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Strongly nonperiodic hyperbolic tilings using single vertex configuration

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Abstract. A strongly nonperiodic tiling is defined as a tiling that does not admit infinite cyclic symmetry. The purpose of this article is to construct, up to isomorphism, uncountably many strongly nonperiodic hyperbolic tilings with a single vertex configuration by a hyperbolic rhombus tile. We use a tile found by Margulis and Mozes [5], which admits tilings, but no tiling with a compact fundamental domain.

1. Introduction

In this paper, we are concerned with tilings in the hyperbolic plane H in the Poincaré model. We fix a finite set of hyperbolic polygons whose element is called a *prototile*. A *tiling* is a covering of H by prototiles and their images by isometry, without interior overlaps. Hereafter we always assume that the tiling is edge to edge, that is, every edge of the prototile exactly matches with the one of the other prototiles in the tiling.

We give several definitions (see for e.g., [1, 2]). A patch P is defined to be a set $P = \{T_\alpha\}_{\alpha \in A}$ of finitely many prototiles so that $\bigcup_{\alpha \in A} T_\alpha$ is simply connected. A vertex configuration of a vertex x is a patch $P = \{T_\alpha\}_{\alpha \in A}$ ($x \in T_\alpha$) having minimal cardinality such that x is in the interior of $\bigcup_{\alpha \in A} T_\alpha$. A tiling is called *weakly nonperiodic* if it does not admit a compact fundamental domain as a quotient by its symmetry group. A tiling is *strongly nonperiodic* if it has no infinite cyclic symmetry. *Weakly aperiodic prototiles* are defined to be sets of prototiles that admit a tiling, but none with a compact fundamental domain. *Strongly aperiodic prototiles* are defined to be sets of tiles that admit a tiling, but none with infinite cyclic symmetry. This distinction of weak/strong non-periodicity of hyperbolic tiling emerged from the pioneering work by R. Penrose [6], who gave a nonperiodic tiling in H by a weakly aperiodic prototile.

A tiling is *Archimedean* if every prototile in the set is a regular polygon and all vertex configurations are congruent. In the Euclidean case, it is well

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known that there are exactly 11 Archimedean tilings, all of which are periodic and *uniform*, that is, all vertex configurations are congruent by the symmetry group of the tiling [3]. Because hyperbolic plane has more freedom than Euclidean one, we may not expect such an easy classification. Goodman-Strauss [2] gave a construction of uncountably many nonperiodic Archimedean tilings by two prototiles: a regular pentagon and an equilateral triangle[1]. In this paper we are interested in presenting a further curious example: strongly nonperiodic tilings by using a **single rhombus** (not a regular polygon) and a **single** vertex configuration. This result suggests a difficulty with the above classification.

We call the following procedure for laying tiles the *ringed expansion* (cf. [4] in the case of the Euclidean plane): First, we prepare vertex configurations to be associated. Starting from a patch, we then associate a vertex configuration to each vertex in the boundary of the patch. If we can compatibly associate vertex configurations to all vertices in the boundary, we get another larger patch. We call this larger patch the 1st expanded patch. We define inductively the k-th expanded patch to be a patch obtained by associating compatibly vertex configurations to all verties in the boundary of the $(k-1)$-th expanded patch for $k = 2, 3, \dots$. And we say that the k-th ringed expansion is *complete* when the k-th expanded patch is obtained. If a similar expansion can be repeated ad infinitum, we obtain a tiling. Each step of the expansion can be represented by a word of angles in the boundary of the patch.

We use a weakly aperiodic prototile found by Margulis and Mozes [5]. In general, it is difficult to construct a tiling with the desired symmetry for a given prototype. Using the ringed expansion, we construct strongly nonperiodic hyperbolic tilings with a trivial symmetry group:

THEOREM 1. *There exist uncountable many strongly nonperiodic hyperbolic tilings with only one vertex configuration by a weakly aperiodic prototile.*

To prove the theorem, we will use a symbolic expression called *substitution rules* to specify how the ringed expansion is performed in the next step.

2. Proof of Theorem

In [5], Margulis and Mozes show Lemma that a prototile consisting of a single tile whose area is not a rational multiple of π is weakly aperiodic. By using Lemma they construct a weakly aperiodic prototile which consists of a single hyperbolic rhombus tile as shown in Figure 1. In Figure 1, the

[1] He also pointed out that it is not known whether there is a set of weakly aperiodic prototiles consisting only of regular polygons.

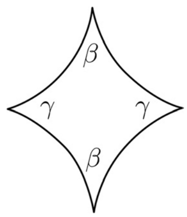

Fig. 1. Hyperbolic rhombus tile

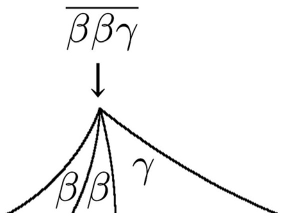

Fig. 2. The vertex $\overline{\beta\beta\gamma}$ with degree 4

symbol β or γ denotes a vertex with angle $\beta = (2 - \sqrt{2})\pi/6$ or $\gamma = \sqrt{2}\pi/7$, respectively. Note that $6\beta + 7\gamma = 2\pi$ is the only $\mathbf{Z}$-linear relation among β, γ, and π. We use this hyperbolic rhombus tile. We prepare 14 symbols:

$$\beta, \gamma, \overline{\beta\beta}, \overline{\beta\gamma}, \overline{\gamma\beta}, \overline{\gamma\gamma}, \overline{\beta\beta\beta}, \overline{\beta\beta\gamma}, \overline{\beta\gamma\beta}, \overline{\beta\gamma\gamma}, \overline{\gamma\beta\beta}, \overline{\gamma\beta\gamma}, \overline{\gamma\gamma\beta} \text{ and } \overline{\gamma\gamma\gamma},$$

where a, $\overline{ab}$, or $\overline{abc}$ denotes a vertex with degree 2, 3, or 4, respectively in the boundary of the patch. For example, $\overline{\beta\beta\gamma}$ denotes the vertex with degree 4 in the boundary where the angles β, β, and γ gather counterclockwise, as shown in Figure 2.

We use the vertex configuration with the center $(\beta, \gamma, \beta^2, \gamma, \beta^3, \gamma^5)$ as shown in Figure 3. And, for convenience, we add indices ①–⑬ as shown in Figure 3. Let $C = \{①, \dots, ⑬\}$ be a set of the indices, and an index in C is called *a color*.

When we associate a vertex configuration to a vertex in the boundary of a patch, we add colors to the symbols of the angles as they appear in the

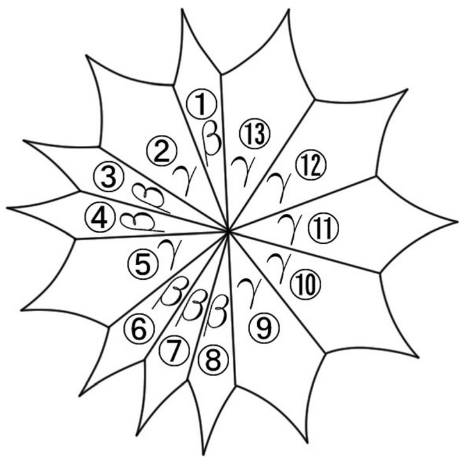

Fig. 3. The vertex configuration

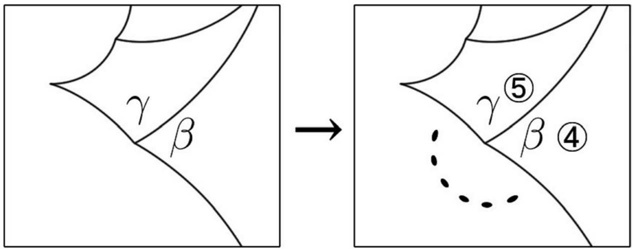

Fig. 4. A substitution rule

boundary in order to describe this rule in a symbolic manner. For example, in Figure 4, we associate the vertex configuration by overlapping the vertex $\overline{\beta\gamma}$ in the boundary of a patch with colors ④⑤. For a symbol at a vertex, there might be several ways to associate the vertex configuration. Adding colors as in Figure 4 is designated by $\beta\gamma =$ ④⑤, which we call *a substitution rule*.

To obtain a tiling on the hyperbolic plane by ringed expansion, we have to associate the vertex configuration to all vertices in the boundary of a patch. *An indexed tile* is a tile for which all the angles are assigned colors in C. Hereafter clockwise (or counterclockwise) order is defined with respect to the curve that gives the boundary of a patch, being homeomorphic to a ball. We consider specific indexed tiles α and $\bar{\alpha}$ as shown in Figure 5, that is, an indexed α has indices ④②③⑩ in clockwise order and an indexed $\bar{\alpha}$ has indices ④②③⑪ in clockwise order.

We start the ringed expansion from one α. Because of the vertex configuration in Figure 3, we know the result of the first ringed expansion is as shown in Figure 6. Note that we did not yet assign colors to the angles in the boundary of this patch. We call such tiles *incomplete*. The second ringed expansion is specified by assigning indices to the incomplete tiles appearing at the boundary of the patch in Figure 6.

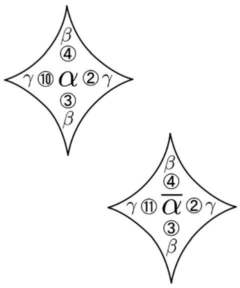

Fig. 5. Indexed tiles α and $\bar{\alpha}$

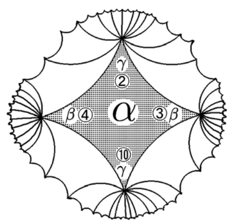

Fig. 6. The patch of first step

Our strategy is to construct a tiling, as shown in Figure 11, which has a sequence of α and $\bar{\alpha}$ connecting at angles γ. We call this sequence (of α and $\bar{\alpha}$) *a spiral sequence*. The adjacent tiles in this sequence are conjoined by two angles, one angle with color ② and the other with either ⑩ or ⑪. Furthermore, we require that all α tiles in the tiling should be contained in the spiral sequence, that is, the outside of the spiral sequence must consist of tiles other than α. Hereafter, we construct such a tiling by successive ringed expansion.

Let us start from one α and assume that its k-th ringed expansion under the above constraints is complete. All vertices in the boundary are of degree 2 or 3, and any vertex of degree 2 is isolated, that is, both the neighboring vertices are of degree 3 (for example, see Figure 6). The k-th expanded patch contains a spiral sequence beginning with the initial α. Then, on the boundary of the k-th expanded patch, there is an incomplete tile in conjunction with the spiral sequence having a vertex with color ② and degree 2 vertex in the boundary, as shown in Figure 7. We call this tile a termination tile.

Step 1.

Here, note that only one angle in the termination tile is assigned a color ② as on the vertex a in Figure 7. In this Step 1, we will associate the vertex configuration to three vertices of the termination tile in the boundary of the k-th expanded patch and index the termination tile with α or $\bar{\alpha}$.

First, on vertices of degree 3 in the termination tile, we apply the following two substitution rules $\beta\gamma =$ ④⑤ and $\gamma\beta =$ ②③ as on the vertices b, c in Figure 8.

Next, we choose either α or $\bar{\alpha}$. If we choose α, we apply the substitution rule $\gamma\gamma\gamma =$ ⑨⑩⑪ on the remaining vertex. Using the substitution rule $\gamma\gamma\gamma =$ ⑨⑩⑪ on the vertex d in Figures 9, we obtain an indexed tile α which emerges from the termination tile. If we choose $\bar{\alpha}$, we apply $\gamma\gamma\gamma =$ ⑩⑪⑫ instead of $\gamma\gamma\gamma =$ ⑨⑩⑪ on the remaining vertex. Using the substitution rule

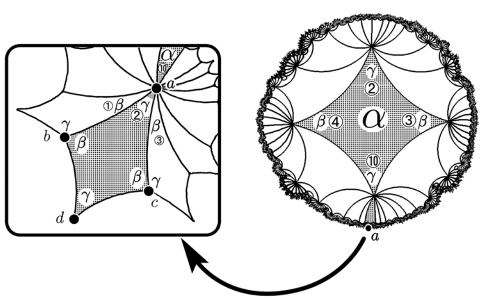

Fig. 7. A termination tile with a color ②

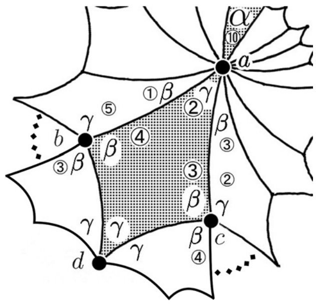

Fig. 8. $\beta\gamma =$ ④⑤, $\gamma\beta =$ ②③

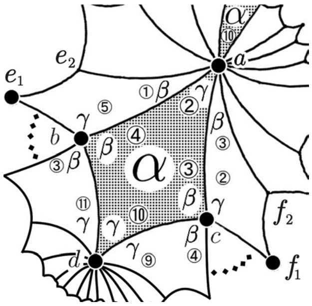

Fig. 9. $\gamma\gamma\gamma =$ ⑨⑩⑪, a termination tile with α

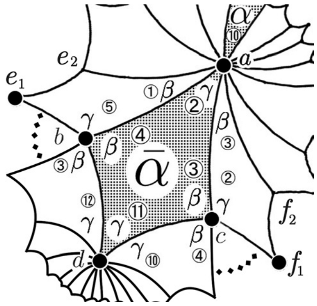

Fig. 10. $\gamma\gamma\gamma =$ ⑩⑪⑫, a termination tile with $\bar{\alpha}$

$\gamma\gamma\gamma =$ ⑩⑪⑫ on the vertex d in Figures 10, we obtain an indexed tile $\bar{\alpha}$ which emerges from the termination tile.

Then we obtain the spiral sequence of α and $\bar{\alpha}$ one longer.

Step 2.

Since the adjacent tiles of α, $\bar{\alpha}$ are conjoined by two angles, one angle with color ② and the other with either ⑩ or ⑪, any vertex in the boundary of the k-th expanded patch has degree 2 or 3 in the patch obtained in Step 1, except the vertices b, c, d of the termination tile that we processed in Step 1. In this Step 2, we apply another set of substitution rules only on vertices of degree 3. If a vertex of degree 3 is isolated, that is, both the neighboring vertices are of degree 2, we apply the following substitution rules $(*)$.

$\beta\beta =$ ⑥⑦, $\beta\gamma =$ ④⑤, $\gamma\beta =$ ②③, $\gamma\gamma =$ ⑪⑫. $\cdots (*)$

If the sequence of consecutive vertices of degree 3 appears in the boundary, we start from the application of list (*) to one of the consecutive vertices of degree 3. Then, the next vertex (vertices) of the sequence will be of degree 4. So we apply from the list $(**)$ to the next vertex (vertices) of degree 4. If the next vertex (vertices) of the sequence to this vertex (these vertices) is (are) of degree 4 again, we apply from the list $(**)$ to the next vertex (vertices) of degree 4. By doing this proccessure repeatedly, we can apply from the list $(**)$ to the rest of the vertices of the sequence.

$\beta\beta\beta =$ ⑥⑦⑧, $\beta\beta\gamma =$ ⑦⑧⑨, $\beta\gamma\beta =$ ④⑤⑥, $\beta\gamma\gamma =$ ⑧⑨⑩,
$\gamma\beta\beta =$ ⑤⑥⑦, $\gamma\beta\gamma =$ ⑬①②, $\gamma\gamma\beta =$ ⑫⑬①, $\gamma\gamma\gamma =$ ⑩⑪⑫. $\cdots (**)$

In fact, there are the sequences $\{e_1, e_2\}$ and $\{f_1, f_2\}$ that emerged from Step 1 as in Figure 9 and 10. For example, we apply to e_1 from list $(*)$ and then to e_2 from list $(**)$, and to f_1 from list $(*)$ and then to f_2 from list $(**)$.

Step 3.

In this Step 3, we will complete the $(k+1)$-th ringed expansion. All the remaining vertices to be substituted are of degree 4, which had been of degree 2 at the beginning of this construction. We apply the same list $(**)$.

Through steps 2–3 of this construction, if a vertex of degree 2 in the boundary has an angle γ, then the list $(**)$ gives the angle a color of ⑤, ⑨, ⑬ or ⑪. Hence, this angle will never be assigned ② or ⑩, and the tile will never be an α tile. It is clear that on the $(k+1)$-th expanded new patch, all vertices in the boundary are of degree 2 or 3 again.

By Step 1 of the construction, in each step of ringed expansions we can select α or $\bar{\alpha}$ per our preference. Hence, we have a tiling with a spiral sequence of α and $\bar{\alpha}$, as shown in Figure 11, and we can choose spiral sequences having an infinite number of α's. Recall that all α tiles in the tiling are contained in the spiral sequence. A congruence transformation on the hyperbolic plane H, which preserves this tiling must then map the initial α tile to itself, and consequently, the symmetry group is trivial, which shows its strong nonperiodicity. Our construction shows that there is a surjective map from the set of tilings having such a spiral sequence, to the set of a one-sided infinite sequence of α and $\bar{\alpha}$ having an infinite number of α's. The latter set is clearly uncountable. Hence, we have uncountably many number of strongly nonperiodic tilings up to isomorphism.

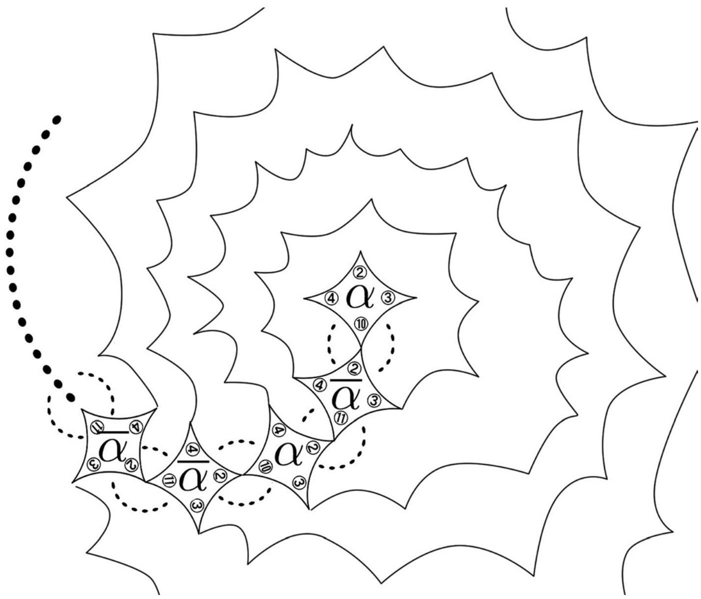

Fig. 11. A spiral sequence of α and $\bar{\alpha}$

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