z^{(t+1)}, \mathbb{Q}_\ell(r-s)) \arrow[r] & H_c^{2(r-s)}(U^{(t)}, \mathbb{Q}_\ell(r-s)) \\
H^{2(r-s)-1}(D^{(t+1)}, \mathbb{Q}_\ell(r-s)) \arrow[r] \arrow[u, "h^{(t+1)*}"] & H_c^{2(r-s)}(V^{(t)}, \mathbb{Q}_\ell(r-s)) \arrow[u, "h^{(t)*}"']
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^{2(r-s)}(Z^{(t)}, \mathbb{Q}_\ell(r-s)) \arrow[r] & H^{2(r-s)}(Z^{(t+1)}, \mathbb{Q}_\ell(r-s)) \\
{} \arrow[r] & H^{2(r-s)}(D^{(t)}, \mathbb{Q}_\ell(r-s)) \arrow[r] \arrow[u, "h^{(t)*}"] & H^{2(r-s)}(D^{(t+1)}, \mathbb{Q}_\ell(r-s)). \arrow[u, "h^{(t+1)*}"']
\end{tikzcd} \tag{5.2}$$

Let $0 \leq t \leq s-j$. We show, by descending induction on t, that $h^{(t)*} : H^{2(r-s)}(D^{(t)}, \mathbb{Q}_\ell(r-s))^{\Pi} \to H^{2(r-s)}(Z^{(t)}, \mathbb{Q}_\ell(r-s))^{\Pi}$ is injective, which will imply that $(h_j)^* = h^{(0)*}$ is injective.

In fact, let $t = s-j$. Then, as $\dim Z^{(t+1)} = \dim D^{(t+1)} = r-j-(t+1) = r-s-1 < (r-s)-\frac{1}{2}$, we have $H^{2(r-s)-1}(Z^{(t+1)}) = H^{2(r-s)}(Z^{(t+1)}) = H^{2(r-s)-1}(D^{(t+1)}) = H^{2(r-s)}(D^{(t+1)}) = 0$. Moreover there is a morphism $v^{(t)} : V^{(t)} = \coprod_{\substack{J\subset I_{j-1}\\ |J|=r-j-t}} X_f^{\bullet}(J) \longrightarrow U^{(t)} = \coprod_{I_j\subset I_{j-1}} \coprod_{\substack{J\subset I_j\\ |J|=r-j-t}} X_f^{\bullet}(J)$ such that $u^{(t)}v^{(t)} = id_{V^{(t)}}$. Therefore $id_{H_c^{2(r-s)}(V^{(t)})} = (id_{V^{(t)}})^* = (u^{(t)}v^{(t)})^* = v^{(t)*}u^{(t)*}$, and $u^{(t)*}$ is injective. Therefore $h^{(t)*}$ is injective.

Let $0 \le t < s-j$. Then, by Lemma 2, we have $H_c^{2(r-s)}(U^{(t)}, \mathbb{Q}_\ell(r-s))^\Pi = H_c^{2(r-s)}(V^{(t)}, \mathbb{Q}_\ell(r-s))^\Pi = 0$. Therefore we obtain from (5.2) the following commutative diagram whose rows are exact:

$$\begin{array}{ccc} 0 \to H^{2(r-s)}(Z^{(t)}, \mathbb{Q}_\ell(r-s))^\Pi & \longrightarrow & H^{2(r-s)}(Z^{(t+1)}, \mathbb{Q}_\ell(r-s))^\Pi \\ \uparrow{\scriptstyle h^{(t)*}} & & \uparrow{\scriptstyle h^{(t+1)*}} \\ 0 \to H^{2(r-s)}(D^{(t)}, \mathbb{Q}_\ell(r-s))^\Pi & \longrightarrow & H^{2(r-s)}(D^{(t+1)}, \mathbb{Q}_\ell(r-s))^\Pi. \end{array}$$

By induction hypothesis, $h^{(t+1)*}$ is injective. Therefore $h^{(t)*}$ is injective.

We see from the above proof that the map

$$(g_j)_* : H^{2(s-j)}\Big(\coprod_{I_j\subset I_{j-1}} \bar{X}_f^{\bullet}(I_j), \mathbb{Q}_\ell(s-j)\Big)^\Pi \longrightarrow H^{2(s-j+1)}\big(\bar{X}_f^{\bullet}(I_{j-1}), \mathbb{Q}_\ell(s-j+1)\big)^\Pi$$

is surjective for $1 \le s \le r$ and $1 \le j \le s$. Therefore the composite

$$\begin{array}{l} H^0 = H^{2(s-s)}\Big(\coprod_{I_j\subset I_{j-1}} \coprod_{I_{j+1}\subset I_j} \cdots \coprod_{I_s\subset I_{s-1}} \bar{X}_f^{\bullet}(I_s), \mathbb{Q}_\ell(s-s)\Big)^\Pi \\ \qquad\qquad \downarrow \\ H^{2(s-(s-1))}\Big(\coprod_{I_j\subset I_{j-1}} \coprod_{I_{j+1}\subset I_j} \cdots \coprod_{I_{s-1}\subset I_{s-2}} \bar{X}_f^{\bullet}(I_{s-1}), \mathbb{Q}_\ell(1)\Big)^\Pi \\ \qquad\qquad \downarrow \\ \qquad\qquad \vdots \\ \qquad\qquad \downarrow \end{array} \quad g_* \big\downarrow$$

$$\begin{tikzcd}
H^{2(s-j-1)}\Big(\coprod_{I_j\subset I_{j-1}}\coprod_{I_{j+1}\subset I_j}\bar{X}_f^{\bullet}(I_{j+1}),\mathbb{Q}_\ell(s-j-1)\Big)^{\Pi} \arrow[d] \arrow[dd, bend right] \\
H^{2(s-j)}\Big(\coprod_{I_j\subset I_{j-1}}\bar{X}_f^{\bullet}(I_j),\mathbb{Q}_\ell(s-j)\Big)^{\Pi} \arrow[d] \\
H = H^{2(s-j+1)}\big(\bar{X}_f^{\bullet}(I_{j-1}),\mathbb{Q}_\ell(s-j+1)\big)^{\Pi}
\end{tikzcd}$$

is surjective. We have the following commutative diagram

$$\begin{tikzcd}
Z^0\Big(\coprod_{I_j\subset I_{j-1}}\cdots\coprod_{I_s\subset I_{s-1}}\bar{X}_f^{\bullet}(I_s)_0\Big) \arrow[r, "\mathrm{cl}^0"] \arrow[d, "g_{0*}"'] & H^0 \arrow[d, "g_*"] \\
Z^{s-j+1}\big(\bar{X}_f^{\bullet}(I_{j-1})_0\big) \arrow[r, "\mathrm{cl}^{s-j+1}"'] & H
\end{tikzcd}$$

where g_{0*} is the composite of

$$\begin{aligned}
Z^0\Big(\coprod_{I_j\subset I_{j-1}}\cdots\coprod_{I_s\subset I_{s-1}}\bar{X}_f^{\bullet}(I_s)_0\Big) &\longrightarrow Z^1\Big(\coprod_{I_j\subset I_{j-1}}\cdots\coprod_{I_{s-1}\subset I_{s-2}}\bar{X}_f^{\bullet}(I_{s-1})_0\Big)\\
&\longrightarrow\cdots\longrightarrow Z^{s-j+1}\big(\bar{X}_f^{\bullet}(I_{j-1})_0\big).
\end{aligned}$$

Clearly $\mathrm{cl}^0\otimes\mathbb{Q}_\ell$ is an isomorphism. Therefore

$$\begin{aligned}
\mathrm{cl}^{s-j+1}_{\bar{X}_f^{\bullet}(I_{j-1})_0}\otimes\mathbb{Q}_\ell : Z^{s-j+1}\big(\bar{X}_f^{\bullet}(I_{j-1})_0\big)\otimes\mathbb{Q}_\ell&\\
\longrightarrow H^{2(s-j+1)}\big(\bar{X}_f^{\bullet}(I_{j-1}),\mathbb{Q}_\ell(s-j+1)\big)^{\Pi}&
\end{aligned}$$

is surjective for $1\le s\le r$ and $1\le j\le s$. Therefore, for any $J\subset S_F$ with $1\le|J|\le r$, and for any integer t, $1\le t\le|J|$, the map

$$\mathrm{cl}^t_{\bar{X}_f^{\bullet}(J)_0}\otimes\mathbb{Q}_\ell : Z^t\big(\bar{X}_f^{\bullet}(J)_0\big)\otimes\mathbb{Q}_\ell\longrightarrow H^{2t}\big(\bar{X}_f^{\bullet}(J),\mathbb{Q}_\ell(t)\big)^{\Pi}$$

is surjective. This is also true for $1\le|J|\le r$ and for $0\le t\le|J|$, and true for $J=\emptyset$ and for $t=0$. Therefore, for any $J\subset S_F$ and for any integer t,

$1 \leq t \leq |J|$, we have

$$\mathbb{Q}_\ell \cdot A^t\big(\bar{X}_f^\bullet(J)_0\big) = H^{2t}\big(\bar{X}_f^\bullet(J), \mathbb{Q}_\ell(t)\big)^{\Pi}.$$

In view of Propositiion 3, we see from the non-degenerateness of the pairing $\langle\ ,\ \rangle_{\bar{X}_f^\bullet(J),1}$, that

$$N^t\big(\bar{X}_f^\bullet(J)_0\big) = 0$$

for any $J \subset S_F$ any for any integer t, $0 \leq t \leq |J|$.

This completes the proof of Theorem 1.

The corollary follows from Theorem 1 by [Ta II, Proposition (5.1), Theorem (5.2)].

References

[Bo] Borel A., *Linear algebraic groups.* Benjamin, 1969.

[Bour] Bourbaki N., *Groupes et algèbres de Lie*, Chapitres 4, 5 e 6. 1st ed., Hermann, Paris, 1968.

[Ca] Carter R.W., *Finite groups of Lie type-Conjugacy classes and complex characters.* John Wiley and Sons, 1985.

[DL] Deligne P. and Lusztig G., *Representations of reductive groups over finite fields.* Ann. of Math. **103** (1976), 103–161.

[Fu] Fulton W., *Intersection theory. Ergebniss der Mathematik und ihren Grenzgebiete*, 3. Folge, Band 2, Springer-Verlag, 1984.

[Lu] Lusztig G., *Coxeter orbits and eigenspaces of Frobenius.* Invent. Math. **38** (1976), 101–159.

[Mi I] Milne J.S., *Étale cohomology.* Princeton, 1980.

[Mi II] Milne J.S., *Motives over finite fields.* Proceed. of Symposia in Pure Math. **55-1** (1994), 401–459.

[Oh] Ohmori J., *Rationality of certain cuspidal unipotent representations in crystalline cohomology groups.* Hokkaido Math. J. **35** (2006), 547–563.

[SGA 4] Grothendieck A., et al., *Séminaire de géométrie algébrique du Bois-Marie* 4, Part 3. Lecture Notes in Math. **305**, Springer.

[SGA 4 1/2] Deligne P., et al., *Séminaire de géométrie algébrique du Bois-Marie* $4\frac{1}{2}$. Lecture Notes in Math. **569**, Springer-Verlag, 1977.

[St] Steinberg R., *Endomorphisms of linear algebraic groups.* Mem. Amer. Math. Soc. **80**, 1968.

[Ta I] Tate J.T., *Algebraic cycles and poles of zeta functions.* In Arithmetic Algebraic Geometry, Harper and Row (1965), 93–110.

[Ta II] Tate J.T., *Conjectures on algebraic cycles in ℓ-adic cohomology.* Proceed. In Symposia in Pure Math. **55-1** (1994), 71–84.

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