Direct Sums and Normal Ideals of Lattices

By Fumitomo MAEDA.

(Received April 30, 1949)

Let L be a lattice with 0 and 1. We say that L is a direct sum of $L(o,z_i)$ $(i=1,\cdots,n)$, if and only if every element a of L is expressible uniquely in the form

$$a=a_1\cup\cdots\cup a_n,\qquad a_i\in L(o,z_i)\qquad (i=1,\cdots,n).$$

And in this case z_i $(i=1,\cdots,n)$ are elements of the center of L. In this paper, we shall extend this notion to the case where the existence of 1 is not assumed. And next, we shall define normal ideals in a modular lattice with some properties, and show that the set of all normal ideals acts as the center. And we shall apply this property to the general continuous geometry, that is, the continuous geometry without the assumption of the existence of 1, and show that the theories of dimension function and subdirect product representation hold also in the general continuous geometry as in the continuous geometry.

§ 1. Direct Sum of Lattices.

In this section, L is a lattice with the zero element 0.

DEFINITION 1·1. By $a\bigtriangledown b$, it is meant that $a\cap b=0$, and $(a,x,b)D$ for every element $x\in L$, that is, $(a\cup x)\cap b=(a\cap b)\cup(x\cap b)=x\cap b$. If S is any subset of L, denote by $S^{\bigtriangledown}$ the set of a such that $a\bigtriangledown b$ for all $b\in S$.

If $a\bigtriangledown b$ aud $b\bigtriangledown a$ simultaneously, we say that a and b is *disjoint*.

REMARK 1·1. When L is modular, since $(a,x,b)D$ implies $(b,x,a)D$, $a\bigtriangledown b$ is equivalent to $b\bigtriangledown a$, and Definition 1·1 is equivalent to that of von Neumann: $a\cap b=0$, $(a,b)D$.[1]

LEMMA 1·1. *If S is any subset of L, $S^{\bigtriangledown}$ is an ideal of L.*

PROOF. (i) If $a_1,a_2\in S^{\bigtriangledown}$, then for all $b\in S$ and $x\in L$,

$$a_1\cap b=0,\qquad a_2\cap b=0,\qquad (a_1\cup x)\cap b=x\cap b,\qquad (a_2\cup x)\cap b=x\cap b.$$

Hence $\qquad (a_1\cup a_2)\cap b=a_2\cap b=0,\qquad (a_1\cup a_2\cup x)\cap b=(a_2\cup x)\cap b=x\cap b,$

and $\qquad a_1\cup a_2\in S^{\bigtriangledown}$.

(ii) If $a\in S^{\bigtriangledown}$, $a_1\leqq a$, then

$$a_1\cap b\leqq a\cap b=0,\ (a_1\cup x)\cap(a\cup x)\cap b=(a_1\cup x)\cap x\cap b=x\cap b,$$

that is $a_1\in S^{\bigtriangledown}$. Consequently $S^{\bigtriangledown}$ is an ideal of L.

DEFINITION 1·2. Let $S_1,\cdots,S_n$ be subsets of L which contain 0. If

(1°) for any element a of L,

$$a=a_1\cup\cdots\cup x_n,\qquad a_i\in S_i\qquad (i=1,\cdots,n),$$

(2°) $i\neq j$ implies $S_j\leqq S_i^{\bigtriangledown}$,

then we say that L is a *direct sum* of $S_1,\cdots, S_n$, and write $L=S_1\oplus\cdots\oplus S_n$. S_i is called the *component* of the direct sum.

(1) v. Neumann [1] I, 41-46. Numbers in brackets refer to the bibliography at the end of the paper.

LEMMA 1·2. *Let L be a direct sum of $S_1,\cdots,S_n$. Then any element $a\in L$ is expressible uniquely as*

$$a=a_1\cup\cdots\cup a_n,\qquad a_i\in S_i\qquad (i=1,\cdots,n).$$

PROOF. If a is expressible as

$$a=b_1\cup\cdots\cup b_n,\qquad b_i\in S_i\qquad (i=1,\cdots,n),$$

then since $b_2\cup\cdots\cup b_n\in S_1^\nabla$ by Lemma 1·1,

$$a_1=a\cap a_1=\{b_1\cup(b_2\cup\cdots\cup b_n)\}\cap a_1=b_1\cap a_1.$$

That is $a_1\leqq b_1$. Similarly $b_1\leqq a_1$. Hence $a_1=b_1$. Generally $a_i=b_i$ $(i=1,\cdots,n)$.

LEMMA 1·3. *The component of a direct sum is an ideal of L.*

PROOF. Let $L=S_1\oplus\cdots\oplus S_n$. If $x,y\in S_1$, then

$$x\cup y=a_1\cup\cdots\cup a_n,\qquad a_i\in S_i\qquad (i=1,\cdots,n).$$

For $i=2,\cdots,n$, $x,y\in S_i^\nabla$, hence $a_i=(x\cup y)\cap a_i=y\cap a_i=0$. Therefore $x\cup y=a_1\in S_1$.

If $x\in S_1$, $y\leqq x$, then

$$y=b_1\cup\cdots\cup b_n,\qquad b_i\in S_i\qquad (i=1,\cdots,n).$$

For $i=2,\cdots,n$, $b_i\leqq y\leqq x$ and $b_i\nabla x$. Hence $b_i=0$. Therefore $y=b_1\in S_1$. Consequently S_1 is an ideal of L.

THEOREM 1·1. *When $L=L_1\cdots L_n$, if we denote by S_i, the set of elements of L which are expressed as $[0_1,\cdots,0_{i-1},a_i^*,0_{i+1},\cdots,0_n]$ $(a_i^*\in L_i)$, then L is a direct sum of $S_1,\cdots,S_n$. Conversely if L is a direct sum of $S_1,\cdots,S_n$, then L is isomorphic to the product $S_1\cdots S_n$.*

PROOF. (i) When $L=L_1\cdots L_n$, if we put $a_i=[0_1,\cdots,0_{i-1},a_i^*,0_{i+1},\cdots,0_n]$, then any element $a=[a_1^*,\cdots,a_n^*]$ of L is expressed as

$$a=a_1\cup\cdots\cup a_n,\qquad a_i\in S_i\qquad (i=1,\cdots,n),$$

and $i\neq j$ implies $a_i\nabla a_j$, that is $S_i\leqq S_j^\nabla$. Consequently L is a direct sum of $S_1,\cdots,S_n$.

(ii) Let L be a direct sum of $S_1,\cdots,S_n$. By Lemma 1·3, $S_1,\cdots,S_n$ are sublattices of L, and by Lemma 1·2, any $a\in L$ is expressible uniquely as

$$a=a_1\cup\cdots\cup a_n,\qquad a_i\in S_i\qquad (i=1,\cdots,n).$$

Hence there exists a one-one correspondence between L and $S_1\cdots S_n$. Let $a=a_1\cup\cdots\cup a_n$, $b=b_1\cup\cdots\cup b_n$ $(a_i,b_i\in S_i)$. If $a\leqq b$, then since $b_2\cup\cdots\cup b_n\in S_1^\nabla$, we have

$$a_1=b\cap a_1=\{b_1\cup(b_2\cup\cdots\cup b_n)\}\cap a_1=b_1\cap a_1,$$

that is $a_1\leqq b_1$. Generally $a_i\leqq b_i$ $(i=1,\cdots,n)$. It is evident that $a_i\leqq b_i$ $(i=1,\cdots,n)$ imply $a\leqq b$. Hence the above correspondence preserves the lattice-order. Therefore L is isomorphic to $S_1\cdots S_n$.

LEMMA 1·4. *If $L=S_1\oplus S_2$, then $S_1=S_2^\nabla$, $S_2=S_1^\nabla$.*

PROOF. By Definition 1·2, $S_1\leqq S_2^\nabla$. Let $a\in S_2^\nabla$, then since

$$a=a_1\cup a_2,\qquad a_1\in S_1,\qquad a_2\in S_2,$$

and $a\nabla a_2$, we have $a_2=a\cap a_2=0$. Therefore $a=a_1\in S_1$. Consequently $S_1=S_2^\nabla$. Similarly $S_2=S_1^\nabla$.

LEMMA 1·5. *Let S be a component of a direct sum of L. Then $L=S\oplus S^\nabla$.*

PROOF. When $L=S\oplus S_1\oplus\cdots\oplus S_n$, denote by T the set of elements which are expressible as $a_1\cup\cdots\cup a_n$ $(a_i\in S_i)$, then as in the proof (ii) of Theorem 1·1, L is isomorphic to the product ST. Therefore by Theorem 1·1, $L=S\oplus T$, and by Lemma 1·4 $T=S^\nabla$.

LEMMA 1·6. *If $L=L(o,z)\oplus L(o,z)^{\triangledown}$, then z is a neutral element of L.*

PROOF. By Theorem 1·1, L is isomorphic to the product $L(o,z)L(o,z)^{\triangledown}$, where z corresponds to $[1_1, 0_2]$. Hence z is a neutral element of L.

LEMMA 1·7. *If z is a neutral element of a relatively complemented lattice L, then $L=L(o,z)\oplus L(o,z)^{\triangledown}$.*

PROOF. Put $S=\{b; b\frown z=0\}$. Let $a\in L(o,z)$, $b\in S$, $x\in L$, then

$$a\frown b\leqq z\frown b=0,$$

$$(a\smile x)\frown b=(a\smile x)\frown(z\smile x)\frown b=(a\smile x)\frown x\frown b=x\frown b.$$

Hence $a\triangledown b$. Consequently $L(o,z)\leqq S^{\triangledown}$. And since

$$(b\smile x)\frown a=(b\smile x)\frown z\frown a=x\frown z\frown a=x\frown a,$$

$b\triangledown a$. Therefore $S\leqq L(o,z)^{\triangledown}$.

For any $x\in L$, let $x=(x\frown z)\oplus y$.(1) Then $x\frown z\in L(o,z)$, $y\in S$. Therefore $L=L(o,z)\oplus S$. By Lemma 1·4 $S=L(o,z)^{\triangledown}$.

§ 2. Subdirect Sum of Conditionally Complete Lattices.

DEFINITION 2·1. Let L_0 be a sublattice of $\Pi(L_\alpha; \alpha\in I)$, and $a=[a_\alpha; \alpha\in I]$ be an arbitrary element of L_0. If, for any $\alpha\in I$, L_α is the image of L_0 under the homomorphism $a\to a_\alpha$, we say that L_0 is a *subdirect product* of L_α $(\alpha\in I)$.

DEFINITION 2·2. Let $\{a_\delta; \delta\in D\}$ be a directed set of a conditionally complete lattice L. When

$$a_\delta\uparrow a \quad \text{implies} \quad a_\delta\frown b\uparrow a\frown b,\text{(2)}$$

we say that L is a *conditionally upper continuous lattice*. Dually, when

$$a_\delta\downarrow a \quad \text{implies} \quad a_\delta\smile b\downarrow a\smile b,$$

we say that L is a *conditionally lower continuous lattice*. When L is a conditionally upper continuous and conditionally lower continuous lattice, we say that L is a *conditionally continuous lattice*.

In this section, we assume that the conditionally complete lattice L has the zero element.

DEFINITION 2·3. Let $\{S_\alpha; \alpha\in I\}$ be a family of subsets with 0 of a conditionally complete lattice L. If

(1°) every $a\in L$ is expressible in the form

$$a=\vee(a_\alpha; \alpha\in I), \qquad a_\alpha\in S_\alpha \ (\alpha\in I).$$

(2°) $\alpha\neq\beta$ implies $S_\beta\leqq S_\alpha^{\triangledown}$,

then we say that L is a *subdirect sum* of S_α $(\alpha\in I)$, and we write $L=\Sigma^*(\oplus S_\alpha; \alpha\in I)$. S_α is called the *component* of the subdirect sum.

If for any $a_\alpha\in S_\alpha(\alpha\in I)$, $\vee(a_\alpha; \alpha\in I)$ exists and belongs to L, then we say that L is a *direct sum* of S_α $(\alpha\in I)$, and write $L=\Sigma(\oplus S_\alpha; \alpha\in I)$.

LEMMA 2·1. *Let L be a conditionally upper continuous lattice.*

---

(1) When $(a_\alpha; \alpha\in I)\perp$, we write $\vee(\oplus a_\alpha; \alpha\in I)$ instead of $\vee(a_\alpha; \alpha\in I)$.

(2) This condition is equivalent to that of von Neumann, i.e. if $\alpha<\beta<\Omega$ implies $a_\alpha\leqq a_\beta$, then $\vee(a_\alpha\frown b; \alpha<\Omega)=\vee(a_\alpha; \alpha<\Omega)\frown b$. Cf. Sasaki [1].

(i) $a_\delta \nabla b$ $(\delta \epsilon D)$, $a_\delta \uparrow a$ *imply* $a \nabla b$.

(ii) *If* $T \leqq S^\nabla$ *and* $\vee(a; a \epsilon T)$ *exists, then* $\vee(a; a \epsilon T) \epsilon S^\nabla$.

(iii) *The components of a subdirect sum* $L = \Sigma^*(\oplus S_a; a \epsilon I)$ *are connditionally complete sublattices of* L.

PROOF. (i) Since $a_\delta \nabla b$, for every $x \epsilon L$,

$$a_\delta \cap b = 0, \qquad (a_\delta \cup x) \cap b = x \cap b.$$

Let $a_\delta \uparrow a$, then

$$a \cap b = 0, \qquad (a \cup x) \cap b = x \cap b.$$

That is $a \nabla b$.

(ii) Denote by ν the finite subset of T, and put $s_\nu = \vee(a; a \epsilon \nu)$, $s = \vee(a; a \epsilon T)$. Since $s_\nu \uparrow s$, and $s_\nu \epsilon S^\nabla$ by Lemma 1·1, we have $s_\nu \nabla b$ for every $b \epsilon S$. Hence by (i) $s \nabla b$ for every $b \epsilon S$. That is, $s \epsilon S^\nabla$.

(iii) As Lemma 1·3, we can prove that S_a is an ideal of L. When $s = \vee(a; a \epsilon T)$ exists where $T \leqq S_a$, put $s_\nu = \vee(a; a \epsilon \nu)$, $s = \vee(a; a \epsilon T)$ as in (ii), then $s_\nu \uparrow s$. If

$$s = \vee(a_a; a \epsilon I), \qquad a_a \epsilon S_a \ (a \epsilon I),$$

then, since $s_\nu \epsilon S_a$, $\beta \neq a$ implies $s_\nu \nabla a_\beta$. Therefore by (i) $s \nabla a_\beta$. Consequently $a_\beta = 0$, that is $s = a_a \epsilon S_a$.

Since S_a is an ideal of L, $\wedge(a; a \epsilon T) \epsilon S_a$.

LEMMA 2·2. *Let a conditionally upper continuous lattice* L *be a subdirect sum of* S_a $(a \epsilon I)$. *Then any element* $a \epsilon L$ *is expressible uniquely as*

$$a = \vee(a_a; a \epsilon I), \qquad a_a \epsilon S_a \ (a \epsilon I).$$

PROOF. We can prove as Lemma 1·2, using Lemma 2·1 (ii).

THEOREM 2·1. (i) *Let* 0_a *be the zero element of a conditionally complete lattice* L_a $(a \epsilon I)$. *Denote by* S_β *the set of elements* $[a_a^*; a \epsilon I]$ *of* $L = \Pi(L_a; a \epsilon I)$, *such that* $a_a^* = 0_a$ *when* $a \neq \beta$. *Then* L *is the direct sum of* S_a $(a \epsilon I)$.

(ii) *If a conditionally upper continuous lattice* L *is a subdirect sum of* S_a $(a \epsilon I)$, *then* L *is isomorphic to a subdirect product of* S_a $(a \epsilon I)$ *as a conditionally complete lattice.*

PROOF. (i) can be proved as the proof (i) of Theorem 1·1.

(ii) By Lemma 2·2, any $a \epsilon L$ is expressible uniquely as

$$a = \vee(a_a; a \epsilon I), \qquad a_a \epsilon S_a \qquad (a \epsilon I). \tag{1}$$

Denote by L_0 the set of $[a_a; a \epsilon I] \epsilon \Pi(S_a; a \epsilon I)$ which correspond to $a \epsilon L$ by (1). Then using Lemma 2·1, we can prove that L is isomorphic to L_0 as a conditionally complete lattice. And L_0 is a subdirect product of S_a $(a \epsilon I)$.

DEFINITION 2·4. When a conditionally upper continuous lattice L is a subdirect sum of S_a $(a \epsilon I)$, identifying the elements of $\bar{L} = \Pi(S_a; a \epsilon I)$, which correspond to the elements of $L = \Sigma^*(\oplus S_a; a \epsilon I)$, we write $\bar{L} = \Sigma(\oplus S_a; a \epsilon I)$, and $\bar{L}$ is called an *extended lattice* of L.

§ 3. Normal Ideals of Modular Lattices.

In this section, we assume that L is a conditionally upper continuous, relatively complemented, modular lattice with 0.

LEMMA 3·1. *The following two propositions* (a) *and* (β) *are equivalent:*

(α) $a \nabla b$,

(β) $a_1 \leqq a$, $b_1 \leqq b$, $a_1 \sim b_1$ *imply* $a_1 = b_1 = 0$.

PROOF. Cf. v.Neumann[1] I 42.

LEMMA 3·2. $a \cap b = 0$, $a \approx b$ *imply* $a \sim b$.

PROOF. Cf. v.Neumann and Halperin[1] 93.

LEMMA 3·3. *Let S be a subset of L. Then S^{∇} is a neutral ideal of L, and if $\bigvee(a; a \in T)$, where $T \leqq S^{\nabla}$, exists, then $\bigvee(a; a \in T) \in S^{\nabla}$. Especially when L has* 1, *there exists an element z of the center, such that $S^{\nabla} = L(o, z)$.*

PROOF. By Lemmas 1·1 and 2·1, S^{∇} is an ideal of L, and if $\bigvee(a; a \in T)$, where $T \leqq S^{\nabla}$, exists, then $\bigvee(a; a \in T) \in S^{\nabla}$. Next let $a \in S^{\nabla}$ and $a \sim c$. For $b \in S$, if there exist b_1 and c_1, such that $c_1 \leqq c$, $b_1 \leqq b$, $c_1 \sim b_1$ then there exists a_1 such that $c_1 \sim a_1 \leqq a$. Since $a_1 \cap b_1 \leqq a \cap b = 0$, by Lemma 3·2 $a_1 \sim b_1$. But since $a \nabla b$, by Lemma 3·1, $b_1 = 0$. Therefore $c \nabla b$, and $c \in S^{\nabla}$. That is, S^{∇} is a neutral ideal.

Especially when L has 1, $z = \bigvee(a; a \in S^{\nabla})$ exists and $z \in S^{\nabla}$. From above $z \sim c$ implies $c \in S^{\nabla}$. Hence $c = z$ and z is an element of the center.

DEFINITION 3·1. Let S be a subset of L. When $S^{\nabla\nabla} = S$, we say that S is a *normal ideal* of L.

LEMMA 3·4. *Let S be a normal ideal of L, then L is the direct sum of S and S^{∇}.*

PROOF. For any $x \in L$, put $x_S = \bigvee(x \cap a; a \in S)$, $x = x_S \oplus x'_S$. Then by Lemma 3·3 $x_S \in S$. If $x_1 \leqq x'_S$, $a_1 \leqq a$, $x_1 \sim a_1$ for $a \in S$, then, since S is a neutral ideal, we have $x_1 \in S$. Hence $x_1 = x \cap x_1 \leqq x_S$, aud $x_1 \leqq x_S \cap x'_S = 0$. Therefore by Lemma 3·1 $x'_S \nabla a$, that is $x'_S \in S^{\nabla}$. Consequently L is the direct sum of S and S^{∇}.

THEOREM 3·1. *A necessary and sufficient condition that L is irreducible is that there exist no normal ideals except* (0) *and L.*

PROOF. (i) Necessary. If there exists a normal ideal S which is different to (0) and L, then by Lemma 3·4 L is the direct sum of S and S^{∇}, and by Theorem 1·1, L is isomorphic to the product SS^{∇}. Hence L is reducible.

(ii) Sufficient. If L is reducible, then L is isomorphic to a product $L_1 L_2$. By Theorem 1·1, L is a direct sum of S_1 and S_2 which correspond to L_1 and L_2 respectively. By Lemma 1·4, S_1 and S_2 are normal ideals of L.

THEOREM 3·2. *The famiiy $\mathbf{Z}$ of all normal ideals in L is a complete Boolean algebra, where lattice-order means set-inclusion.*

PROOF. Since $S \to S^{\nabla\nabla}$ is a closure operation, $\mathbf{Z}$ is a complete lattice. Since $S \cap S^{\nabla} = (o)$ and by Lemma 3.4 $S \cup S^{\nabla} = L$, S^{∇} is a complement of S.

Next, let S and T be normal ideals such that $S \cap T = (0)$. If $T \nleqq S^{\nabla}$, then there exists $a \in L$, such that $a \in T$, $a \notin S^{\nabla}$. Then by Lemma 3·1, for an element $b \in S$, there exist a_1, b_1 such that $0 < a_1 \leqq a$, $0 < b_1 \leqq b$, $a_1 \sim b_1$. But by Lemma 3·3, $a_1 \in S$. This is contradictory, since $a_1 \in S \cap T$. Consequently $S \cap T = (0)$ implies $T \leqq S^{\nabla}$, and $\mathbf{Z}$ is a Boolean algebra.(1)

LEMMA 3·5. *A necessary and sufficient condition that $L(o, z)$ is a normal ideal is that*

---

(1) Cf. Ogasawara[1], 5.

z is a neutral element.

PROOF. (i) If z is a neutral element, then by Lemma 1·7 $L=L(o,z)\oplus L(o,z)^{\triangledown}$. Hence by Lemma 1·4 $L(o,z)^{\triangledown\triangledown}=L(o,z)$.

(ii) If $L(o,z)$ is a normal ideal, by Lemma 3·4 $L=L(o,z)\oplus L(o,z)^{\triangledown}$. Hence by Lemma 1·6 z is a neutral element.

THEOREM 3·3. *In the complete Boolean algebra $\mathbf{Z}$ of normal ideals, the set $\mathbf{Z}_0$ of normal ideals expressed in the form $L(o,z)$ is an ideal of $\mathbf{Z}$, and is isomorphic to the set Z_0 of all neutral elements of L.*

PROOF. By Lemma 3·5, between Z_0 and $\mathbf{Z}_0$, there exists a one-one correspondence $Z\longleftrightarrow L(o,z)$, and we can easily prove that Z_0 and $\mathbf{Z}_0$ is lattice-isomorphic by the relation:

$$L(o,z_1)\cup L(o,z_2)=L(o,z_1\cup z_2),\qquad L(o,z_1)\cap L(o,z_2)=L(o,z_1\cap z_2).$$

Let S be a normal ideal such that $S\leqq L(o,z)$. Then $z_1=\vee(a;a\in S)$ exists, and by Lemma 3·3 $z_1\in S$. Hence $S=L(o,z_1)$. Therefore $\mathbf{Z}_0$ is an ideal of $\mathbf{Z}$.

THEOREM 3·4. *Let Z_0 be the set of all neutral elements of L, and put $Z_0^{\triangledown\triangledown}=L_1$, $Z_0^{\triangledown}=L_\infty$. Then $L=L_1\oplus L_\infty$, where L_∞ has no neutral elements except 0, and L_1 is embedded in an upper continuous, complemented, modular lattice $\bar{L}_1$, whose center contains Z_0.*

PROOF. Since $Z_0^{\triangledown}$ is a normal ideal, by Lemma 3·4 $L=L_1\oplus L_\infty$. It is evident that $L_\infty=Z_0^{\triangledown}$ contains no neutral elements of L except 0. Since the neutral element of L_∞ is the neutral element of L, L_∞ has no neutral elements except 0.

Since $L_1=Z_0^{\triangledown\triangledown}$ is the smallest normal ideal which contains all neutral elements of L, by Lemma 3·5 we have $L_1=\vee(L(o,z);z\in Z_0)$. Applying v.Neumann[1] III, 32 Lemma 3·3 to $\mathbf{Z}$, there exist neutral elements $z_a\,(a\in I)$, such that $L_1=\vee(\oplus L(o,z_a);a\in I)$. This means that L_1 is a subdirect sum of $L(o,z_a)$ $(a\in I)$. And the extended lattice $\bar{L}_1=\Sigma(\oplus L(o,z_a);a\in I)$ is an upper continuous, complemented, modular lattice with the unit element $\vee(\oplus z_a;a\in I)$, and to the neutral element z of L corresponds $z=\vee(\oplus(z\cap z_a);a\in I)$, which is an element of the center of $\bar{L}$.

§ 4. Dimension Function and Subdirect Product Representation of General Continuous Geometry.

By a *general continuous geometry* L, we shall mean a conditionally continuous, relatively complemented, modular lattice with 0. If L has 1, then it is a continuous geometry. For a (reducible) continuous geometry L, Iwamura[1] and Kawada-Matsushima-Higuchi[1] gave complete results about the dimension function and the subdirect product sepresentation of L. But these theories can be applied to the general continuous geometry with slight modifications. Since the whole discussions are complicated, we shall give brief explanations and notices.

By Theorem 3·2, the set $\mathbf{Z}$ of all normal ideals in a general contiuous geometry L is a complete Boolean algebra, and by Lemma 3·3, when L has 1, that is, L is a continuous geometry, $\mathbf{Z}$ is isomorphic to the center of L. Hence we may expect that in the general continuous geometry, $\mathbf{Z}$ has the same rôle as the center of the continuous geometry.

Let $S\in\mathbf{Z}$. By Lemma 3·4, any $x\in L$ is expressed uniquely as

$$x=x_S\oplus x'_S,\qquad x_S\in S,\qquad x'_S\in S^{\triangledown}.$$

If we denote x_S by Sx, then we have

$$S(x\cup y)=Sx\cup Sy,\qquad S(x\cap y)=Sx\cap Sy,$$

$$(S_1\cup S_2)x=S_1x\cup S_2x,\qquad (S_1\cap S_2)x=S_1x\cap S_2x.$$

Therefore the operation Sx corresponds to $z\cap x$, where z is an element of the center of the continuous geometry. Instead of the central envelope $e(a)$, we must use $a^{\triangledown\triangledown}$, which is the smallest normal ideal which contains a. And we have the fundamental theorems about the general continuous geometry, as in the case of the continuous geometry discussed in v.Neumann[1] III.

Hereafter, L is a general continuous geometry.

An element $a\in L$ is *minimal* in case $x\ll a$ implies $x=0$. Denote by L_{I} the join of normal ideals $a^{\triangledown\triangledown}$, where a runs over all minimal elements of L, and let L_{II} be the complement of L_{I} in $\mathbf{Z}$. Then $L=L_{\mathrm{I}}\oplus L_{\mathrm{II}}$. There exists a minimal element h of the extended lattice $\bar{L}_{\mathrm{I}}$ of L_{I} such that $h^{\triangledown\triangledown}=\bar{L}_{\mathrm{I}}$, which we shall call a *basic minimal element* of $\bar{L}_{\mathrm{I}}$.

By Theorem 3·4. $L=L_1\oplus L_\infty$. Hence if we put $L_{\mathrm{II}_1}=L_1\cap L_{\mathrm{II}}$, $L_{\mathrm{I}_\infty}=L_\infty\cap L_{\mathrm{I}}$, $L_{\mathrm{II}_\infty}=L_\infty\cap L_{\mathrm{II}}$, then

$$L=(L_1\cap L_{\mathrm{I}})\oplus L_{\mathrm{II}_1}\oplus L_{\mathrm{I}_\infty}\oplus L_{\mathrm{II}_\infty}.$$

An element $e\in L$ such that $e^{\triangledown\triangledown}=L$ is called a *subunit element of* L. Since $L=\bigvee(a^{\triangledown\triangledown};a\in L)$, there exist a_α $(\alpha\in I)$ such that $L=\bigvee(\oplus a_\alpha^{\triangledown\triangledown};\alpha\in I)$. Then $e=\bigvee(\oplus a_\alpha;\alpha\in I)$ is a subunit element of $\bar{L}=\Sigma(\oplus a_\alpha{}^{\triangledown\triangledown};\alpha\in I)$, which is the extended lattice of L.

Hereafter, we take the subunit element e of the extended lattice $\bar{L}$ of $L=L_1\oplus L_{\mathrm{I}_\infty}\oplus L_{\mathrm{II}_\infty}$, such that

$$e=e_1\oplus h_\infty\oplus e_{\mathrm{II}_\infty},$$

where e_1 is the unit element of $\bar{L}_1$, h_∞ is a basic minimal element of $\bar{L}_{\mathrm{I}_\infty}$, and e_{II_∞} is a subunit element of $\bar{L}_{\mathrm{II}_\infty}$.

The set $\bar{Z}$ of $\bar{S}e$, where $\bar{S}$ runs over the normal ideals of $\bar{L}$, is called the *subcenter* of $\bar{L}$. $\bar{Z}$ is isomorphic to $\mathbf{Z}$.

We can prove as v.Neumann[1] III, 34 Theorem 3·2, that

$L_1\cap L_{\mathrm{I}}=\overset{*}{\underset{1\leqq k<\infty}{\Sigma}}\oplus L_{\mathrm{I}_k}$. Hence we have

$$L=\overset{*}{\underset{1\leqq k<\infty}{\Sigma}}\oplus L_{\mathrm{I}_k}\oplus L_{\mathrm{II}_1}\oplus L_{\mathrm{I}_\infty}\oplus L_{\mathrm{II}_\infty}.$$

Denote by Ω the set of all maximal ideals $\mathfrak{p}$ of $\mathbf{Z}$ (*or* $\bar{Z}$), and by $E(S)$ the set of all maximal ideals $\mathfrak{p}$ which do not contain S. Then by the correspondence $S\to E(S)$, $\mathbf{Z}$ is isomorphic to a set-lattice $\{E(S);S\in\mathbf{Z}\}$ in the Boolean space Ω.

As Iwamura[1] did, we can obtain a dimension $D(a)$ $(a\in\bar{L})$, which is a continuous function $\delta_a(\mathfrak{p})$ defined in Ω. The range $\triangle$ of $\delta_a(\mathfrak{p})$ is as follows:

when $\mathfrak{p}\in E(L_{\mathrm{I}_k})$, $\triangle$ consists of all m/k, $m=0,1,\ldots,k$;

when $\mathfrak{p}\in E(L_{\mathrm{II}_1})$, $\triangle$ consists of all λ, $0\leqq\lambda\leqq 1$;

when $\mathfrak{p}\in E(L_{\mathrm{I}_\infty})$, $\triangle$ consists of all m, $m=0,1,\ldots$;

when $\mathfrak{f} \in E(L_{\mathrm{II}_\infty})$, $\triangle$ consists of all λ, $0 \leqq \lambda < \infty$.

$D(a)$ has the same properties of dimension functions as the continuous geometry, except the normalizasion, which is expressed as follows:

For any element z of $\bar{Z}$, $\delta_z(\mathfrak{f})=0$ or 1 according as $z \in \mathfrak{f}$ or not.

And as Kawada-Matsushima-Higuchi[1] did, we have the following subdirect product representation:

The extended lattice $\bar{L}$ of a general continuous geometry L is isomorphic to a subdirect product of $\bar{L}/J$ ($J \in \Omega$), where Ω is the set of all maximal neutral ideals of $\bar{L}$, and $\bar{L}/J$ are simple general continuous geometries.

Bibliography.

T.Iwamura[1], *On continuous geometries* I, Jap. Jour. of Math. **19** (1944), 57-71.

Y.Kawada, Y.Matsushima and K.Higuchi[1], *About Iwamura's paper on continuous geometries*, Shijô-Sûgaku-Danwakai **264** (1944), 134-143. (In Japanese)

J.v.Neumann [1], *Lectures on Continuous Geometries*, I, II, III, Princeton, 1936-7.

J.v.Neumann and I.Halperin [1], *On the transitivity of perspective mappings*. Annals of Math. **41** (1940), 87-93.

T.Ogasawara[1], *Lattice Theory* II, Tôkyô, 1948. (In Japanese)

U.Sasaki[1], *On the axioms of continuous geometries*, Shijô-Sûgaku-Danwakai, (2) **10** (1948), 303-305. (In Japanese)

HIROSHIMA UNIVERSITY.