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A new example of the dissipative wave equations with the total energy decay

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ABSTRACT. This note gives a new sufficient condition of the total energy decay for the solutions of the initial-boundary value problems to the dissipative wave equations in exterior domains with non-compactly supported initial data. That condition provides an example of the damping terms of the dissipative wave equations with the total energy decay which has a smaller amplitude than those of all examples derived from a sufficient condition in Mochizuki and Nakazawa [Publ. Res. Inst. Math. Sci. **32** (1996), 401–414].

1. Introduction and the result

We shall consider the initial-boundary value problems for the dissipative wave equations

$$\begin{cases} L[w] = \partial_t^2 w(t,x) - \Delta_x w(t,x) + b(t,x)\partial_t w(t,x) = 0 \quad \text{in } [0,\infty) \times \Omega, \\ w(t,x) = 0 \quad \text{in } [0,\infty) \times \partial\Omega, \\ w(0,x) = w_1(x), \quad \partial_t w(0,x) = w_2(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is an unbounded domain of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ such that $0 \notin \bar{\Omega}$ and b is a smooth function in $[0,\infty) \times \bar{\Omega}$ with bounded derivatives of all orders in $[0,\infty) \times \bar{\Omega}$.

$\mathbf{N}$ indicates the set of all natural numbers. For each $x \in \mathbf{R}^n$, r denotes the distance of the point x from the origin. For each solution w of (1), $E(t;w)$ designates the total energy of w at time t, that is, $E(t;w) = \|w_t(t,\cdot)\|_{L^2(\Omega)}^2 + \| |\nabla_x w(t,\cdot)| \|_{L^2(\Omega)}^2$. In Section 3, we shall use the j times iterated exponential e_j and the j times iterated logarithm $\log^{[j]}(s)$, $j = 0, 1, \ldots,$ videlicet, $e_0 = 1$, $e_l = e^{e_{l-1}}$ and $\log^{[0]}(s) = s$, $\log^{[l]}(s) = \log(\log^{[l-1]}(s))$.

Sufficient conditions of the total energy decay for the solutions of the Cauchy problems or the initial-boundary value problems to the equation $L[w] = 0$ in unbounded domains with compactly supported initial data have been studied by many authors, for example see [2], [8], [7] and [1]. We would like to refer to a series of works by Wirth about wave equations with time-

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dependent dissipation, for example see [9] and [10]. After the work [4], Mochizuki and Nakazawa [5] proved that the total energy of the solutions for the equation (1) with non-compactly supported initial data decays, if $b_t \le 0$ in $[0, \infty) \times \Omega$ and $b_0 / \prod_{l=0}^{k} \log^{[l]}(e_k + t + r) \le b(t,x) \le b_1$ with some $k \in \mathbf{N}$ and positive constants b_0, b_1. The purpose of this paper is to improve their result. We remark that taking over the work [5], Mochizuki and Nakazawa [6] showed that if $n \ge 3$, the complement of Ω is star-shaped and the damping term b satisfies conditions similar to those stated above only near infinity, then the total energy for the equation (1) decays.

For simplicity we shall suppose compactness and smoothness of initial data, but our results hold for each initial data in some weighted energy spaces, see Remarks 1 and 3.

THEOREM 1. *Let $b_t \le 0$ hold in $[0, \infty) \times \Omega$ and let one can find a positive and decreasing function $f \in C^2([0, \infty))$ so that $f^2 + f' \le 0$ on $[0, \infty)$ and $b(t,x) \ge 2f(t+r)$ in $[0, \infty) \times \Omega$. Then for each compactly supported smooth initial data we have with a positive constant C depending only on initial data and f*

$$E(t; w) \le C \exp\left(- \int_0^t f \, ds \right) \qquad \text{on } [0, \infty). \tag{2}$$

In particular, $E(t; w)$ decays as time tends to infinity, if f is not integrable on $(0, \infty)$.

We note that the same inequality as (2) is shown in [5] by assuming some conditions on the second derivative of f besides the hypothesis of Theorem 1, see [5, Section 2]. We mention that [3], [4] and [5] give sufficient conditions of non-decay of the total energy for the solutions to the equation $L[w] = 0$, see [3, Theorem 1], [4, Theorem 27.3] and [5, Theorem 2].

2. Proof of the theorem

We shall first show the following lemma in order to obtain an weighted energy inequality to the equation (1) needed for the proof of Theorem 1.

LEMMA 1. *Under the assumption in Theorem 1, for each solution w of (1) with compactly supported smooth initial data we have*

$$\int_\Omega X(t,x) dx \le \int_\Omega X(0,x) dx \qquad \text{for all } t \in [0, \infty) \tag{3}$$

where $X = 2^{-1}\varphi(w_t^2 + |\nabla_x w|^2) + \varphi_t w_t w + 2^{-1}\varphi_t(b - \varphi_{tt}/\varphi_t) w^2$ *and* $\varphi(t,x) = \exp(\int_0^{t+r} f \, ds)$ *with the function f in Theorem 1.*

Proof. We see from the antecedents of b and f that

$$\begin{cases} \varphi_t(t,x) = \varphi(t,x)f(t+r) > 0, \quad \nabla_x\varphi(t,x) = \varphi_t(t,x)x/r, \\ \varphi_{tt}(t,x) = \varphi(t,x)(f^2+f')(t+r) \leq 0, \quad (\varphi_t b)_t \leq 0, \\ \Delta_x\varphi_t(t,x) = \varphi_{ttt}(t,x) + \varphi_{tt}(t,x)(n-1)/r \end{cases} \tag{4}$$

in $[0,\infty)\times\Omega$. By using $L[w]=0$ and the last equality of (4) we have

$$\partial_t X = \nabla_x \cdot Y + Z \qquad \text{in } [0,\infty)\times\Omega \tag{5}$$

where $Y = \varphi w_t\nabla_x w + \varphi_t w\nabla_x w - 2^{-1}w^2\nabla_x\varphi_t$ and $Z = \varphi\{(3\varphi_t)/(2\varphi) - b\}w_t^2 - 2^{-1}\varphi_t|\nabla_x w|^2 - (\nabla_x\varphi\cdot\nabla_x w)w_t + 2^{-1}(\varphi_t b)_t w^2 + (n-1)(2r)^{-1}\varphi_{tt}w^2$. On the other hand, (4), the Schwarz inequality and the hypothesis about b yield $Z\leq 0$ in $[0,\infty)\times\Omega$. Hence, by integrating both sides of (5) over Ω and applying the finite propagation speed for the equation (1) and the divergence theorem, it follows that the integral of X over Ω monotonically decreases on time. The proof is complete. □

Proof of Theorem 1. The Schwarz inequality and (4) give

$$X \geq \frac{1}{4}\varphi(w_t^2+|\nabla_x w|^2) + \frac{1}{2}\varphi_t\left(b - 2\frac{\varphi_t}{\varphi}\right)w^2 \qquad \text{in } [0,\infty)\times\Omega,$$

$$X(0,x) \leq \varphi(w_t^2+|\nabla_x w|^2)|_{t=0} + \frac{1}{2}\varphi_t\left(b+\frac{\varphi_t}{\varphi}-\frac{\varphi_{tt}}{\varphi_t}\right)w^2\bigg|_{t=0} \qquad \text{in } \Omega.$$

Accordingly, by using the hypothesis about b and (4) we have

$$X \geq \frac{1}{4}\exp\left(\int_0^t f\,ds\right)(w_t^2+|\nabla_x w|^2) \qquad \text{in } [0,\infty)\times\Omega, \tag{6}$$

$$X(0,x) \leq \exp\left(\int_0^r f\,ds\right)[w_2^2 + |\nabla_x w_1|^2 + (b(0,x)f(r) - f'(r))w_1^2] \tag{7}$$

in Ω. Therefore, from Lemma 1 we can conclude that this theorem is true. □

Remark 1. (6) *and* (7) *imply that the estimate* (2) *holds for each initial data in the Hilbert space which is the completion of* $C_0^\infty(\Omega)\times C_0^\infty(\Omega)$ *with respect to the norm*

$$\|(u,v)\| = \|kv\|_{L^2(\Omega)} + \|k|\nabla_x u|\,\|_{L^2(\Omega)} + \|k(b(0,x)f(r)-f'(r))^{1/2}u\|_{L^2(\Omega)}$$

where $k(x) = \exp((1/2)\int_0^r f\,ds)$.

We remark that the same inequalities as that in Lemma 1 are shown in [4, Theorem 27.1] and [5, Lemma 2.1]. In [4], $r^{-(n-1)/2}\nabla_x(r^{(n-1)/2}w)$ is used instead of $\nabla_x w$.

REMARK 2. *Theorem* 1 *holds for the Cauchy problems in the whole space under the same condition as that in Theorem* 1. *Indeed, in the proof of Lemma* 1 *by taking a positive number* ε, *by integrating* $\nabla_x \cdot Y$ *over* $\{r \geq \varepsilon\}$ *instead of* Ω *and by letting* $\varepsilon \to +0$ *we see that* (3) *is correct in the the whole space, because for each* $t \in [0, \infty)$ *the integral of* $\nabla_x \cdot Y$ *over* $\{r \geq \varepsilon\}$ *converges to zero when* $n \geq 2$ *and* $w(t,0)^2 \varphi_{tt}(t,0)$ (≤ 0) *when* $n = 1$ *as* $\varepsilon \to +0$. *Hence the same argument as that of the proof of Theorem* 1 *verifies that* (2) *is valid in the whole space.*

3. An example of the damping terms

We shall state a new example of the damping terms of the equation (1) with the total energy decay below.

PROPOSITION 1. *For each number* $\gamma \in (0, 1]$ *one can find a damping term* b *so that we have*

$$0 < b(t,x) \leq \gamma \prod_{l=0}^{k-1} \frac{1}{\log^{[l]}(t+r)} \tag{8}$$

when $t + r \in [e_k, \infty)$, $k \geq 2$, *and for each compactly supported smooth initial data we have with a positive constant* C *depending only on initial data*

$$E(t; w) \leq C \exp(-\gamma(k-2)/40) \qquad \text{when } t \in [e_k, \infty),\ k \geq 2. \tag{9}$$

In particular, $E(t; w)$ *decays as time tends to infinity. In addition, for each* $\alpha \in \mathbf{N}$ *and* $\beta \in \mathbf{N}^n$ *we have with a positive constant* $\tilde{C}$ *depending only on* α, β

$$|\partial_t^\alpha \partial_x^\beta b(t,x)| \leq \tilde{C}(t+r)^{-\alpha} r^{-|\beta|} \prod_{l=0}^{k-1} \frac{1}{\log^{[l]}(t+r)} \tag{10}$$

when $t + r \in [e_k, \infty)$, $k \geq 2$.

PROOF. Let χ be a smooth function on $\mathbf{R}$ such that $0 \leq \chi \leq 1$ on $\mathbf{R}$, $\chi = 0$ on $(-\infty, 1]$ and $\chi = 1$ on $[2, \infty)$. Let us put

$$g(s) = \left(\int_0^s h\, du + 1\right)^{-1}, \qquad h(s) = \exp\left[\int_0^s \sum_{l=1}^{\infty} \chi(u/e_l) \prod_{j=0}^{l} \frac{1}{\log^{[j]}(u)}\, du\right]$$

on $[0, \infty)$. Then we see that g and h are smooth, that $h \geq 1$ on $[0, \infty)$ and that

$$g' = -hg^2, \qquad h'(s) = h(s) \sum_{l=1}^{\infty} \chi(s/e_l) \prod_{j=0}^{l} \frac{1}{\log^{[j]}(s)} \geq 0 \tag{11}$$

on $[0, \infty)$. Hence integration by parts give

$$\int_0^s h\,du = sh(s) - \int_0^s uh'(u)du \le sh(s) \qquad \text{on } [0, \infty), \tag{12}$$

and by using the second equality of (11) we have

$$\begin{aligned}\int_0^s uh'(u)du &\le \sum_{l=1}^{\infty} \prod_{j=1}^{l} \frac{1}{\log^{[j]}(e_l)} \int_0^s h\,du \\ &\qquad \text{by means of } \log^{[k]}(e_j) = e_{j-k} \text{ for } j \ge k \\ &\le \sum_{l=1}^{\infty} (1/e)^{l-1} \int_0^s h\,du = \frac{e}{e-1} \int_0^s h\,du\end{aligned} \tag{13}$$

on $[0, \infty)$. Consequently it follows from (12) and (13) that we have

$$\frac{1}{2}\frac{1}{sh(s)} \le g(s) \le 3\frac{1}{sh(s)} \qquad \text{on } [1, \infty). \tag{14}$$

On the other hand, the Taylor expansion formula and induction on l induce $\log^{[l]}(2e_k) \le e_{k-l} + \log(2)\prod_{i=k-(l-1)}^{k-1} e_i^{-1}$, $k \ge l \ge 2$, and especially $\log^{[l]}(2e_l) \le 1 + \log(2)e^{-(l-1)}$, $l \in \mathbf{N}$. Accordingly, on $[e_k, e_{k+1}]$, $k \ge 2$, we have

$$\begin{aligned} h(s) &\ge \exp\left[\sum_{l=1}^{k-1} \int_{2e_l}^{s} \prod_{j=0}^{l} \frac{1}{\log^{[j]}(u)}\,du\right] \ge \prod_{l=1}^{k-1} \frac{\log^{[l]}(s)}{1 + \log(2)e^{-(l-1)}} \\ &\ge \exp[-(\log(2)e/(e-1)]\prod_{l=1}^{k-1} \log^{[l]}(s) \ge \frac{3}{10}\prod_{l=1}^{k-1}\log^{[l]}(s), \\ h(s) &\le \exp\left[\sum_{l=1}^{k} \int_{e_l}^{s} \prod_{j=0}^{l} \frac{1}{\log^{[j]}(u)}\,du\right] \le \prod_{l=1}^{k} \log^{[l]}(s). \end{aligned}$$

Therefore (14) deduces

$$\frac{1}{2}\prod_{l=0}^{k} \frac{1}{\log^{[l]}(s)} \le g(s) \le 10 \prod_{l=0}^{k-1} \frac{1}{\log^{[l]}(s)} \qquad \text{on } [e_k, e_{k+1}],\ k \ge 2 \tag{15}$$

and hence we get

$$\int_{e_2}^{s} g\,du \ge \frac{k-2}{2} \quad \text{on } [e_k, \infty), \qquad \int_{e_2}^{s} g\,du \le 20(k-2) \quad \text{on } [e_2, e_k], \tag{16}$$

$k \ge 2$. It follows inductively from (11) that for each $j \in \mathbf{N} \cup \{0\}$ we have $|h^{(j)}(s)| \le C_{j,1}h(s)s^{-j}$ with a positive constant $C_{j,1}$ on $(0, \infty)$. Moreover, we see inductively that $(g^2)^{(j)}$, $j \in \mathbf{N} \cup \{0\}$, can be written in the form of a linear combinations of $g^p \prod_{q=0}^{j} (h^{(q)})^{l_q}$ where $p = 2, \dots, j+2$, $l_0, \dots, l_j \in \mathbf{N} \cup \{0\}$, $l_j = 0$, $\sum_{q=0}^{j} l_q = p-2$ and $\sum_{q=0}^{j} q l_q = j - (p-2)$. Hence for each $i \in \mathbf{N} \cup \{0\}$ we get from (11) and the Leibniz formula

$$|g^{(i+1)}(s)| \le C_{i,2} \sum_{j=0}^{i} h(s) s^{-(i-j)} \sum_{p=2}^{j+2} g(s)^p h(s)^{p-2} s^{-(j-(p-2))}.$$

Consequently, owing to (14) and (15), we obtain

$$|g^{(i+1)}(s)| \le C_{i,3} s^{-(i+1)} \prod_{l=0}^{k-1} \frac{1}{\log^{[l]}(s)} \qquad \text{on } [e_k, \infty),\ k \ge 2, \tag{17}$$

where $C_{i,2}$ and $C_{i,3}$ are positive constants depending only on i. For each number $\gamma \in (0,1]$ let us put $b(t,x) = 2f(t+r)$ in $[0,\infty) \times \bar{\Omega}$ with $f = (\gamma/20)g$. Then we see that b satisfies the hypothesis of Theorem 1. Moreover, (15) implies (8) and (17) implies (10). The estimate (9) is derived from (16) and Theorem 1. The proof is complete. $\square$

Remark 3. *When $n \neq 2$, the estimate* (9) *is valid for each initial data in the Hilbert space which is the completion of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ with respect to the norm $\|(u,v)\| = \|kv\|_{L^2(\Omega)} + \|k|\nabla_x u|\,\|_{L^2(\Omega)}$ where $k(x) = \exp(\int_0^r f\,ds/2)$. Indeed, for each $x \in \Omega$ with $r \in [e_k, e_{k+1}]$, $k \ge 2$, the first equality of* (11) *gives*

$$\begin{aligned} k(x)^2(b(0,x)f(r) - f'(r)) &\le \frac{C_1}{20} \exp\left(\frac{1}{20}\int_{e_2}^{r} g\,ds\right)(g(r)^2 + h(r)g(r)^2) \\ &\quad \textit{by applying } (14) \\ &\le \frac{3}{10} C_1 \exp\left(\frac{1}{20}\int_{e_2}^{r} g\,ds\right)\frac{g(r)}{r} \\ &\quad \textit{by using } (15) \textit{ and the second inequality of } (16) \\ &\le 3C_1 \frac{1}{r^2} \exp(k-1) \prod_{l=1}^{k-1} \frac{1}{\log^{[l]}(e_k)} \le 3C_1 \frac{1}{r^2} \end{aligned}$$

where $C_1 = \exp(\int_0^{e_2} g\,ds/20)$. Hence $0 \le k(x)(b(0,x)f(r) - f'(r))^{1/2} \le C_2 r^{-1}$ holds in Ω for some positive constant C_2. Consequently we get from the Hardy inequality $\|k(b(0,x)f(r) - f'(r))^{1/2} u\|_{L^2(\Omega)} \le 2C_2(n-2)^{-1} \|\,|\nabla_x u|\,\|_{L^2(\Omega)}$ for all $u \in C_0^\infty(\Omega)$ which deduces the conclusion from Remark 1.

Remark 4. *Proposition* 1 *for the Cauchy problems in the whole space is valid. Indeed, let us set $b(t,x) = (\gamma/10)g(t + \langle x \rangle)$ and $f(s) = (\gamma/20)g(s+1)$*

where $\langle x \rangle = \sqrt{1 + r^2}$ *and* g *is the function defined in the proof of Proposition* 1. *Then we see that* $b(t, x) \geq 2f(t + r)$ *in* $[0, \infty) \times \mathbf{R}^n$ *and the other assumptions of Theorem* 1 *(in the whole space) are satisfied. Hence the same statement in the whole space as that of Proposition* 1 *is derived from an argument similar to that of the proof of Proposition* 1.

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