On Integrals of the Certain Ordinary Differential Equations in the Vicinity of the Singularity. I.

By

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§ 1. Introduction.

In this paper, in the vicinity of $x_i=0$, we consider the differential equation

(1.1)
$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\dots\dots=\frac{dx_n}{X_n},$$

where $X_i(x)$ are regular in the vicinity of $x_i=0$ and vanish there. Let the expansions of $X_i(x)$ in the vicinity of $x_i=0$ be $X_i=\sum_{j=1}^{n}a_{ij}x_j+\dots\dots$, and put $\| a_{ij} \|=A$. We assume that the eigen values λ_i of the matrix A satisfy either Poincaré's condition (1) or Picard's condition.(2) Corresponding to the equation (1.1), we consider the equation as follows:

(1.2)
$$\sum_{i=1}^{n}X_i\frac{\partial f}{\partial x_i}=0.$$

Then it is evident that the integrals of (1.1) are obtained by putting $n-1$ independent solutions of (1.2) constants. However, in the previous papers,(3) the equation (1.2) is already integrated. Consequently, making use of the results of those papers, we can determine the integrals of the equation (1.1).

In solving the equation (1.2) we have transformed the matrix A into that of Jordan's form by the suitable linear transformation of the variables x_i. Let the transformation be $y_i=\sum_{k=1}^{n}s_{ik}x_k$. Then the equation (1.2) is transformed into the equation as follows:

$$\sum_{i=1}^{n}Y_i\frac{\partial f}{\partial y_i}=0,$$

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(1) M. Urabe, Jour. Sci. Hiroshima Univ. Vol. 14, No. 2, p. 115. In the following, we denote this paper by I.

(2) M. Urabe, Jour. Sci. Hiroshima Univ. Vol. 14, No. 3, p. 195. In the following, we denote this paper by II.

(3) M. Urabe, ibid. I, II.

where $Y_i=\sum_{k=1}^{n} s_{ik}X_k$. By the transformation $y_i=\sum_{k=1}^{n} s_{ik}x_k$, the equation (1.1) is transformed into the equation as follows:

$$\frac{dy_1}{Y_1}=\frac{dy_2}{Y_2}=\dots\dots=\frac{dy_n}{Y_n}.$$

Thus, without loss of generality, we can assume that the matrix A is of Jordan's form, namely that X_i are of the forms as follows:

$$X_{11}=\lambda_1x_{11}+x_{12}+\dots,\ X_{12}=\lambda_1x_{12}+x_{13}+\dots,\ \dots,\ X_{1k}=\lambda_1x_{1k}+\dots,$$
$$X_{21}=\lambda_1x_{21}+x_{22}+\dots,\ \dots\dots\dots\dots\quad ,\ X_{2l}=\lambda_1x_{2l}+\dots\dots,$$
$$\dots\dots\dots\dots\dots\dots$$
$$X_{r1}=\lambda_2x_{r1}+x_{r2}+\dots,\ X_{r2}=\lambda_2x_{r2}+x_{r3}+\dots,\ \dots,\ X_{rs}=\lambda_2x_{rs}+\dots,$$
$$\dots\dots\dots\dots\dots\dots .$$

§ 2. Integrals of (1.1) under Poincaré's condition.

Let $\lambda_i=\rho_ie^{\omega_i\sqrt{-1}}$. By Poincaré's first condition, there exists a line L passing through the origin O, in one side of which all λ_i's lie. Draw a perpendicular OH to L in the side where λ_i's lie. Let the angle between the half-line OH and the positive side of the real axis be ω. Then $\cos(\omega_i-\omega)>0$. Put $\lambda_i'=\lambda_ie^{-\omega\sqrt{-1}}$. Then $\Re(\lambda_i')=\rho_i\cos(\omega_i-\omega)>0$. Multiplying the denominators of (1.1) by $e^{-\omega\sqrt{-1}}$, we have the equation of the same form as (1.1), where the eigen values of the matrix A become λ_i', consequently all of them have positive real parts. From the second of the initial Poincaré's condition, it is also evident that

$$\lambda_1'p_1+\lambda_2'p_2+\dots\dots+\lambda_n'p_n-\lambda_i'\neq 0$$

for all non-negative integers satisfying $p_1+p_2+\dots+p_n\geqq 2$. Thus, in (1.1), without loss of generality, we can assume that all the eigen values λ_i of the matrix A have positive real parts and that they satisfy Poincaré's second condition. In the following, we shall discuss the problem under this assumption.

Since λ_i satisfy Poincaré's condition, the equation (1.2) has $n-1$ independent integrals as follows[1]:

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(1) M. Urabe, ibid. I.

$$(2.1)\quad \begin{cases} \qquad\quad U-\dfrac{1}{\lambda_1}\log\varphi,\ U_1,\ U_2,\ \ldots\ldots,\ U_{k-2}, \\ \theta/\varphi,\quad V-\dfrac{1}{\lambda_1}\log\theta,\ V_1,\ V_2,\ \ldots\ldots,\ V_{l-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega^{\frac{1}{\lambda_2}}/\varphi^{\frac{1}{\lambda_1}},\ W-\dfrac{1}{\lambda_2}\log\omega,\ W_1,\ W_2,\ \ldots\ldots,\ W_{s-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots . \end{cases}$$

Therefore, the integrals of (1.1) are those obtained by putting above functions constants. Put

$$(2.2)\qquad \varphi=Ct^{\lambda_1},$$

where C is an arbitrarily chosen constant and t is an auxiliary variable. Then, putting the functions of (2.1) constants, we have:

$$(2.3)\quad \begin{cases} \varphi=Ct^{\lambda_1},\ U-\log t=C_0,\ U_1=C_1,\ U_2=C_2,\ \ldots\ldots,\ U_{k-2}=C_{k-2}, \\ \theta=Dt^{\lambda_1},\ V-\log t=D_0,\ V_1=D_1,\ \ldots\ldots\ldots,\ V_{l-2}=D_{l-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega=Kt^{\lambda_2},\ W-\log t=K_0,\ W_1=K_1,\ \ldots\ldots\ldots,\ W_{s-2}=K_{s-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots , \end{cases}$$

where $C_0,\ C_1,\ \ldots,\ C_{k-2};\ D,\ D_0,\ \ldots,\ D_{l-2};\ \ldots;\ K,\ K_0,\ \ldots,\ K_{s-2};\ \ldots\ldots$ are constants.

From (2.3), we shall determine the forms of the integrals x_i of (1.1). From the second of the first row of (2.3), it follows that $U=C_0+\log t$. From $U=\varphi_1/\varphi$ i.e. $\varphi_1=U\varphi$, by the definition of U_p, we have:

$$\varphi_{p+1}=U_p\varphi+\binom{p}{1}U_{p-1}\varphi_1+\binom{p}{2}U_{p-2}\varphi_2+\ldots\ldots+\binom{p}{p-1}U_1\varphi_{p-1}+U\varphi_p.$$

Therefore, from the first row of (2.3), we have:

$$(2.4)\qquad \varphi_{p+1}=C_p\varphi+\binom{p}{1}C_{p-1}\varphi_1+\ldots\ldots+\binom{p}{p-1}C_1\varphi_{p-1}+C_0\varphi_p+\varphi_p\log t.$$

Now $\varphi_1=U\varphi=C_0\varphi+\varphi\log t=C_0Ct^{\lambda_1}+Ct^{\lambda_1}\log t$. Substituting this into (2.4), we can determine successively the functions $\varphi_2,\ \varphi_3,\ \ldots\ldots,\ \varphi_{k-1}$. The consequences are as follows:

(2.5)

$$\varphi_p=A_0t^{\lambda_1}+A_1t^{\lambda_1}\log t+A_2t^{\lambda_1}(\log t)^2+\ldots\ldots+A_{p-1}t^{\lambda_1}(\log t)^{p-1}+Ct^{\lambda_1}(\log t)^p,$$

where $A_0, A_1, \ldots\ldots, A_{p-1}$ are polynomials of $C, C_0, C_1, \ldots\ldots, C_{p-1}$ and A_0 actually contains C_{p-1}. Likewise, from the second and lower rows of (2.3), we obtain the analogous results. Thus, we have:

(2.6)

$$\begin{cases} \varphi = Ct^{\lambda_1} \\ \varphi_p = F_p[t^{\lambda_1}, t^{\lambda_1}\log t, \ldots\ldots, t^{\lambda_1}(\log t)^p\,;\, C, C_0, \ldots, C_{p-1}], \; (p=1, 2, \ldots, k-1); \\ \theta = Dt^{\lambda_1} \\ \theta_p = F_p[t^{\lambda_1}, t^{\lambda_1}\log t, \ldots\ldots, t^{\lambda_1}(\log t)^p\,;\, D, D_0, \ldots, D_{p-1}], \; (p=1, 2, \ldots, l-1); \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,; \\ \omega_p = Kt^{\lambda_2} \\ \omega_p = F_p[t^{\lambda_2}, t^{\lambda_2}\log t, \ldots\ldots, t^{\lambda_2}(\log t)^p\,;\, K, K_0, \ldots\ldots, K_{p-1}], \; (p=1, 2, \ldots\ldots, s-1); \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,, \end{cases}$$

where $F_p[t^{\lambda_1}, t^{\lambda_1}(\log t), \ldots\ldots, t^{\lambda_1}(\log t)^p\,;\, C, C_0, \ldots\ldots, C_{p-1}]$ denote the functions of the right-hand side of (2.5). Now, by our assumptions, the real parts of λ_i are all positive, therefore, when $t \to 0$ (Arg t=finite), t^{λ_i}, $t^{\lambda_i}\log t$, $t^{\lambda_i}(\log t)^2, \ldots\ldots, t^{\lambda_i}(\log t)^p, \ldots\ldots$ tend to zero. And, from the forms of the functions $\varphi, \varphi_1, \ldots\ldots, \varphi_{k-1}\,;\, \theta, \theta_1, \ldots\ldots, \theta_{l-1}\,;\, \ldots\ldots\,;\, \omega, \omega_1, \ldots\ldots, \omega_{s-1}\,;\, \ldots\ldots$, it is evident that the Jacobian of these functions with respect to the variables x_i is not zero for $x_i=0$. Therefore, for the sufficiently small value of $|t|$ (Arg t=finite), we can solve the equation (2.6) with regard to x_i, and there the solutions $x_i=x_i(t)$ are expressed as the regular functions of the following functions of t:

$$(2.7) \quad \begin{cases} t^{\lambda_1}, t^{\lambda_1}\log t, t^{\lambda_1}(\log t)^2, \ldots\ldots, t^{\lambda_1}(\log t)^{\nu_1}, & (\nu_1=\text{max.}\,(k, l, \ldots)-1), \\ t^{\lambda_2}, t^{\lambda_2}\log t, \ldots\ldots, t^{\lambda_2}(\log t)^{\nu_2}, & (\nu_2=\text{max.}\,(s, \ldots)-1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,. \end{cases}$$

Besides, these solutions $x_i=x_i(t)$ actually contain $n-1$ constants $C_0, C_1, \ldots\ldots, C_{k-1}\,;\, D, D_0, \ldots\ldots, D_{l-1}\,;\, \ldots\ldots\,;\, K, K_0, \ldots\ldots, K_{s-1}\,;\, \ldots\ldots$, and evidently $x_i \to 0$ when $t \to 0$ (Arg t=finite), therefore there arises no restriction upon these constants, namely $x_i=x_i(t)$ actually contain $n-1$ arbitrary constants.

Conversely, when we elliminate t from $x_i=x_i(t)$, we obtain the relations where the functions of (2.1) are put constants, namely we obtain the integrals of the equation (1.1). In this sense, we can say that $x_i=x_i(t)$ which are obtained by solving (2.6) with regard to x_i, are the integrals of of parametric form of the equation (1.1).

Thus we arrive at the following conclusion:

The integrals x_i of the eqution (1.1) are expressed as the regular functions of the functions of (2.7) for sufficiently small value of $|t|$ (Arg $t=$ finite) and they actually contain $n-1$ arbitrary constants.

Up to the present it is known that there exist integrals x_i of the equation (1.1) which have the above properties. Here, by making use of the theory of linear homogeneous partial differential equations, we have been able to conclude that there exist no other integrals.

§ 3. Integrals of (1.1) under Picard's condition.

In this paragraph, we assume that the eigen values λ_i satisfy Picard's condition, namely, that among λ_i, there exist λ_α ($\alpha=1, 2, \ldots\ldots, m$) so that

(I) on a complex plane there exists a convex domain Ω which contains all λ_α's but not the origin,

(II) $\lambda_1 p_1+\lambda_2 p_2+\ldots\ldots+\lambda_m p_m-\lambda_i\neq 0 \quad (i=1, 2, \ldots\ldots, n)$

for all non-negative integers $p_1, p_2, \ldots\ldots, p_m$ satisfying $p_1+p_2+\ldots\ldots+p_m\geqq 2$.

As in § 2, without loss of generality, we can replace the condition (I) by the following:

(I′) the real parts of all λ_α's are positive.

In the following, we assume the conditions (I′) and (II). By the results of the previous paper [1], there exist regular fnnctions

$$g_\lambda(x_\alpha, x_\lambda)\equiv x_\lambda-x_\lambda(x_\alpha), \tag{3.1}$$

where $x_\lambda(x_\alpha)$ are sums of the terms of the second and higher orders of x_α, and under the condition that $g(x_\alpha, x_\lambda)=0$, the equation (1.2) has $n-1$ independent integrals $g_\sigma(x_\alpha)$ and $g_\lambda(x_\alpha, x_\lambda)$. Namely, for x_i satisfying $g_\lambda(x_\alpha, x_\lambda)=0$, it is valid that

$$\sum_i X_i\frac{\partial g_\lambda}{\partial x_i}=0 \quad \text{and} \quad \sum_i X_i\frac{\partial g_\sigma}{\partial x_i}=0. \tag{3.2}$$

Now, for x_i satisfying

$$g_\lambda(x_\alpha, x_\lambda)=0 \quad \text{and} \quad g_\sigma(x_\alpha)=\text{const.}, \tag{3.3}$$

it is valid that

(1) M. Urabe, ibid. II. In this paragraph, we make use of the notations explained in that paper.

$$(3.4) \qquad \sum_i dx_i \frac{\partial g_\lambda}{\partial x_i}=0 \quad \text{and} \quad \sum_i dx_i \frac{\partial g_\sigma}{\partial x_i}=0.$$

However, the rank of the Jacobian matrix $\frac{\partial(g_\sigma, g_\lambda)}{\partial(x_\alpha, x_\lambda)}$ is $n-1$ in the vicinity of $x_i=0$. Then, comparing (3.2) with (3.4), we have (1.1). Namely, x_i satisfying (3.3) are integrals of the equation (1.1).

But, it happens that there exist integrals which do not satisfy (3.3). For example, the equation

$$(\mathrm{E}) \qquad \frac{dx}{x}=\frac{dy}{ay}=\frac{dz}{-z}$$

where a is an irrational positive number. The eigen values of A are evidently as follows:

$$\lambda_1=1, \quad \lambda_2=a, \quad \lambda_3=-1$$

These values evidently satisfy the conditions (I′) and (II). Solving the equation

$$x\frac{\partial z}{\partial x}+ay\frac{\partial z}{\partial y}=-z$$

we have: $g_3=z$. Then we have $g_1=y/x^a$. Therefore the integral satisfying (3.3) is as follows:

$$z=0 \quad \text{and} \quad y/x^a=\text{const.}$$

But, integrating directly (E), we have

$$xz=0 \quad \text{and} \quad y/x^a=\text{const.}$$

Thus the set composed of $x=0$ and $y=0$ is also an integral and this does not satisfy (3.3).

Now, substituting $x_\lambda=x_\lambda(x_\alpha)$ into $X_\alpha(x_i)$, let the results be $X_\alpha(x_\beta)$. Then, for x_i satisfying (3.3), from (3.2), we have:

$$\sum_\alpha X_\alpha(x_\beta)\frac{\partial g_\sigma}{\partial x_\alpha}=0.$$

Now $x_\lambda(x_\alpha)$ are sums of the terms of the second and higher orders of x_α, therefore the eigen values of the matrix, the elements of which are the coefficients of the terms of the first order in the expansions of $X_\alpha(x_\beta)$, are λ_α, consequently they satisfy Poincaré's second condition because of the condition (II) and moreover evidently their real parts are all positive. Therefore, for $g_\sigma=$const., the discussions of § 2 are valid, namely, if we introduce an auxiliary variable t, for the sufficiently small value of $|t|$ (Arg t=finite), x_α satisfying g_σ=const. are expressed as regular functions of the functions of the same forms as (2.7) and they contain $m'-1$ arbitrary constants, where m' denotes the number of the variables x_α. Substituting these $x_\alpha = x_\alpha(t)$ into $g_\lambda(x_\alpha, x_\lambda)=0$, it is evident that x_λ are also regular functions of the arguments of $x_\alpha(t)$. Thus, x_i satisfying (3.3) are expressed as regular functions of the functions of the forms of (2.7). Namely we see that there exist integrals of the equation (1.1), which are expressed as regular functions of the functions of the forms of (2.7) and contain $m'-1$ arbitrary constants.

Here we must note that, under Poincaré's condition, there exists no intergrals other than those which are expressed as regular functions of the functions of the forms of (2.7), but, under Picard's condition, as indicated in the example, it may happen that there exist other integrals.

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