Hokkaido Mathematical Journal Vol. 38 (2009) p. 303–314

The simple and multiple zero points of meromorphic functions

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(Received February 26, 2008; Revised May 12, 2008)

Abstract. The paper studies the counting functions of the simple and multiple zero points of meromorphic functions that share three values with finite weight. The results in this paper improve some results of H.X. Yi, X.M. Li and H.X. Yi and other authors. Examples show that the results in this paper are best possible.

Key words: Meromorphic function, Weighted sharing, Small function.

1. Introduction and results

Two meromorphic functions are said to share a value a CM if they have the same a-points with the same multiplicities, and if we do not consider the multiplicities then they are said to share a value a IM. One of the main tools that has been used in the study of functions that share values is Nevanlinna's theory on the distribution of values (see [4] or [8]).

Throughout this paper we denote by f, g two nonconstant meromorphic functions defined on the open complex plane. The symbol $S(r,f)$ is quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to+\infty$ possibly outside a set E of finite Lebesgue measure.

Definition 1 Let k be a positive integer. We denote by $N_{k)}(r,f)$ (or $\overline{N}_{k)}(r,f)$) the counting function of the poles of f with multiplicities less than or equal to k (ignoring multiplicities), and $N_{(k}(r,f)$ (or $\overline{N}_{(k}(r,f)$) the counting function of the poles of f with multiplicities greater than or equal to k (ignoring multiplicities).

In 1995, Yi [10] proved the following theorem:

Theorem A ([10]) *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1, ∞ CM, and let a $(\neq 0,1)$ be a finite complex number. If $N_{2)}\left(r,\frac{1}{g-a}\right)\neq T(r,g)+S(r,g)$, then a is a Picard exceptional value of g, and f and g satisfy one of the following three relations*:

(i) $(g-a)(f+a-1)\equiv a(1-a)$; (ii) $g+(a-1)f\equiv a$; (iii) $g\equiv af$.

2000 Mathematics Subject Classification : 30D35, 30D30.

According to Theorem A that it is necessary to investigate the properties of the simple a-points of g. Li and Yi [7] proved the following:

Theorem B ([7]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If there exists a finite complex number a ($\neq 0,1$) such that a is not a Picard value of f, and $N_{1)}\big(r, \frac{1}{g-a}\big) \leq uT(r,g) + S(r,g)$, where $u < 1/3$, then $N_{1)}\big(r, \frac{1}{g-a}\big) = 0$, and f and g assume one of the following forms*:

(i) $g = \frac{e^{3\gamma}-1}{e^{\gamma}-1}, \quad f = \frac{e^{-3\gamma}-1}{e^{-\gamma}-1}$, *with* $a = \frac{3}{4}$;
(ii) $g = \frac{e^{3\gamma}-1}{e^{2\gamma}-1}, \quad f = \frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}$, *with* $a = -3$;
(iii) $g = \frac{e^{\gamma}-1}{e^{3\gamma}-1}, \quad f = \frac{e^{-\gamma}-1}{e^{-3\gamma}-1}$, *with* $a = \frac{4}{3}$;
(iv) $g = \frac{e^{2\gamma}-1}{e^{3\gamma}-1}, \quad f = \frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}$, *with* $a = -\frac{1}{3}$;
(v) $g = \frac{e^{2\gamma}-1}{e^{-\gamma}-1}, \quad f = \frac{e^{-2\gamma}-1}{e^{\gamma}-1}$, *with* $a = \frac{1}{4}$;
(vi) $g = \frac{e^{\gamma}-1}{e^{-2\gamma}-1}, \quad f = \frac{e^{-\gamma}-1}{e^{2\gamma}-1}$, *with* $a = 4$;
(vii) $g = \frac{e^{2\gamma}-1}{\delta e^{\gamma}-1}, \quad f = \frac{e^{-2\gamma}-1}{(1/\delta)e^{-\gamma}-1}$, *with* $\delta^2 \neq 1$, $a^2\delta^2 = 4(a-1)$;
(viii) $g = \frac{e^{\gamma}-1}{\delta e^{2\gamma}-1}, \quad f = \frac{e^{-\gamma}-1}{(1/\delta)e^{-2\gamma}-1}$, *with* $\delta \neq 1$, $4a(1-a)\delta = 1$;
(ix) $g = \frac{e^{\gamma}-1}{\delta e^{-\gamma}-1}, \quad f = \frac{e^{-\gamma}-1}{(1/\delta)e^{\gamma}-1}$, *with* $\delta \neq 1$, $(1-a)^2 + 4a\delta = 0$,

where γ is a nonconstant entire function.

In [1] and [3], the present author discussed Theorem A when a is not a constant by the following result:

Theorem C ([1], [3]) *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM, and let a ($\neq \infty$) be a nonconstant small meromorphic function of f and g. Then $N_{(2}\big(r, \frac{1}{f-a}\big) = S(r,f)$ and $N_{(2}\big(r, \frac{1}{g-a}\big) = S(r,g)$. Furthermore, if $N_{1)}\big(r, \frac{1}{g-a}\big) \neq T(r,g) + S(r,g)$, then f and g satisfy one of the following three relations* (i)–(iii) *in Theorem A.*

It is evident from Theorem B that Theorem C is not true when a is a constant.

Question 1 *Can one be relaxed the nature of sharing* 0, 1 *and* ∞ *in Theorem B? What happens in Theorem B, if a is a small meromorphic function of f and g?*

In fact, Lahiri [5] gave an accurate concept for the CM sharing which provides opportunity to relax a CM shared value. We explain this notion by the following definition.

Definition 2 ([5]) Let k be a nonnegative integer or infinity. For any $a \in \mathbb{C}\bigcup\{\infty\}$ we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\le k)$, and z_0 is a zero of $f-a$ with multiplicity m $(> k)$ if and only if it is a zero of $g-a$ with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for all integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

In this paper we exploit the idea of weighted sharing to answer the question 1 by the following result which is a improvement Theorem B.

Theorem 1 *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$, (∞,k_3), where k_j $(j = 1,2,3)$ are positive integers satisfying*

$$k_1k_2k_3 > k_1 + k_2 + k_3 + 2, \tag{1.1}$$

and let a $(\not\equiv 0,\ 1,\ \infty)$ be a small meromorphic function of f and g such that

$$T(r,g) < \left(\frac{3}{2} + o(1)\right) N_{(2}\left(r, \frac{1}{g-a}\right). \tag{1.2}$$

(I) *If a is a constant, then f and g share $0,\ 1,\ \infty$ CM, and the conclusions of Theorem B still hold.*

(II) *If a is not a constant and f, g do not satisfy the relations* (i)–(iii) *in Theorem A, then $N_{1)}\big(r, \frac{1}{g-a}\big) = S(r,g)$, and f, g satisfy one of the following three relations*:

(i) $g = a\frac{2(a-1)h+1}{(a-1)h^2+a}$, $\quad f = \frac{h}{2}\frac{2(a-1)h+1}{(a-1)h^2+a}$;
(ii) $g = \frac{a}{h}\frac{h^2-(a-1)}{ah-2(a-1)}$, $\quad f = 2\frac{h^2-(a-1)}{ah-2(a-1)}$;
(iii) $g = \frac{a}{h}\frac{(a-1)+2h}{2a-(a-1)h}$, $\quad f = -h\frac{(a-1)+2h}{2a-(a-1)h}$,

where h is a nonconstant meromorphic such that $N(r,h)+N(r,1/h) = S(r,f)+S(r,g)$.

Theorem 2 *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$, (∞,k_3), where k_j $(j = 1,2,3)$ are positive integers satisfying* (1.1)*, and let a $(\not\equiv 0,\ 1,\ \infty)$ be a nonconstant small meromorphic function of f and g. Suppose further that f and g do not satisfy the relations* (i)–(iii) *in Theorem A. If one of $\{\overline{N}(r,g),\ \overline{N}(r,\frac{1}{g}),\ \overline{N}(r,\frac{1}{g-1})\}$ is equal to $S(r,g)$, then $T(r,g) = N_{1)}\ (r,\frac{1}{g-a}) + S(r,g)$, that means $N_{(2}(r,\frac{1}{g-a}) = S(r,g)$.*

Corollary 1 *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$, (∞,k_3), where k_j $(j = 1,2,3)$ are positive integers satisfying* (1.1)*, and let a $(\not\equiv 0,\ 1,\ \infty)$ be a small meromorphic function of f and g such that*

$$N_{1)}\left(r,\frac{1}{g-a}\right) < \left(\frac{1}{3}+o(1)\right)T(r,g). \tag{1.3}$$

If f and g do not satisfy the relations (i)–(iii) *in Theorem A, then the conclusions of Theorem* 1 *still hold.*

Proof. Since f and g do not satisfy the relations (i)–(iii) in Theorem A then, from Lemma 4 and Lemma 5, we have $N(r,\frac{1}{g-a}) + S(r,g) = T(r,g)$. Consequently, the condition (1.3) implies to the inequality (1.2), that means, the conclusions of Theorem 1 still hold. This proves Corollary 1. $\square$

Corollary 2 *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$, (∞,k_3), where k_j $(j = 1,2,3)$ are positive integers satisfying* (1.1)*, and let a $(\not\equiv 0,\ 1,\ \infty)$ be a nonconstant small meromorphic function of f and g. If one of $\{\overline{N}(r,g),\ \overline{N}(r,\frac{1}{g}),\ \overline{N}(r,\frac{1}{g-1})\}$ is equal to $S(r,g)$, then $\overline{N}_{(2}(r,\frac{1}{g-a}) = S(r,g)$.*

Proof. If f and g satisfy the relation (i) in Theorem A then, by using Lemma 1, we see that $\overline{N}_{(2}(r,1/(g-a)) \le \overline{N}_{(2}(r,g)+S(r) = S(r)$. Similarly, one can prove that if f and g satisfy the relations (ii) or (iii) in Theorem A, then the conclusion of Corollary 2 is true. Consequently, the Corollary 2 follows from the Theorem 2. This proves Corollary 2. $\square$

The following example shows that the condition (1.2) in Theorem 1 is best possible.

Example 1 Let $f = 1+e^{-\beta}+e^{-2\beta}+e^{-3\beta}$, $g = 1+e^{\beta}+e^{2\beta}+e^{3\beta}$ and $a = \frac{20\pm 4\sqrt{2}i}{27}$, where β is a non-constant entire function. It is easy to verify that f and g share $(0,3)$, $(1,4)$ and $(\infty,4)$ and $T(r,g) = (3/2)N_{(2}\big(r, \frac{1}{g-a}\big)+S(r,g)$. But f and g do not satisfy any one of the forms (i)–(ix) in Theorem B.

The following example shows that Theorem 2 is not true when a is a constant.

Example 2 Let $g = e^{2z}+e^{z}+1$ and $f = e^{-2z}+e^{-z}+1$, and $a = 3/4$. Then f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞) and $N(r,g) = 0$, further that f and g do not satisfy the relations (i)–(iii) in Theorem A, but $T(r,g) = N_{(2}\big(r, \frac{1}{g-a}\big) + S(r,g)$ and $N_{1)}\big(r, \frac{1}{g-a}\big) = 0$.

Remark 1 ([8]) If f and g are two nonconstant meromorphic functions sharing three distinct values IM, then $S(r,f) = S(r,g)$. We use $S(r)$ to express either $S(r,f)$ or $S(r,g)$, unless otherwise stated.

2. Some lemmas

In this section we discuss some lemmas which will be required in the sequel.

Lemma 1 ([9]) *Let f and g be two distinct meromorphic functions sharing $(0,k_1)$, $(1,k_2)$, (∞,k_3), where k_j $(j = 1,2,3)$ are positive integers satisfying (1.1). Then $\overline{N}_{(2}\big(r, \frac{1}{f-a}\big) + \overline{N}_{(2}\big(r, \frac{1}{g-a}\big) = S(r)$, for any $a = 0,1,\infty$.*

Lemma 2 ([2]) *Under the assumptions of Lemma 1, if $\alpha = \frac{f-1}{g-1}$ and $H = \frac{f}{g}$, then $\overline{N}\big(r, \frac{1}{\alpha}\big) + \overline{N}(r,\alpha) + \overline{N}\big(r, \frac{1}{H}\big) + \overline{N}(r,H) = S(r)$.*

Lemma 3 ([6]) *Let f_1 and f_2 be distinct nonconstant meromorphic functions satisfying $\overline{N}(r,f_i) + \overline{N}\big(r, \frac{1}{f_i}\big) = S(r,f_1,f_2)$, $i = 1,2$. If $f_1^s f_2^t - 1$ is not identically zero for all integers s, t $(|s|+|t|>0)$, then for any positive number ϵ, we have*

$$N_0(r,1,f_1,f_2) \le \epsilon T(r) + S(r,f_1,f_2),$$

where $N_0\ (r,1,f_1,f_2)$ *denotes the reduced counting function of* f_1 *and* f_2 *related to the common* 1*-points,* $T(r) = T(r,f_1) + T(r,f_2)$ *and* $S(r,f_1,f_2) = \max\{S(r,f_1),\ S(r,f_2)\}$.

Lemma 4 ([2]) *Let* f *and* g *be satisfying the assumptions of Lemma* 1*, and let* a $(\not\equiv 0,\ 1,\ \infty)$ *be a small meromorphic function of* f *and* g*. If* f, g *do not satisfy the relations* (i)–(iii) *in Theorem A, then* $N_{(3}\big(r, \frac{1}{g-a}\big) = S(r,g)$.

Lemma 5 ([2]) *Under the assumptions of Lemma* 1*, if* a $(\not\equiv 0,\ 1,\ \infty)$ *is a small meromorphic function of* f *and* g *such that* $N_{2)}\big(r, \frac{1}{g-a}\big) \ne T(r,g) + S(r)$ *then* $\overline{N}\big(r, \frac{1}{g-a}\big) = S(r)$*, and* f *and* g *satisfy one of the three relations* (i)–(iii) *in Theorem A.*

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We first suppose that f is a fractional linear transformation of g. Then f and g share 0, 1, ∞ CM, and there are two distinct Picard values of g. Indeed, we see that the condition (1.2) tells us that a is not a Picard value of g, that is, if we apply the second fundamental theorem of Nevanlinna we get $T(r,g) = \overline{N}(r, 1/(g-a)) + S(r,g)$, which yields that $N_{(2}(r, 1/(g-a)) = S(r,g)$, this and (1.2) imply to $T(r,g) = S(r,g)$, which is impossible. Therefore, f is not a fractional linear transformation of g. From the definition of α and H in Lemma 2, we have

$$f = \frac{1-\alpha^{-1}}{H^{-1} - \alpha^{-1}}, \quad g = \frac{1-\alpha}{H-\alpha}, \tag{3.1}$$

and

$$g - a = \frac{\alpha(a-1) - aH + 1}{H - \alpha}. \tag{3.2}$$

From (3.1) and Remark 1, we see that $S(r) = \max\{S(r,H), S(r,\alpha)\}$. □

Assume that $T(r,\alpha) = S(r)$. The possibility $\alpha(a-1)+1 \equiv 0$ gives us (ii) in Theorem A, and a is not a constant. Therefore, $\alpha(a-1)+1 \not\equiv 0$, and by the second fundamental theorem of Nevanlinna and Lemma 2, we see that

$$T(r,g) = T(r,H) + S(r) = \overline{N}\left(r, \frac{1}{\alpha(a-1) - aH + 1}\right) + S(r)$$
$$= \overline{N}\left(r, \frac{1}{g-a}\right) + S(r),$$

which implis $N_{(2}\big(r, \frac{1}{g-a}\big) = S(r)$, this is a contradiction with (1.2). Thus, $T(r,\alpha) \neq S(r)$. In the same way, we can prove that $T(r,H) \neq S(r)$ and $T(r,\alpha/H) \neq S(r)$. Since f and g do not satisfy (i)–(iii) in Theorem A (whatever a is a constant or not), by Lemma 4 we have $N_{(3}\big(r, \frac{1}{g-a}\big) = S(r)$. Set

$$\alpha_1 = \frac{a'}{a} + \frac{H'}{H}, \quad \alpha_2 = \frac{\alpha'}{\alpha} + \frac{a'}{a-1}. \tag{3.3}$$

Since a is a small function of f and so g, then we have $\alpha_i \not\equiv 0$ and $T(r,\alpha_i) = S(r)$, $i = 1,2$. Let z_0 be a multiple zero of $g-a$, which is neither any zero of α', H', $\alpha' - H'$, a, $a-1$, α_1, α_2, $\alpha_1 - \alpha_2$, nor the pole of a. From (3.2) we obtain

$$\begin{aligned}
&(\alpha(a-1) - aH + 1)(z_0) = 0,\\
&\left(\left(\frac{\alpha'}{\alpha}(a-1) + a'\right)\alpha - \left(a' + a\frac{H'}{H}\right)H\right)(z_0) = 0.
\end{aligned} \tag{3.4}$$

Let us now define the following two functions:

$$f_1 = (a-1)\frac{\alpha_2 - \alpha_1}{\alpha_1}\alpha, \quad f_2 = a\frac{\alpha_2 - \alpha_1}{\alpha_2}H, \tag{3.5}$$

and consider $T_0(r) = T(r,f_1) + T(r,f_2)$, $S_0(r) = o(T_0(r))$. From this, Lemma 2 and $(3,5)$, we get $S_0(r) = S(r)$ and

$$\overline{N}(r,f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r) \quad (j = 1,2). \tag{3.6}$$

In view of (3.4) and $(3,5)$, it can be verified that z_0 is a common zero of $f_1 - 1$ and $f_2 - 1$. By this and Lemma 5, one deduces that

$$N_{(2}\left(r, \frac{1}{f-a}\right) \le N_0(r,1,f_1,f_2) + S(r), \tag{3.7}$$

where $N_0(r,1,f_1,f_2)$ is defined as in Lemma 3.

According to the condition (1.2), from (3.7) we observe $N_0(r,1,f_1,f_2) \neq S(r)$. Then from (3.6), Lemma 2 and by using Lemma 3, there exist two integers p and q ($|p|+|q|>0$) such that

$$f_1^p f_2^q \equiv 1. \tag{3.8}$$

Combining (3.5) and (3.8) we obtain $pq \neq 0$ and

$$(a-1)^p a^q \alpha^p H^q = \left(\frac{\alpha_1}{\alpha_2-\alpha_1}\right)^p \left(\frac{\alpha_2}{\alpha_2-\alpha_1}\right)^q. \tag{3.9}$$

Since $N_0(r,1,f_1,f_2) \neq S(r)$, from (3.8) we can assume that p is a positive integer, and p and $|q|$ are relatively primes. If $p+q=0$ then from (3.5) and (3.8), we get $T(r,\alpha/H)=S(r)$, which is a contradiction. Differentiating the equation (3.9) and using (3.3), we can easily obtain that

$$q\alpha_1 + p\alpha_2 = q\frac{\left(\frac{\alpha_1}{\alpha_2}\right)'}{1-\frac{\alpha_1}{\alpha_2}} - p\frac{\left(\frac{\alpha_2}{\alpha_1}\right)'}{\frac{\alpha_2}{\alpha_1}-1} = \frac{q\alpha_1+p\alpha_2}{\alpha_1-\alpha_2}\frac{\left(\frac{\alpha_2}{\alpha_1}\right)'}{\frac{\alpha_2}{\alpha_1}}.$$

If $q\alpha_1+p\alpha_2 \not\equiv 0$ then, from the last equation and (3.3), we get

$$\frac{a'}{a}+\frac{H'}{H}-\frac{\alpha'}{\alpha}-\frac{a'}{a-1} = \frac{\left(\frac{\alpha_2}{\alpha_1}\right)'}{\frac{\alpha_2}{\alpha_1}},$$

which implies that $H/\alpha = c((a-1)/a)(\alpha_2/\alpha_1)$, where c is a constant. This equation gives us $T(r,H/\alpha)=S(r)$, which is impossible. Thus $q\alpha_1+p\alpha_2 \equiv 0$; this relation and (3.3) yield

$$p\left\{\frac{\alpha'}{\alpha}+\frac{a'}{a-1}\right\}+q\left\{\frac{H'}{H}+\frac{a'}{a}\right\} \equiv 0. \tag{3.10}$$

By taking integration on the above equation to conclude $(a-1)^p a^q H^q \alpha^p = A$, where $A \neq 0$ is a constant. Since $q\alpha_1+p\alpha_2 \equiv 0$, then $p+q \neq 0$, and then the last equation and (3.9) give us

$$(a-1)^p a^q H^q \alpha^p = (\lambda-1)^p \lambda^q, \tag{3.11}$$

where $\lambda = \alpha_2/(\alpha_2 - \alpha_1) = q/(q+p)$. From (1.2) and Lemma 5, we get

$$N_{1)}\left(r, \frac{1}{g-a}\right) < \left(\frac{1}{3} + o(1)\right) T(r,g). \tag{3.12}$$

From (3.11), we observe that if a is a constant then f and g share 0, 1, ∞ CM, and then (I) of Theorem 1 follows from (3.12) and Theorem B.

We suppose that a is not a constant. Since p and $|q|$ are relatively primes, there exist two integers u and v such that $uq + vp = 1$. We let $h = f_1^{-u} f_2^{v}$. By utilizing (3.8), we find that $f_1 = h^{-q}$ and $f_2 = h^p$. One can write (3.5) as

$$\alpha = \frac{1}{a-1}(\lambda - 1)h^{-q}, \quad H = \frac{1}{a}\lambda h^p. \tag{3.13}$$

Consequently, from (3.1) and (3.13) we deduce

$$\begin{aligned} g &= a\frac{(a-1) - (\lambda-1)h^{-q}}{(a-1)\lambda h^p - a(\lambda-1)h^{-q}}, \\ f &= \lambda\frac{(a-1)h^q - (\lambda-1)}{(a-1)\lambda h^q - a(\lambda-1)h^{-p}} \end{aligned} \tag{3.14}$$

Let z be a double zero of $g - a$ such that it is not any zero or pole of $H - \alpha$, a, $a-1$, H, α, h and h'. Then from (3.14), we get that z is a common zero of the following equations

$$1 + (\lambda-1)h^{-q} - \lambda h^p = 0 \quad \text{and} \quad q(\lambda-1)h^{-q-1} + p\lambda h^{p-1} = 0. \tag{3.15}$$

By solving these two equation, we obtain that z is a common zero of $h^p - 1 = 0$ and $h^q - 1 = 0$, which means that z is a zero of $h - 1 = 0$, because $g.c.d(p, |q|) = 1$. Conversely, if z is a zero of $h - 1 = 0$, then z is a common zero of the two equations in (3.15). From (3.5) and Lemma 2, it is clear

$$\begin{aligned} \overline{N}\left(r, \frac{1}{h'}\right) &= \overline{N}\left(r, \frac{1}{f_2'}\right) + S(r) \le \overline{N}\left(r, \frac{1}{\frac{H'}{H} + \frac{a'}{a}}\right) + S(r) \\ &\le T\left(r, \frac{H'}{H} + \frac{a'}{a}\right) + S(r) = S(r). \end{aligned}$$

Consequently, from the above discussion, Lemma 2 and (3.6), and by applying the second fundamental theorem of Nevanlinna, we deduce that

$$\overline{N}_{(2}\left(r, \frac{1}{g-a}\right) = \overline{N}\left(r, \frac{1}{h-1}\right) + S(r) = T(r,h) + S(r). \tag{3.16}$$

From (3.13) and by using the second fundamental theorem of Nevanlinna, it is easy to verify that

$$\begin{aligned} T(r,H) &= pT(r,h) + S(r) = \overline{N}\left(r, \frac{1}{H-1}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{f-1}\right) + S(r), \end{aligned} \tag{3.17}$$

$$\begin{aligned} T(r,\alpha) &= |q|T(r,h) + S(r) = \overline{N}\left(r, \frac{1}{\alpha-1}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + S(r) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} T\left(r, \frac{\alpha}{H}\right) &= |p+q|T(r,h) + S(r) \\ &= \overline{N}\left(r, \frac{1}{\frac{\alpha}{H}-1}\right) + S(r) = \overline{N}(r,f) + S(r). \end{aligned} \tag{3.19}$$

From (3.14) and (3.17)–(3.19), we note

$$\begin{aligned} T(r,g) &\le \max\{p, |q|, |p+q|\}T(r,h) + S(r) \\ &= \max\left\{\overline{N}\left(r, \frac{1}{f-1}\right), \overline{N}\left(r, \frac{1}{f}\right), \overline{N}(r,f)\right\} + S(r) \le T(r,g) + S(r) \end{aligned}$$

from this and (3.16) we obtain

$$T(r,g) = \frac{1}{2}\max\{p, |q|, |p+q|\}N_{(2}\left(r, \frac{1}{g-a}\right) + S(r).$$

This equation and the assumption (1.2) give us that $\max\{p, |q|, |p+q|\} < 3$, which implies that $\max\{p, |q|, |p+q|\} = 2$, because $p + q \neq 0$. Therefore $(p, q) \in \{(1, 1), (1, -2), (2, -1)\}$. It follows, from (3.14), we get the cases (i)–(iii) of Theorem 1. It remains to show that

$$N(r, h) + N\left(r, \frac{1}{h}\right) = S(r). \tag{3.20}$$

Let z be a common zero of f with multiplicity $i(f)$ and g with multiplicity $i(g)$ such that $i(g) < i(f)$. If $q > 0$ ($q < 0$), then from (3.11) we see that z is a pole (zero) of $(a-1)^p a^q$ with multiplicity $q(i(f) - i(g))(-q(i(f) - i(g)))$. It is not difficult by utilizing Lemma 1 to show that

$$N_{(2}\left(r, \frac{1}{f}\right) = S(r, f). \tag{3.21}$$

Similarly, one can be derived

$$N_{(2}(r, f) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, g) + N_{(2}\left(r, \frac{1}{g-1}\right) + N_{(2}\left(r, \frac{1}{g}\right) = S(r). \tag{3.22}$$

From the inequalities (3.21) and (3.22), we see

$$N\left(r, \frac{1}{H}\right) \leq N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}(r, g) + S(r) = S(r). \tag{3.23}$$

Similarly, from (3.21) and (3.22) we obtain

$$N(r, H) + N\left(r, \frac{1}{\alpha}\right) + N(r, \alpha) = S(r). \tag{3.24}$$

Consequently, we will arrive at (3.20) from (3.23) and (3.24). This proves Theorem 1.

Proof of Theorem 2. As we have seen in the proof of Theorem 1 that if $N_{(2}\left(r, \frac{1}{g-a}\right) \neq S(r, g)$ then, from (3.7), we have (3.11), (3.13) and (3.17)–(3.19). Then, from the assumption of Theorem 2 and (3.17)–(3.19), we obtain that one of $\{T(r, \alpha),\ T(r, H),\ T(r, \alpha/H)\}$ is equal to $S(r)$, and this requires that f and g must satisfy one of the relations (i)–(iii) in Theorem

A, which is impossible. Therefore $N_{(2}\left(r, \frac{1}{g-a}\right) = S(r, g)$, and Theorem 2 follows from Lemma 5. This proves Theorem 2. □

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