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On Certain Continuous and Equicontinuous Collections of Compact Continua

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Introduction

In this paper we shall prove some theorems concerning the continuous and equicontinuous collections of compact continua in a separable metric space. In § 1 we shall prove a theorem on the cross-sections of a continuous and equicontinuous collection of arcs which is a generalization of the case of R. L. Moore (see Proposition 1 in § 2). In § 2 we shall apply the theorems in § 1 to the set-theoretical characterization of closed n-cells ($n \leqq 3$). From the same standpoint as in § 2, the closed n-cells were considered by M. E. Hamstrom and E. Dyer (see [4][1]). In the final section we shall consider the topological properties of the sum of all the sets of a continuous and equicontinuous collection of compact continua which are induced from the properties of the decomposition space and each set of the collection.

Throughout this paper all spaces are separable and metric, and the distance functions are denoted by ρ[2].

§ 1. Theorems on the cross-sections of a continuous and equicontinuous collection of arcs

DEFINITION. A collection $\{Q\}$ of compact sets is said to be *continuous* if, for each $Q \in \{Q\}$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that $Q' \frown U_\delta(Q) \neq \phi$ implies both

$$Q' \subset U_\varepsilon(Q) \quad \text{and} \quad Q \subset U_\varepsilon(Q'), \quad Q' \in \{Q\}.$$

DEFINITION. A collection $\{Q\}$ of compact continua is said to be *equicontinuous* if, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if x and y are two points belonging to a Q of $\{Q\}$ and with $\rho(x, y) < \delta$ then there exists an arc xy of Q of diameter less than ε.

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1) Numbers in brackets refer to the references at the end of this paper.

2) Most of the terminologies and notations in this paper are due to R. L. Moore's book [1] and G. T. Whyburn's [2].

The following proposition is an immediate consequence of a theorem of R. L. Moore (Cf. [1], p. 397).

PROPOSITION 1. *Suppose a, b, c and d are four distinct points, ab, bc, cd, da are arcs of which no two have any point in common other than a common end point and f is a topological mapping of ab onto dc. Suppose $\{l_\alpha\}$ is a collection of mutually exclusive arcs l_α with x_α and $f(x_\alpha)$ as its end points, $x_\alpha \in ab$, such that*

(1) *$\{l_\alpha\}$ is continuous and equicontinuous,*

(2) *bc and ad are arcs of $\{l_\alpha\}$.*

Let M denote the set which is the sum of all the arcs of $\{l_\alpha\}$.

Then there exists a collection $\{m_\beta\}$ of mutually exclusive arcs m_β with y_β and $g(y_\beta)$ as its end points, where $y_\beta \in bc$ and g is a topological mapping of bc onto ad, such that

(a) *$\{m_\beta\}$ is continuous and equicontinuous,*

(b) *ab and cd are arcs of $\{m_\beta\}$,*

(c) *M is the sum of all the arcs of $\{m_\beta\}$,*

(d) *if $l_\alpha \in \{l_\alpha\}$ and $m_\beta \in \{m_\beta\}$, then $l_\alpha \frown m_\beta$ is exactly one point.*

COROLLARY 1. *The set M is a closed 2-cell.*

PROOF. Let S be the solid square with vertices A(0, 1), B(0, 0), C(1, 0), D(1, 1) in Euclidean 2-space, let φ be a topological mapping of the arc ab onto the straight line interval AB and ψ be a topological one of the arc bc onto the interval BC. Let $\{m'_\beta\}$ be the collection of all the intervals of S which have the end points on $BC \smile AD$ and are parallel to AB, and let $\{l'_\alpha\}$ be the collection of all the intervals of S which have the end points on $AB \smile CD$ and are parallel to BC. Then ψ induces the one-to-one transformation Ψ of $\{m_\beta\}$ onto $\{m'_\beta\}$ under which each m_β of $\{m_\beta\}$ is transformed to the interval m'_β of $\{m'_\beta\}$ having the image of $m_\beta \frown bc$ under ψ as one end point; and, in the same way, φ induces the one-to-one transformation Φ of $\{l_\alpha\}$ onto $\{l'_\alpha\}$. Moreover, for each point $l_\alpha \frown m_\beta$ of M if we make correspond the point $\Phi(l_\alpha) \frown \Psi(m_\beta)$ of S, then the correspondence h between M and S is one-to-one. Since $\{l_\alpha\}, \{l'_\alpha\}, \{m_\beta\}, \{m'_\beta\}$ are continuous and φ and ψ are topological, it is easily known that h is topological.

Thus Corollary 1 is proved.

It is known by the above corollary that Proposition 1 is a result for 2-dimensional sets. Next we shall prove the following theorem for 3-dimensonal sets which corresponds to Proposition 1.

THEOREM 1. *Suppose D_0 and D_1 are two disjoint closed 2-cells and f is a topological mapping of D_0 onto D_1. Suppose $\{n_\alpha\}$ is a collection of mutually exclusive arcs n_α with x_α and $f(x_\alpha)$ as its end points, $x_\alpha \in D_0$, such that $\{n_\alpha\}$ is continuous and equicontinuous. Let N denote the set which is the sum of all the arcs of $\{n_\alpha\}$.*

Then there exists a collection $\{D_\beta\}$ of mutually exclusive closed 2-cells *such that*

(a) $\{D_\beta\}$ *is continuous and equicontinuous,*

(b) D_0 *and* D_1 *are closed 2-cells of* $\{D_\beta\}$,

(c) N *is the sum of all the closed 2-cells of* $\{D_\beta\}$,

(d) *if* n_α *and* D_β *are an arc of* $\{n_\alpha\}$ *and a closed 2-cell of* $\{D_\beta\}$, *respectively, then* $n_\alpha \frown D_\beta$ *is exactly one point.*

Theorem 1 will be proved with the help of some lemmas. These lemmas will be proved first. In this section we suppose that N, D_0, D_1 and $\{n_\alpha\}$ are as defined in Theorem 1.

NOTATIONS. Let S be the solid square defined in the proof of Corollary 1, and T be a topological mapping of S onto D_0. In the following of this section we shall use the notations as follows:

n'_β, $0\leqq\beta\leqq1$, (n''_γ, $0\leqq\gamma\leqq1$) denotes the image of the interval from $(\beta, 0)$ to $(\beta, 1)$ (from $(0,\gamma)$ to $(1,\gamma)$) under T,

$n_{(\beta,\gamma)}$ denotes the arc of $\{n_\alpha\}$ with the point $n'_\beta \frown n''_\gamma$ as one end point,

$D'_\beta(D''_\gamma)$ denotes the sum of all the arcs of $\{n_\alpha\}$ whose one end point lies on $n'_\beta(n''_\gamma)$.

We remark here that, by Corollary 1, the sets D'_β and D''_γ are closed 2-cells. Then the following lemma holds.

LEMMA 1.1. *Each one of* $\{D'_\beta\}$ *and* $\{D''_\gamma\}$ *is a continuous and equicontinuous collection of mutually exclusive closed 2-cells*[3].

PROOF. We shall only prove that $\{D'_\beta\}$ has the above properties. Let x_0 be any point and let $\{x_i\}$ be any sequence of points converging to x_0, where $x_0 \in n_{\alpha_0} \subset D'_{\beta_0}$ and $x_i \in n_{\alpha_i} \subset D'_{\beta_i}$. Then, by the continuity of $\{n_\alpha\}$, we have $\lim n_{\alpha_i} = n_{\alpha_0}$. Hence, by the continuity of $\{n'_\beta\}$, we have $\lim n'_{\beta_i} = n'_{\beta_0}$. Again by the continuity of $\{n_\alpha\}$ we have $\lim D'_{\beta_i} = D'_{\beta_0}$. Thus $\{D'_\beta\}$ is continuous.

Now to prove the equicontinuity of $\{D'_\beta\}$, suppose, on the contrary, that $\{D'_\beta\}$ is not so. Then for some $\varepsilon>0$ there exist two sequences of points $\{x_i^1\}$ and $\{x_i^2\}$ such that (1) for each i $\rho(x_i^1, x_i^2)<\dfrac{1}{i}$ and (2) there is no arc $x_i^1x_i^2$ of D'_{β_i} of diameter less than ε, where $x_i^1 \in n_{\alpha_i^1} \subset D'_{\beta_i}$ and $x_i^2 \in n_{\alpha_i^2} \subset D'_{\beta_i}$. We may suppose $\{x_i^1\}$ and $\{x_i^2\}$ converge to the same point $x_0 \in n_{\alpha_0}$. On the other hand, by the continuity of $\{n_\alpha\}$ we have $\lim n_{\alpha_i^1} = n_{\alpha_0}$ and $\lim n_{\alpha_i^2} = n_{\alpha_0}$. Hence, by the continuity and equicontinuity of $\{n_\alpha\}$, for i sufficiently large there exists a points $y_i^1 \in n_{\alpha_i^1}$ such that y_i^1 can be joined to x_i^1 and x_i^2 by arcs of $n_{\alpha_i^1}$ and D'_{β_i} of diameter less than $\dfrac{1}{2}\varepsilon$, respectively. Therefore x_i^1 and x_i^2 can be joined by an arc of D'_{β_i} of diameter less than ε, contrary to the

3) We note that, by virtue of the compactness of D_0 together with the continuity of $\{n_\alpha\}$, N is compact, and hence that the continuity of a collection in N is equivalent to the continuity in the limit sense of the one (Cf. [2], p. 130).

condition (2) of x_i^1 and x_i^2. Thus $\{D'_\beta\}$ is equicontinuous.

LEMMA 1.2. *Suppose x_0y_0 is an arc of D'_{β_0}, $\{x_i\}$ and $\{y_i\}$ are two sequences of points converging to the end points x_0 and y_0 respectively, where $x_i, y_i \in D'_{\beta_i}$, and ε is any positive number. Then there exists an integer N such that for each $i>N$ we can take an arc x_iy_i of D'_{β_i} joining x_i to y_i and lying in the ε-neighborhood of x_0y_0.*

PROOF. By the equicontinuity of $\{D'_\beta\}$, there exists a $\delta>0$ such that every two points of a D'_β of $\{D'_\beta\}$ whose distance apart is less than δ can be joined by an arc of D'_β of diameter less than $\frac{1}{3}\varepsilon$. Let $z_0^1, \cdots, z_0^m$ be a finite number of points on x_0y_0 such that all the distances $\rho(x_0, z_0^1)$, $\rho(z_0^1, z_0^2)$, $\cdots$, $\rho(z_0^{m-1}, z_0^m)$, $\rho(z_0^m, y_0)$ are less than $\frac{1}{3}\delta$. By the continuity of $\{D'_\beta\}$ we have $\lim D'_{\beta_i}=D'_{\beta_0}$. Therefore there exist an integer N and points $z_N^1, \cdots$, z_N^m of D'_{β_N} such that all the distances $\rho(x_0, x_N)$, $\rho(z_0^1, z_N^1)$, $\rho(z_0^2, z_N^2)$, $\cdots$, $\rho(z_0^m, z_N^m)$ and $\rho(y_0, y_N)$ are less than $\frac{1}{3}\delta$. Hence all the distances $\rho(x_N, z_N^1)$, $\rho(z_N^1, z_N^2)$, $\cdots$, $\rho(z_N^m, y_N)$ are less than δ. Therefore, there exist in D'_{β_N} arcs $x_Nz_N^1$, $z_N^2z_N^2$, $\cdots$, $z_N^{m-1}z_N^m$, $z_N^my_N$ each of which is of diameter less than $\frac{1}{3}\varepsilon$. It is easy to see that the sum of the arcs $x_Nz_N^1\smile z_N^1z_N^2\smile\cdots\smile z_N^my_N$ contains an arc desired in Lemma 1.2. Thus Lemma 1.2 is proved.

LEMMA 1.3. *Suppose x_0y_0 and x_iy_i $(i=1, 2, \cdots)$ are subarcs of arcs n_{α_0} and n_{α_i} of $\{n_\alpha\}$, respectively, such that the two sequences of end points of them $\{x_i\}$ and $\{y_i\}$ converge to the end points x_0 and y_0 respectively. Then the sequence of arcs $\{x_iy_i\}$ converges to the arc x_0y_0.*

PROOF. First we shall show that in the case $x_0=y_0$ Lemma 1.3 holds. Let ε be any positive number and let δ be a number defined for ε by the equicontinuity of $\{n_\alpha\}$. Now, since $\{x_i\}$ and $\{y_i\}$ converge to the same point x_0, there exists an integer N such that for each $i>N$ $\rho(x_i, x_0)<\frac{1}{3}\varepsilon$, $\rho(y_i, x_0)<\frac{1}{3}\varepsilon$ and $\rho(x_i, y_i)<\delta$. Therefore each arc x_iy_i, $i>N$, lies in the ε-neighborhood of x_0. Thus we have $\lim x_iy_i=x_0=x_0y_0$.

Next, to show that in general Lemma 1.3 holds, suppose, on the contrary, that there exists a point $z_0\in n_{\alpha_0}$ belonging to $\limsup x_iy_i$ but not to x_0y_0. We may suppose, without loss of generality, that we have the order z_0, x_0, y_0 on n_{α_0}, Let $\{z_{i'}\}$ be a sequence of points converging to z_0, where $z_{i'}\in x_{i'}y_{i'}$. Let us take a sequence $\{y'_{i'}\}$ of points converging to x_0, where each $y'_{i'}$ belongs to the subarc $z_{i'}y_{i'}$ of $n_{\alpha_{i'}}$. Then, by the result of the preceding paragraph, we have $\lim x_{i'}y'_{i'}=x_0$, contrary to the assumption that $\{z_{i'}\}$ converges to z_0. Thus Lemma 1.3 is proved.

DEFINITIONS. A set K is said to be a *simple set of type 1* if there exists a number β_0, $0\leqq\beta_0\leqq1$, such that

(1′) K is a connected subset of D'_{β_0} which is open in D'_{β_0},

(2′) for any γ, $0\leqq\gamma\leqq1$, if x and y are any two points of $K\frown n_{(\beta_0,\gamma)}$, K contains the subarc xy of $n_{(\beta_0,\gamma)}$,

(3′) $K\frown n_{(\beta_0,i)}$ contains a non-vacuous open arc ($i=0,1$).

A set K is said to be a *simple set of type 2* if there exist two numbers β_1 and β_2, $0\leqq\beta_1<\beta_2\leqq1$, such that

(1″) K is a connected open subset of N which lies between D'_{β_1} and D'_{β_2},

(2″) for any pair (β,γ), $\beta_1<\beta<\beta_2$ and $0\leqq\gamma\leqq1$, if x and y are any two points of $K\frown n_{(\beta,\gamma)}$, K contains the subarc xy of $n_{(\beta,\gamma)}$,

(3″) for any β, $\beta_1<\beta<\beta_2$, $K\frown D'_\beta$ is a simple set of type 1,

(4″) $\overline{K}\frown D'_{\beta_i}$ contains a simple set K^i of type 1 such that no point of K^i is a limit point of any set which lies between D'_{β_1} and D'_{β_2} and contains no point of K ($i=1,2$).

The simple set K^1 of type 1 will be called the *upper base*, and the simple set K^2 of type 1 called the *lower base* of K.

A simple set K of type 1 or 2 is said to be *of rank n* if, for each n_α of $\{n_\alpha\}$, the diameter of $K\frown n_\alpha$ is less than $\dfrac{1}{n}$.

If there exists a simple set of type 2 and of rank n whose upper base contains one of two arcs and whose lower base contains another one, then we say that the two arcs can *be joined by a simple set* of type 2 and of rank n.

An arc t lying in D'_β will be called an S-*arc* if for each γ, $0\leqq\gamma\leqq1$, $t\frown n_{(\beta,\gamma)}$ is exactly one point.

LEMMA 1.4. *Let s be an* S-*arc of D'_{β_1} and n be any positive integer. Then there exists a simple set K of type 2 and of rank n whose upper base* (*or lower base*) *contains the arc s.*

PROOF. Let S be any point of s and let D''_γ be the closed 2-cell of $\{D''_\gamma\}$ containing S. Let δ be the minimum of $\dfrac{1}{4n}$ and a positive number defined for $\dfrac{1}{2n}$ by the equicontinuity of $\{n_\alpha\}$, and let $U_\delta(S)$ and $U_{\frac{1}{2n}}(S)$ be the δ- and $\dfrac{1}{2n}$ -neighborhoods of S in D''_γ, respectively. Then for each β, $0\leqq\beta\leqq1$, $U_\delta(S)\frown n_{(\beta,\gamma)}$ is contained in one component of $U_{\frac{1}{2n}}(S)\frown n_{(\beta,\gamma)}$. If $c_{(\beta,\gamma)}$ denotes the component of $U_{\frac{1}{2n}}(S)\frown n_{(\beta,\gamma)}$ containing $U_\delta(S)\frown n_{(\beta,\gamma)}$, $V(S)$ is the component of $U_\delta(S)$ containing S and $W(S)$ is the sum of all sets $c_{(\beta,\gamma)}$ such that $c_{(\beta,\gamma)}$ intersects $V(S)$, then $W(S)$ has the following properties: (i′) $W(S)$ is connected, (ii′) for each β, $0\leqq\beta\leqq1$, $W(S)\frown n_{(\beta,\gamma)}$ is either an open arc of diameter less than $\frac{1}{n}$ or an empty set and (iii′) $W(S)$ is open in D''_γ. (i′) and (ii′) are obvious. To prove (iii′), suppose, on the contrary, that $W(S)$ is not open in D''_γ. Then there exist a point $x_0 \in W(S)$ and a sequence $\{x_i\}$ of points in $D''_\gamma - W(S)$ converging to x_0. Let $n_{(\beta_0, \gamma)}$ and $n_{(\beta_i, \gamma)}$ be the arcs of $\{n_\alpha\}$ containing x_0 and x_i respectively. Let y_0 be a point of $V(S) \frown n_{(\beta_0, \gamma)}$ and $\{y_i\}$ be a sequence of points in D''_γ converging to y_0, where $y_i \in n_{(\beta_i, \gamma)}$. By the continuity of $\{n_\alpha\}$, such a sequence $\{y_i\}$ exists. Since $V(S)$ is open in D''_γ, we may suppose that every point of $\{y_i\}$ is in $V(S)$. By Lemma 1.3 we have $\lim x_i y_i = x_0 y_0$, where $x_0 y_0$ and $x_i y_i$ are the subarcs of $n_{(\beta_0, \gamma)}$ and $n_{(\beta_i, \gamma)}$ respectively. Hence, it is readily seen by the definition of $W(S)$ that for each i sufficiently large the arc $x_i y_i$ lies in $W(S)$, contrary to the assumption that $x_i \in D''_\gamma - W(S)$. Thus (iii′) is proved.

For every point S of s, we construct the sets $V(S)$ and $W(S)$ described above, and denote by V and W the sum of $V(S)$ and the sum of $W(S)$, respectively. Then the set V is a connected open set in N. For, suppose, on the contrary, that V is not open in N, then there exist a point z_0 and a sequence $\{z_i\}$ converging to z_0, where $z_0 \in V(S_0) \subset V$ and $z_i \in N - V$. Let D''_{γ_i} be the closed 2-cell of $\{D''_\gamma\}$ containing z_i and S_i be the point $s \frown D''_{\gamma_i}$. Let $z_0 S_0$ be an arc of $V(S_0)$ joining z_0 to S_0. Now, by Lemma 1.2, we can take arcs $z_i S_i$ of D''_{γ_i} joining z_i to S_i such that $\lim z_i S_i = z_0 S_0$. Hence, it is easily known that for i sufficiently large, the arc $z_i S_i$ lies in $V(S_i)$, contrary to the assumption that $z_i \in N - V$. Thus V is open in N.

The set W also has the following properties: (i″) W is connected, (ii″) for each n_α of $\{n_\alpha\}$, $W \frown n_\alpha$ is either an open arc of diameter less than $\frac{1}{n}$ or an empty set and (iii″) W is open in N. For, (i″) and (ii″) are obvious, and (iii″) can be proved by the same method as in (iii′) since V is open in N.

Now, by virtue of Lemma 1.1 together with the fact that V is open in N, we can choose a set D'_{β_2} of $\{D'_\beta\}$ near to D'_{β_1}, so that for any pair (β, γ), $\beta_1 \leqq \beta \leqq \beta_2$ (or $\beta_2 \leqq \beta \leqq \beta_1$) and $0 \leqq \gamma \leqq 1$, $V \frown n_{(\beta, \gamma)}$ is not empty. Finally, let K denote the common part of W and the part of N lying between D'_{β_1} and D'_{β_2}. Then it is easily shown that K satisfies all the conditions required in the statement of Lemma 1.4. Thus Lemma 1.4 is proved.

LEMMA 1.5. *Suppose K is a simple set of type 2 whose bases lie in D'_{β_1} and D'_{β_2}, s_1 and s_2 are* S*-arcs in the bases of K lying in D'_{β_1} and D'_{β_2} respectively, and n is a positive integer. Then there exists a simple set K_n of type 2 and of rank n such that* (1) *$\overline{K}_n$ lies in the sum of K and its bases and* (2) *K_n joins s_1 to s_2.*

PROOF. Without loss of generality we may suppose that $\beta_1 < \beta_2$. The proof of the lemma will be divided into the following two steps.

(I) Suppose t' is an S-arc such that (1) t' lies either in K or in the base of K not containing s_1 and (2) t' and s_1 can be joined by a simple set of type 2 and of rank n whose closure lies in the sum of K and its bases, and suppose t'' is any S-arc of D'_β lying in the sum of K and its bases, where D'_β is the closed 2-cell of $\{D'_\beta\}$ containing t'. Then t'' and s_1 also can be joined by a simple set of type 2 and of rank n whose closure lies in the sum of K and its bases.

For, let $\{t_x\}$, $0 \leqq x \leqq 1$, be a continuous deformation from t' to t'' in the common part of D'_β and the sum of K and its bases such that each t_x is an S-arc, $t_0 = t'$ and $t_1 = t''$. Now suppose, on the contrary, that s_1 and t'' can not be joined by a set satisfying the conditions in (I). Let $\{t_{x'}\}$ be the collection of all S-arcs of $\{t_x\}$ which can be joined to s_1 by a set satisfying the conditions of (I), and let $\{t_{x''}\}$ be the collection $\{t_x\} - \{t_{x'}\}$. Furthermore, let $\{x'\}$ and $\{x''\}$ be the sets of indices of all the arcs of $\{t_{x'}\}$ and $\{t_{x''}\}$, respectively. Then it follows at once from the definition of joining two S-arcs by a simple set that $\{x'\}$ is open in the unit interval $[0,1]$. Hence, by the connectedness of intervals, there exist a point $x'' \in \{x''\}$ and a sequence $\{x'_i\}$ in $\{x'\}$ converging to x''. By Lemma 1.4, we can take a simple set $K_n^{(1)}$ of type 2 and of rank n such that the lower base of $K_n^{(1)}$ contains $t_{x''}$ and $\overline{K}_n^{(1)}$ lies in the sum of K and its bases. Then, for i sufficiently large, $t_{x'_i}$ lies in the lower base of $K_n^{(1)}$. Now let $K_n^{(2)}$ be a simple set of type 2 and of rank n such that $\overline{K}_n^{(2)}$ lies in the sum of K and its bases and $K_n^{(2)}$ joins s_1 to $t_{x'_i}$. By Lemma 1.2, we can take a γ, $\beta_1 < \gamma < \beta$, such that D'_γ contains an S-arc in $K_n^{(1)} \frown K_n^{(2)}$. Let $K_n^{(3)}$ denote the sum of the part of $K_n^{(2)}$ lying between D'_{β_1} and D'_γ, the part of $K_n^{(1)}$ lying between D'_γ and D'_β and the part of $K_n^{(1)} \frown K_n^{(2)}$ lying on D'_γ. Then it is easily shown that $K_n^{(3)}$ is a simple set of type 2 and of rank n such that $K_n^{(3)}$ joins s_1 to $t_{x''}$ and $K_n^{(3)}$ lies in the sum of K and its bases, contrary to the assumption that $t_{x''} \in \{t_{x''}\}$. Thus (I) is proved.

(II) If we denote by δ the least upper bound of the set of all indices β, $\beta_1 \leqq \beta \leqq \beta_2$, such that each D'_β contains an S-arc which can be joined to s_1 by a simple set satisfying the conditions required in (I), then δ is equal to β_2.

For, suppose, on the contrary, that δ is not equal to β_2. Then, it results from Lemma 1.4 together with (I) that any S-arc in $K \frown D'_\delta$ can not be joined to s_1 by a simple set satisfying the conditions required in (I). On the other hand, let t be an S-arc in $K \frown D'_\delta$ and let $K_n^{(4)}$ be a simple set of type 2 and of rank n such fhat $\overline{K}_n^{(4)}$ lies in the sum of K and its bases and the lower base of $K_n^{(4)}$ contains t. For any S-arc s in the upper base of $K_n^{(4)}$, there exists a simple set $K_n^{(5)}$ of type 2 and of rank n which joins s_1 to s and whose closure lies in the sum of K and its bases. So it is easily seen that the set $K_n^{(6)}$, where $K_n^{(6)}=K_n^{(4)}\smile K_n^{(5)}\smile$ (the common part of the upper base of $K_n^{(5)}$ and the lower base of $K_n^{(4)}$), is a simple set of type 2 and of rank n which joins s_1 to t and whose closure in the sum of K and its bases. Therefore (II) is proved.

Now, we have Lemma 1.5 from (II) and Lemma 1.4.

Proof of Theorem 1. Let s_0 and s_1 be any two S-arcs of D_0' and D_1', respectively, each of which is disjoint from $D_0\smile D_1$. Clearly the set $N-(D_0'\smile D_1')$ is a simple set of type 2. Hence, applying Lemma 1.5 repeatedly, we obtain a sequence $\{K_n\}$ of simple sets such that

(i) for each n, K_n is a simple set of type 2 and of rank n which joins s_0 to s_1,

(ii) $N-(D_0\smile D_1)\supset \overline{K}_1\supset$ (the sum of K_1 and its bases) $\supset\overline{K}_2\supset$ (the sum of K_2 and its bases) $\supset \overline{K}_3\supset \cdots$.

Now, if we denote by K_0 the intersection of all the simple sets of $\{K_n\}$, then K_0 has the following properties:

(1) $K_0\frown n_\alpha$ is one point,

(2) K_0 is a closed 2-cell whose boundary contains both s_0 and s_1,

(3) $N-K_0$ consists of two components such that the intersection of the closures of them is K_0 and each closure of them has the same properties as are assumed for N.

For, (1) is obvious and (2) results from the fact that the transformation of D_0 onto K_0 under which a point $D_0\frown n_\alpha$ is transformed to a point $K_0\frown n_\alpha$ is topological. And it is easily shown that two components of $N-K_0$ containing D_0 and D_1, respectively, satisfy all the conditions of (3).

Next, by the help of the manner used to construct the closed 2-cell K_0, we shall construct the collection $\{D_\beta\}$ required in Theorem 1. Let $\{x_i\}$ be a countable subset of N which is dense in N, i_1 the smallest integer such that the point x_{i_1}, is not contained in $D_0\smile D_1$, and $D'_{\beta_{i_1}}$ the closed 2-cell of $\{D'_\beta\}$ containing x_{i_1}. And let t_1 be an S-arc through x_{i_1} and not intersecting $D_0\smile D_1$, and let t_1' and t_1'' be two S-arcs of D_0' and D_1' not intersecting $D_0\smile D_1$, respectively. If N'_{i_1} and N''_{i_1} denote the closures of components of $N-D'_{\beta_{i_1}}$ containing t_1' and t_1'' respectively, then N'_{i_1} and N''_{i_1} have the same properties as are assumed for N. Hence, by the same manner as in the construction of K_0 we obtain a closed 2-cell S_1' of N'_{i_1} whose boundary contains t_1' and t_1, and a closed 2-cell S_1'' of N''_{i_1} whose boundary contains t_1 and t_1''. Let S_1 denote the sum of S_1' and S_1''. Then S_1 has the following properties:

(1′) $S_1\frown n_\alpha$ is one point,

(2′) S_1 is a closed 2-cell containing x_{i_1},

(3′) $N-S_1$ consists of two components such that the intersection of the closures of them is S_1 and each of the closures of them has the same properties as are assumed for N.

Let i_2 be the smallest integer such that x_{i_2} is not contained in $D_0 \smile D_1 \smile S_1$. By applying the manner used to construct S_1 for x_{i_1} and N, to x_{i_2} and the closure of the component of $N-S_1$ containing x_{i_2}, we have a closed 2-cell S_2. Continuing this process indefinitly, we obtain a collection $\{S_i\}$ of mutually exclusive closed 2-cells S_i such that for any i and α, $S_i \frown n_\alpha$ is one point and the sum of all the sets of $\{S_i\}$ is dense in N. Moreover, it is easily shown that each component T_α of $N- \smile S_i$ is a closed 2-cell. The collection $\{D_\beta\}$, which is composed of S_i and T_α, satisfies all the conditions required in the statement of Theorem 1.

Thus Theorem 1 is proved.

COROLLARY 2. *The set N is a closed 3-cell.*
The corollary is proved in the same way as in the proof of Corollary 1.

§ 2. A set-theoretical characterization of closed cells

THEOREM 2. *In order that a separable metric space C be a closed n-cell ($n \leqq 3$) it is necessary and sufficient that there exists a collection $\{l_\alpha\}$ of mutually exclusive arcs such that*:

(1) *$\{l_\alpha\}$ is continuous and equicontinuous,*
(2) *the sum of all the arcs of $\{l_\alpha\}$ is C,*
(3) *the decomposition space of $\{l_\alpha\}$ is a closed $(n-1)$-cell.*

Theorem 2 will be proved with the help of three lemmas.

LEMMA 2.1. *Suppose $\{m_\alpha\}$ is a collection of mutually exclusive arcs such that (1) the sum of all the arcs of $\{m_\alpha\}$ is a compact, connected set, (2) $\{m_\alpha\}$ is continuous and equicontinuous.*

Then the set E of end points of all the arcs of $\{m_\alpha\}$ consists of at most two components.

PROOF. First, we see that from the condition (2) of the lemma, any limit point of any subset of E also belongs to E, and hence E is compact. Let f be the transformation of E onto itself which transforms one of two end points of each arc of $\{m_\alpha\}$ to another. For any subset K of E, K' denotes the image of K under f. If a subset K of E is connected, then K' is also connected. Hence, if K_1 is a component of E, then K_1' is also a component of E. To prove this lemma, suppose, on the contrary, that E contains a component K_2 distinct from both K_1 and K_1'. Then there exists a separation $E=A_1 \smile A_2$, where $K_1 \smile K_1' \subset A_1$ and $K_2 \smile K_2' \subset A_2$. Let A_{12} denote the sum of all components K_β of E such that $K_\beta \subset A_2$ and $K_\beta' \subset A_1$, and let B_1 and B_2 denote the sets $A_1 \smile A_{12}$ and $A_2 - A_{12}$, respectively. Next we shall show that the decomposition $E=B_1 \smile B_2$ is a separation. If, on the contrary, the decomposition $E=B_1 \smile B_2$ were not a separation, we would have a sequence $\{x_i\}$ of points converging to a point to x_0, where either

$x_0 \in B_1$ and $x_i \in B_2$ or $x_0 \in B_2$ and $x_i \in B_1$. In the case $x_0 \in B_1$ and $x_i \in B_2$, it results at once from the fact that $E = A_1 \smile A_2$ is a separation, that $x_0 \in A_2$, $x_0' \in A_1$, $x_i \in A_2$ and $x_i' \in A_2$. Furthermore, in the same way as in the proof of Lemma 1.3, it is readily shown that the sequence $\{x_i'\}$ coverges to x_0', contrary to the fact that $E = A_1 \smile A_2$ is a separation. Hence the decomposition $E = B_1 \smile B_2$ is a separation.

On the other hand, it is easily shown by the conditions (1) and (2) in the lemma, that such a decomposition $E = B_1 \smile B_2$, where $B_1 = B_1'$ and $B_2 = B_2'$, can not be a separation. Therefore we have a contradiction, and for the other case we obtain the same result.

Thus we have Lemma 2.1.

LEMMA 2.2. *If the collection $\{m_\alpha\}$ in Lemma 2.1 satisfies the additional condition that E consists of exactly two components, then the following two facts hold:*

(a) *for any arc m_α of $\{m_\alpha\}$, the two end points of m_α belong to distinct components of E;*

(b) *each component of E is homeomorphic to the decomposition space of $\{m_\alpha\}$.*

PROOF. As is obvious from the proof of Lemma 2.1, if K denotes one component of E, another component of E is K', where K' is the image of K under f in the proof of Lemma 2.1. Accordingly, if the two end points of some arc of $\{m_\alpha\}$ belong to the same component of E, then we have $K = K'$, contrary to the condition that E has exactly two components. Thus (a) is proved.

Next if, to each point of one component of E, we make correspond the arc of $\{m_\alpha\}$ containing the point, then, by (a) together with the definition of neighborhoods of decomposition spaces, this correspondence between one component of E and the decomposition space of $\{m_\alpha\}$ is topological. Thus (b) is proved.

LEMMA 2.3. *Suppose $\{m_\alpha\}$ is a collection of mutually exclusive arcs such that* (1) *$\{m_\alpha\}$ is continuous and equicontinuous, and* (2) *the decomposition space M' of $\{m_\alpha\}$ is a closed n-cell. Then the set E of end points of all the arcs of $\{m_\alpha\}$ consists of exactly two components.*

PROOF. Since M' is a closed n-cell, we may suppose without loss of generality that M' is the solid $(n-1)$-sphere with center at the origin and radius 1 in Euclidean n-space E^n. Hence M' is the sum of the sets M_r', $0 \leqq r \leqq 1$, where M_0' is the origin of E^n and M_r' $(0 < r \leqq 1)$ is the solid $(n-1)$-sphere with center at the origin and radius r. Let R denote the set of all the numbers r such that, for each $r' \leqq r$, the set of end points of all the arcs of $\{m_\alpha\}$ corresponding to points of $M_{r'}'$ consists of exactly two components. We shall prove the lemma by showing that the least upper bound a of R is equal to 1. For this purpose, suppose, on the contrary, that a is not equal to 1. Then, since R is clearly a non-vacuous open set in the unit interval [0,1] containing 0, a does not belong to R. Hence the set of end points of all the arcs of $\{m_\alpha\}$ corresponding to the points of M'_a has one component, but the set of end points of all the arcs of $\{m_\alpha\}$ corresponding to the points of the interior of M'_a has exactly two components, which are denoted by A and B in the following. By (i) and (ii) in the proof of Theorem 3, we can define the two topological correspondences between A and B and between A and the interior of M'_a in the same way as in the proof of Lemma 2.2. Let x_0 be a limit of both A and B. Clearly such a point x_0 exists and corresponds to a point of the boundary of M'_a. It results at once from the assumption (1) of the lemma that if a sequence of points in B converges to x_0 then the sequence of points in A corresponding to the sequence converges to either x_0 or another end point y_0 of the arc of $\{m_\alpha\}$ with x_0 as one end point. That the first case is impossible, is easily shown in the same way as in the proof of Lemma 1.3. That the second case, where $\bar{A}-A$ contains two end points of some arc of $\{m_\alpha\}$, also is impossible, results at once from the following two facts each of which is easily shown:

(a) each point x of $\bar{A}-A$ is a limit point of the subset l_x of A corresponding to the comon part L_x of the interior of M'_a and the radius of M'_a with the point corresponding to x as one end point;

(b) the set $\bar{l}_x-l_x$ is connected, and each point of $\bar{l}_x-l_x$ is one end point of the arc of $\{m_\alpha\}$ with x as one end point.

Therefore a is equal to 1. Moreover, in the same way as above, it is shown that 1 belongs to R.

Thus Lemma 2.3 is proved.

Proof of Theorem 2. The necessity is obvious. To show the sufficiency, we first note that since, by Theorem 3 in § 3, the set C in Theorem 2 is compact and connected, we may apply Lemmas 2.1, 2.2, 2.3 to the proof of Theorem 2. By Lemma 2.3, the set of end points of all the arcs of $\{l_\alpha\}$ consists of exactly two components. By Lemma 2.2, these two components are closed $(n-1)$-cells, the two end points of each arc of $\{l_\alpha\}$ belong to distinct components respectively, and if, for each point x of one of these components, we make correspond another end point of the arc of $\{l_\alpha\}$ with x as one end point, then this correspondence between two components is topological. Therefore, it results at once from Corollary 1 or Corollary 2 that C is a closed n-cell.

Thus Theorem 2 is proved.

§ 3. Topological relations between spaces and decomposition spaces

As is seen by the example in the paper by R. F. Williams [3], there exists a compact continuum M of Euclidean 3-space such that M is the sum of all the arcs of a continuous collection of mutually exclusive arcs whose decomposition space is an arc, but such that M is not locally connected. In connection with the example, the following theorem will be proved.

THEOREM 3. *Suppose $\{m_\alpha\}$ is a collection of mutually exclusive compact continua such that*

(1) *$\{m_\alpha\}$ is continuous and equicontinuous,*

(2) *the decomposition spaces M' of $\{m_\alpha\}$ is a locally connected, compact continuum.*

Then the sum M of all the compact continua of $\{m_\alpha\}$ is a locally connected, compact continuum.

PROOF. Let f be the continuous transformation associated with the collection $\{m_\alpha\}$ of M (Cf. [2], p. 125, (3.1)).

(i) M is compact. For, let $\{x_i\}$ be any countable infinite subset of M, and let m_{α_i} denote the continuum of $\{m_\alpha\}$ containing x_i, then we may suppose without loss of generality that any two points of $\{x_i\}$ do not belong to the same continuum of $\{m_\alpha\}$. Since M' is compact and the set $\{f(x_i)\}$ is infinite, there exists a point $f(m_{\alpha_0})\in M'$ which is a limit point of $\{f(x_i)\}$. We may suppose that $\{f(x_i)\}$ converges to $f(m_{\alpha_0})$. So, by the definition of neighborhoods of decomposition spaces, every neighborhood of m_{α_0} contains all but a finite number of continua of $\{m_{\alpha_i}\}$. Hence, there exists a sequence $\{y_i\}$ of points of m_{α_0} such that for each i, $\rho(x_i, y_i)<\frac{1}{i}$. Since m_{α_0} is compact, there exists a point $x_0\in m_{\alpha_0}$ which is a limit point of $\{y_i\}$. Clearly x_0 is also a limit point of $\{x_i\}$. Thus (i) is proved.

(ii) M is connected. Since M is compact, M' is connected and the transformation f of M onto M' is monotone, it follows at once that M is connected (Cf. [2], p. 138, (2,2)).

Here, we note that the equicontinuity of $\{m_\alpha\}$ is not employed in the above proofs (i) and (ii).

(iii) M is locally connected. Let ε be any positive number, and let δ be a positive number such that if x and y are any two points belonging to a compact continuum m_α of $\{m_\alpha\}$ and with $\rho(x, y)<2\delta$ then there exists an art xy of m_α of diameter less than $\frac{1}{2}\varepsilon$. By the equicontinuity of $\{m_\alpha\}$, such a $\delta>0$ exists. Now let p be any point of M and let $U_\varepsilon(p)$ and $U_\delta(p)$ be the ε- and δ-neighborhoods of p, respectively. Then, for each m_α of $\{m_\alpha\}$, $U_\delta(p)\frown m_\alpha$ lies in a component of $U_\varepsilon(p)\frown m_\alpha$, which we denote by C_α. Let N denote the sum of all the sets C_α such that C_α intersects $U_\delta(p)$. Clearly, N contains $U_\delta(p)$ and $f(N)=f(U_\delta(p))$. Since M is compact and $\{m_\alpha\}$ is continuous, it follows that f is an open mapping (Cf. [2], p. 130, (4.31)). Therefore $f(U_\delta(p))$ is open in M'. Since M' is locally connected, the component W' of $f(U_\delta(p))$ containing $f(p)$ is open in M'. Therefore $f^{-1}(W')$

$\frown U_\delta(p)$ is open in M. Hence, we obtain a neighborhood V of p in M which lies in $f^{-1}(W')\frown U_\delta(p)$. The common part W of $f^{-1}(W')$ and N has the two properties as follows: (a) $p \in V \subset W \subset U_\varepsilon(p)$ and (b) W is connected. The truth of (a) is obvious. To prove (b), suppose, on the contrary, that there exists a separation $W = W_1 \smile W_2$. The separation $W = W_1 \smile W_2$ yields the decomposition $f(W) = f(W_1) \smile f(W_2)$. Then, obviously $f(W) = W'$, and $f(W_1)$ and $f(W_2)$ are mutually exclusive. Hence we may suppose that there exist a point $f(m_{\alpha_0}) \in f(W_1)$ and a sequence $\{f(m_{\alpha_i})\}$ of points of $f(W_2)$ converging to $f(m_{\alpha_0})$. Then we have $C_{\alpha_0} \frown \liminf C_{\alpha_i} \neq \phi$. For, let x_0 be a point of $C_{\alpha_0} \frown U_\delta(p)$ then, since $\{f(m_{\alpha_i})\}$ converges to $f(m_{\alpha_0})$ and hence $\{m_{\alpha_i}\}$ converges to m_{α_0}, we have a sequence $\{x_i\}$ of points converging to x_0 such that $x_i \in m_{\alpha_i} \frown U_\delta(p) \subset C_{\alpha_i}$. This contradicts the assumption that $W = W_1 \smile W_2$ is a separation. Hence (b) is proved.

It follows from (a) and (b) that M is locally connected at p. Accordingly M is locally connected.

Thus Theorem 3 is proved.

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