Hiroshima Math. J.
44 (2014), 223–234

Complete classification of genus-1 simplified broken Lefschetz fibrations

Kenta Hayano

(Received January 7, 2013)
(Revised February 17, 2014)

Abstract. We complete the classification of smooth 4-manifolds which admit genus-1 simplified broken Lefschetz fibrations.

1. Introduction

Broken Lefschetz fibrations are smooth maps from oriented 4-manifolds to oriented 2-manifolds which are allowed to have two specific types of critical points, namely *indefinite folds* and *Lefschetz critical points*. Auroux, Donaldson and Katzarkov [2] introduced broken Lefschetz fibrations as generalizations of Lefschetz fibrations in order to understand near-symplectic structures on 4-manifolds. They proved that the total space of a broken Lefschetz fibration such that every component of any fiber represents a non-trivial homology class admits a near-symplectic structure. Afterwards, Perutz [16, 17] gave a new invariant, which is a candidate for a geometric interpretation of the Seiberg–Witten invariant, using broken Lefschetz fibrations together with near-symplectic structures.

On the other hand, there exist broken Lefschetz fibrations whose total spaces would not admit near-symplectic structures. Indeed, it turns out that for any smooth map f from a 4-manifold X to the 2-sphere there exists a broken Lefschetz fibration homotopic to f such that f has connected set of indefinite folds which are embedded into S^2 by f, every fiber of f is connected and all the Lefschetz critical points are contained in fibers with the highest genera. Such a broken Lefschetz fibration is called a *simplified broken Lefschetz fibration*[1]. It was introduced by Baykur [4] and he further gave a combinatorial description of simplified broken Lefschetz fibrations via mapping

The author was supported by JSPS Research Fellowships for Young Scientists (24-993).

2010 *Mathematics Subject Classification*. Primary 57M50, 57R45; Secondary 30F99.

Key words and phrases. Broken Lefschetz fibrations, Indefinite folds, Hurwitz systems.

[1] Baykur [4] first proved the existence of simplified broken Lefschetz fibrations on (blow-ups of) near-symplectic 4-manifolds, and then Williams [18] generalized it to all closed oriented 4-manifolds relying on the general existence results for broken Lefschetz fibrations [1, 3, 7, 12].

class groups of surfaces. This description is a generalization of monodromy factorizations of Lefschetz fibrations.

In this paper, we discuss the classification problem of simplified broken Lefschetz fibrations with small fiber genera. The *genus* of a simplified broken Lefschetz fibration is defined to be a genus of a fiber with the highest genus. Kas [11] and Moishezon [14] classified genus-1 Lefschetz fibrations over the 2-sphere. They proved that a 4-manifold X admits a genus-1 Lefschetz fibration over S^2 (with critical points) if and only if X is diffeomorphic to an elliptic surface $E(n)$ or a manifold obtained by blowing up $E(n)$. Baykur, Kamada [6] and the author [9, 10] also studied the classification problem. Baykur and Kamada classified total spaces of genus-1 simplified broken Lefschetz fibrations up to blow-ups, while the author provided a classification of the total spaces of genus-1 simplified broken Lefschetz fibrations with small number of Lefschetz critical points or spin structures. The main result of this paper gives the complete classification of total spaces of genus-1 simplified broken Lefschetz fibrations:

THEOREM 1. *Let $f : X \to S^2$ be a genus-1 simplified broken Lefschetz fibration. Suppose that f has an indefinite fold and l Lefschetz critical points (l is possibly equal to 0). Then the total space X of f is diffeomorphic to one of the following 4-manifolds:*

- $\#k\mathbf{CP}^2\#(l-k)\overline{\mathbf{CP}^2}$, *where* $0 \leq k \leq l-1$*;*
- $S^4\#\frac{l}{2}S^2 \times S^2$ *(this manifold appears only if l is even);*
- $S^1 \times S^3\#S\#l\overline{\mathbf{CP}^2}$, *where S is an S^2-bundle over S^2;*
- $L\#l\overline{\mathbf{CP}^2}$, *where L is either of the manifolds L_n or L_n' ($n \geq 2$) which is defined by Pao* [15].

The total spaces of genus-1 simplified broken Lefschetz fibrations without Lefschetz critical points have already been classified by Baykur and Kamada [6] and independently the author [9]. Note that Theorem 1 gives the affirmative answer to [9, Conjecture 5.3] and the negative answer to [5, Problem 24].

Unlike the arguments in [6] and [9], instead of attempting to completely classify monodromies, we modify a given genus-1 simplified broken Lefschetz fibration using homotopies and surgeries, which allows us to induct on the number of Lefschetz critical points. In particular, the author does not classify the monodromies, but only classifies the diffeomorphism types of the total spaces of genus-1 simplified broken Lefschetz fibrations.

2. Preliminaries

2.1. Simplified broken Lefschetz fibrations. Let X and B be oriented, compact, smooth manifolds of dimension 4 and 2, respectively, and $f : X \to B$ a smooth map. Assume that f satisfies the condition $f^{-1}(\partial B) = \partial X$. Denote the set of critical points of f by $\mathrm{Crit}(f) \subset X$. A point $p \in \mathrm{Crit}(f)$ is called an *indefinite fold* if there exist a real coordinate neighborhood $(U, \varphi : U \to \mathbf{R}^4)$ of X centered at p and a real coordinate neighborhood $(V, \psi : V \to \mathbf{R}^2)$ of B centered at $f(p)$ such that these coordinates satisfy the following condition:

$$\psi \circ f \circ \varphi^{-1}(t, x, y, z) = (t, x^2 + y^2 - z^2).$$

By the definition, the set of indefinite folds of f is a 1-dimensional submanifold of X. We denote this submanifold by Z_f. A point $p \in \mathrm{Crit}(f)$ is called a *Lefschetz critical point* if there exist a complex coordinate neighborhood $(U, \varphi : U \to \mathbf{C}^2)$ of X centered at p and a complex coordinate neighborhood $(V, \psi : V \to \mathbf{C})$ of B centered at $f(p)$ such that these coordinates are compatible with the orientations of X and B, respectively, and satisfy the following condition:

$$\psi \circ f \circ \varphi^{-1}(z, w) = z^2 + w^2.$$

The set of Lefschetz critical points of f is a discrete set. We denote this set by $\mathscr{C}_f$.

Definition 1. A map f is called a *broken Lefschetz fibration* if the set $\mathrm{Crit}(f) \subset X$ consists of indefinite folds and Lefschetz critical points. We will refer to broken Lefschetz fibrations as BLFs for short. A BLF $f : X \to B$ is said to be *relatively minimal* if no fibers of f contain a sphere with self-intersection -1 (in other words, no vanishing cycles of Lefschetz critical points of f are null-homotopic in a fiber).

We define other critical points. For a smooth map $f : X^4 \to B^2$, a point $p \in \mathrm{Crit}(f)$ is called an *indefinite cusp* if there exist a real coordinate neighborhood $(U, \varphi : U \to \mathbf{R}^4)$ of X centered at p and a real coordinate neighborhood $(V, \psi : V \to \mathbf{R}^2)$ of B centered at $f(p)$ such that these coordinates satisfy the following condition:

$$\psi \circ f \circ \varphi^{-1}(t, x, y, z) = (t, x^3 + 3tx + y^2 - z^2).$$

In this paper, we will call this critical point a cusp for short. In the coordinate neighborhood (U, φ) the set of critical points is a 1-dimensional submanifold. It is known that all the critical points except for the origin are indefinite folds. Thus, there always exist indefinite folds around a cusp point. A point $p \in \mathrm{Crit}(f)$ is called an *achiral Lefschetz critical point* if there exist a complex coordinate neighborhood $(U, \varphi : U \to \mathbf{C}^2)$ of X centered at p and a complex coordinate neighborhood $(V, \psi : V \to \mathbf{C})$ of B centered at $f(p)$ such that these coordinates are compatible with the orientations of X and B, respectively, and satisfy the following condition:

$$\psi \circ f \circ \varphi^{-1}(z, w) = z^2 + \bar{w}^2.$$

Although a BLF does not have these critical points, they are still important when we consider homotopies involving indefinite folds and Lefschetz critical points (see Subsection 2.3).

Let $f : X \to S^2$ be a BLF over the 2-sphere. Suppose f satisfies the following conditions:

(a) the set Z_f is connected, that is, Z_f is either the empty set or an embedded circle in X;
(b) the restriction $f|_{\mathrm{Crit}(f)}$ is injective;
(c) every fiber of f is connected.

If Z_f is the empty set, f is nothing but a Lefschetz fibration over S^2. If Z_f is not empty, the image $f(Z_f)$ is an embedded circle in S^2. Thus, $S^2 \backslash \nu f(Z_f)$ consists of two open disks $D_1, D_2 \in S^2$, where $\nu f(Z_f)$ is a regular neighborhood of $f(Z_f)$. By the definition of indefinite folds, the genus of a regular fiber of the fibration $f|_{f^{-1}(D_1)} : f^{-1}(D_1) \to D_1$ differs by 1 from that of the fibration $f|_{f^{-1}(D_2)} : f^{-1}(D_2) \to D_2$. We call the preimage $f^{-1}(D_1)$ the *higher side* of f and $f^{-1}(D_2)$ the *lower side* of f.

DEFINITION 2. A BLF $f : X \to S^2$ is said to be *simplified* if f satisfies the above conditions (a), (b) and (c) and the following condition:

(d) the set $\mathscr{C}_f$ is contained in the higher side of f.

In this paper we will refer to simplified BLFs as SBLFs for short. The *genus* of an SBLF is defined to be the genus of a regular fiber in the higher side if Z_f is non-empty, and the genus of any fiber if Z_f is empty.

REMARK 1. *Every* 4-*manifold which appears in Theorem* 1 *admits a genus*-1 *SBLF. Indeed, the author* [9] *constructed a genus*-1 *SBLF for each manifold explicitly. (Also see* [5, 6].*)*

2.2. Vanishing cycles. For an SBLF $f : X \to S^2$, we put $f(\mathscr{C}_f) = \{p_1, \dots, p_l\}$. We take a regular value p_0 in the image of the higher side of f. We also take paths $\gamma, \gamma_1, \dots, \gamma_l$ so that γ connects p_0 to a point on the image of indefinite folds, while γ_i connects p_0 to p_i, and that these paths are mutually disjoint except on the point p_0. Suppose that the indices of the paths are given so that $\gamma, \gamma_1, \dots, \gamma_l$ appear in this order when we go around p_0 counterclockwise. These paths determine *vanishing cycles* of Lefschetz critical points and indefinite folds of f (the reader can refer to [8] or [9], for example, for details of vanishing cycles). We denote these vanishing cycles by $c, c_1, \dots, c_l \subset f^{-1}(p_0)$.

We identify the fiber $f^{-1}(p_0)$ with the standard closed surface Σ_g and regard vanishing cycles as simple closed curves in Σ_g. We call a sequence $W_f = (c; c_1, \dots, c_l)$ a *Hurwitz cycle system* of f.

We denote the mapping class group of Σ_g, that is, the set of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g by $\mathrm{Mod}(\Sigma_g)$. In order to make the notation coincide with those in the author's previous papers [9, 10], we define the multiplication in $\mathrm{Mod}(\Sigma_g)$ as the *opposite* of that induced by the composition as maps, that is, if an element $\phi_i \in \mathrm{Mod}(\Sigma_g)$ is represented by a diffeomorphism $T_i : \Sigma_g \to \Sigma_g$, the product $\phi_1 \cdot \phi_2$ is defined as follows:

$$\phi_1 \cdot \phi_2 = [T_2 \circ T_1].$$

For a simple closed curve $c \subset \Sigma_g$, we define a subgroup $\mathrm{Mod}(\Sigma_g)(c)$ of $\mathrm{Mod}(\Sigma_g)$ as follows:

$$\mathrm{Mod}(\Sigma_g)(c) = \{[T] \in \mathrm{Mod}(\Sigma_g) \mid T : \Sigma_g \xrightarrow{\cong} \Sigma_g, T(c) = c\}.$$

Proposition 1 ([2]). *Let $f : X \to S^2$ be a genus-g SBLF and $W_f = (c; c_1, \dots, c_l)$ its Hurwitz cycle system. Then, the following holds:*

$$t_{c_1} \cdot \cdots \cdot t_{c_l} \in \mathrm{Mod}(\Sigma_g)(c),$$

where $t_{c_i} \in \mathrm{Mod}(\Sigma_g)$ is the right-handed Dehn twist along c_i (for the definition of the Dehn twist, see [8, Definition 8.2.3]*).*

Remark 2. *It is known that the product $t_{c_1} \cdot \cdots \cdot t_{c_l}$ is contained in the kernel of some homomorphism defined on $\mathrm{Mod}(\Sigma_g)(c)$. The reader can refer to* [4] *for details on this homomorphism.*

We will introduce two modifications of Hurwitz cycle systems. The first one is called an *elementary transformation*, which changes a Hurwitz cycle system as follows:

$$(c; c_1, \dots, c_i, c_{i+1}, \dots, c_l) \mapsto (c; c_1, \dots, c_{i+1}, t_{c_{i+1}}(c_i), \dots, c_l).$$

It is known that this modification can be realized by replacing paths γ and γ_i (for details, see [8]). The second one is called a *simultaneous conjugation* by an element $\phi \in \mathrm{Mod}(\Sigma_g)$, which changes a Hurwitz cycle system as follows:

$$(c; c_1, \dots, c_l) \mapsto (\phi(c); \phi(c_1), \dots, \phi(c_l)).$$

This modification can be realized by replacing an identification of $f^{-1}(p_0)$ with Σ_g. Two sequences $(c; c_1, \dots, c_l)$ and $(d; d_1, \dots, d_l)$ of simple closed curves in Σ_g are said to be *Hurwitz equivalent* if one can be obtained from the other by successive application of simultaneous conjugations, elementary transformations and their inverse. Note that for a given SBLF f, any sequence W which is Hurwitz equivalent to W_f can be realized as a Hurwitz cycle system of f by replacing reference paths and an identification $f^{-1}(p_0) \cong \Sigma_g$.

2.3. Homotopies. In this subsection we will give a quick review for several homotopies involving indefinite folds and Lefschetz critical points which will be used in the proof of Theorem 1. The reader can refer to [12], for example, for details of the homotopies in this subsection.

Merging: We define a homotopy $F_s : \mathbf{R}^4 \to \mathbf{R}^2$ $(s \in [-1,1])$ as follows:

$$F_s(t,x,y,z) = (t, x^3 + 3(s - t^2)x + y^2 - z^2).$$

This homotopy is called a *fold merge* and the inverse of this homotopy F_{-s} is called a *cusp merge*. The critical point set $\mathrm{Crit}(F_s)$ is equal to $\{(t,x,0,0) \in \mathbf{R}^4 \mid x^2 - t^2 + s = 0\}$. For $s < 0$, the set $\mathrm{Crit}(F_s)$ consists of indefinite folds and the set of critical values are two parallel arcs (see the left side of Figure 1). However, when s is equal to 0, another kind of critical point (which is called *beak-to-beak*) appears at the origin and the critical value set consists of two arcs which are tangent to each other at the origin (see the middle figure in Figure 1). For $s > 0$, two cusp points appear at $(\pm\sqrt{s},0,0,0)$ and the critical value set consists of two cusped arcs (the right side of Figure 1).

Let $f : X \to B$ be a smooth map from a 4-manifold to a 2-manifold. Suppose that f has arcs of indefinite folds whose images are two parallel arcs as shown in the left side of Figure 1. It is known that we can apply a fold merge to these two indefinite folds if and only if the middle region in the figure (the region with the dot) is the higher-genus side of both of the indefinite folds and two vanishing cycles of the indefinite folds (vanishing cycles derived from two dotted arcs in Figure 1) intersect at one point transversely.

Wrinkling: We define a homotopy $H_s : \mathbf{R}^4 \to \mathbf{R}^2$ $(s \in [0,1])$ as follows:

$$H_s(t,x,y,z) = (t^2 - x^2 + y^2 - z^2 + st, 2tx + 2yz).$$

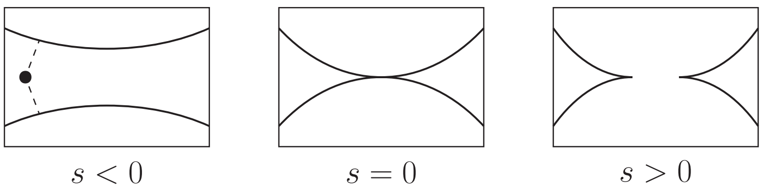

Fig. 1. The critical value sets of the map F_s.

This homotopy is called a *wrinkle*. We give complex coordinates of $\mathbf{R}^4$ and $\mathbf{R}^2$ by the maps $(t, x, y, z) \mapsto (t + \sqrt{-1}x, y + \sqrt{-1}z)$ and $(u, v) \mapsto u + \sqrt{-1}v$, respectively. In these coordinates, H_s has the following description:

$$H_s(z, w) = z^2 + w^2 + s\,\mathrm{Re}(z).$$

In particular, H_0 has only one critical point at the origin which is a (possibly achiral) Lefschetz critical point. For $s > 0$, the set $\mathrm{Crit}(H_s)$ is a circle which consists of indefinite folds and three cusps and this circle is mapped to a triangle whose corners are the images of cusps (see the right side of Figure 2).

For a smooth map $f : X^4 \to B^2$, we can always apply a wrinkle to any Lefschetz critical points of f. Suppose that f has a circle consisting of indefinite folds with three cusps such that it is mapped to a triangle as shown in the right side of Figure 2. We further assume that a regular fiber of f inside the triangle is a torus. Then it is not hard to see that we can always apply the inverse of a wrinkle which changes the circle into a (possibly achiral) Lefschetz critical point.

Sinking: There is a homotopy which changes a Lefschetz critical point near fold arcs into a cusp point (see Figure 3). This homotopy is called a *sink* and the inverse of a sink is called an *unsink*. This homotopy was introduced in [12] and named in [18]. A sink is a sequence of several homotopies (see [12, Figure 8]) and we can apply this homotopy if a vanishing cycle of the Lefschetz critical point intersects that of indefinite folds at one point transversely. We can apply an unsink to any cusp point of a map $f : X^4 \to B^2$.

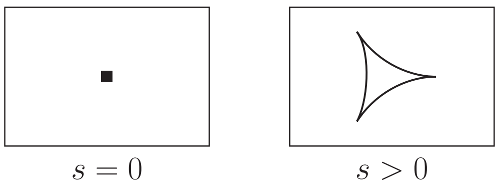

Fig. 2. The critical value set of H_s. The square dot describes the image of a Lefschetz critical point.

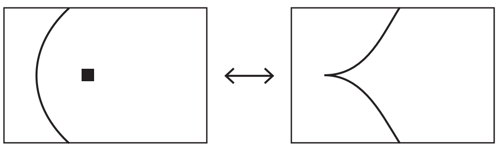

Fig. 3. Sink (left to right) and unsink (right to left).

3. The classification of genus-1 simplified broken Lefschetz fibrations

In this section we give the proof of Theorem 1. We begin by reviewing basic properties of Hurwitz cycle systems of genus-1 SBLFs. We take elements $\alpha, \beta \subset H_1(T^2;\mathbf{Z})$ so that the algebraic intersection $\alpha \cdot \beta$ is 1. Since a primitive element in $H_1(T^2;\mathbf{Z})$ uniquely determines the isotopy class of an oriented simple closed curve in T^2, we represent the isotopy class of an oriented simple closed curve by its homology class.

LEMMA 1 ([9, Theorem 3.11]). *Let $f: X \to S^2$ be a relatively minimal genus-1 SBLF. A Hurwitz cycle system W_f is Hurwitz equivalent to the following sequence:*

$$(\alpha; S_r T(n_1, \dots, n_s)),$$

where $T(n_1, \dots, n_s) = (\beta + n_1\alpha, \dots, \beta + n_s\alpha)$ and $S_r = (\alpha, \dots, \alpha)$ (r α's are contained in this sequence).

LEMMA 2 ([9, Theorem 4.6]). *Let $f: X \to S^2$ be a relatively minimal genus-1 SBLF.*

(1) *Suppose that a Hurwitz cycle system W_f of f is equal to $(\alpha; S_r T(n_1, \dots, n_s))$. Then, there exists a genus-1 SBLF $h: X' \to S^2$ with a Hurwitz cycle system $(\alpha; T(n_1, \dots, n_s))$ such that X is diffeomorphic to $X' \# r\overline{\mathbf{CP}^2}$.*
(2) *Suppose that a Hurwitz cycle system W_f of f is equal to $(\alpha; T(n, n-2) T(n_1, \dots, n_s))$. Then, there exists a genus-1 SBLF $h: X' \to S^2$ with a Hurwitz cycle system $(\alpha; T(n_1, \dots, n_s))$ such that X is diffeomorphic to $X' \# S$, where S is an S^2-bundle over S^2.*

The group $H_1(T^2;\mathbf{Z})$ admits a right action of $\mathrm{Mod}(T^2)$. It is known that the isomorphism $H_1(T^2;\mathbf{Z}) \cong \mathbf{Z}^2$ given by the generator (α, β) induces an isomorphism $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2,\mathbf{Z})$. This isomorphism maps elements t_α and t_β to matrices $X_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, respectively.

LEMMA 3. *Let $f: X \to S^2$ be a relatively minimal genus-1 SBLF and $W_f = (\alpha; S_r T(n_1, \dots, n_s))$ its Hurwitz cycle system. Then the following equality holds in $\mathrm{PSL}(2,\mathbf{Z}) = \mathrm{SL}(2,\mathbf{Z})/\{\pm 1\}$ for some integer $n \in \mathbf{Z}$:*

$$x_1^r \cdot (x_1^{-n_1} x_2 x_1^{n_1}) \cdot \dots \cdot (x_1^{-n_s} x_2 x_1^{n_s}) = x_1^n,$$

where x_i is the element in $\mathrm{PSL}(2,\mathbf{Z})$ represented by X_i.

PROOF. Under the identification $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2,\mathbf{Z})$, the image of the subgroup $\mathrm{Mod}(T^2)(\alpha)$ under the quotient map $\mathrm{SL}(2,\mathbf{Z}) \to \mathrm{PSL}(2,\mathbf{Z})$ is equal to the set $\{x_1^n \in \mathrm{PSL}(2,\mathbf{Z}) \mid n \in \mathbf{Z}\}$. Lemma 3 immediately follows from this observation and Proposition 1. $\square$

LEMMA 4. *Let $f: X \to S^2$ be a relatively minimal genus-1 SBLF and $W_f = (\alpha; S_r T(n_1, \ldots, n_s))$ its Hurwitz cycle system. Assume that s is not equal to 0. Then, s is greater than 1 and $n_i - n_{i+1} = 1, 2$ or 3 for some $i \in \{1, \ldots, s-1\}$.*

PROOF. We put $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$. Then the following relations hold:

$$X_1 = ABA, \quad X_2 = BA^2.$$

Denote the elements in $\mathrm{PSL}(2, \mathbf{Z})$ represented by A and B by a and b, respectively. The group $\mathrm{PSL}(2, \mathbf{Z})$ has the following presentation:

$$\mathrm{PSL}(2, \mathbf{Z}) = \langle a, b \mid a^3, b^2 \rangle.$$

The sequence $(w_1, \ldots, w_n)$ of elements in $\mathrm{PSL}(2, \mathbf{Z})$ is said to be *reduced* if the set $\{w_i, w_{i+1}\}$ is equal to either of the sets $\{a, b\}$ or $\{a^2, b\}$ for each $i \in \{1, \ldots, n-1\}$. Since $\mathrm{PSL}(2, \mathbf{Z})$ is isomorphic to the free product $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ generated by a and b, for any element $g \in \mathrm{PSL}(2, \mathbf{Z})$ there uniquely exists a reduced sequence $(w_1, \ldots, w_n)$ such that the product $w_1 \cdot \cdots \cdot w_n$ is equal to g ([13, Theorem 4.1]).

Suppose that s is equal to 1. By Lemma 3 the following relation holds for some integer n:

$$\begin{aligned} & x_1^r \cdot x_1^{-n_1} x_2 x_1^{n_1} = x_1^n \\ \Leftrightarrow\ & x_2 = x_1^{n-r} \\ \Leftrightarrow\ & ba^2 = (aba)^{n-r}. \end{aligned}$$

This contradicts the uniqueness of a factorization by a reduced sequence. Thus s would not be equal to 1.

Suppose that $n_i - n_{i+1}$ is not equal to 1, 2 or 3 for any $i \in \{1, \ldots, s-1\}$. By Lemma 3 the following relation holds for some integer n:

$$\begin{aligned} & x_1^r \cdot (x_1^{-n_1} x_2 x_1^{n_1}) \cdot \cdots \cdot (x_1^{-n_s} x_2 x_1^{n_s}) = x_1^n \\ \Leftrightarrow\ & x_2 x_1^{n_1 - n_2} x_2 \cdot \cdots \cdot x_1^{n_{s-1} - n_s} x_2 = x_1^m, \end{aligned} \tag{1}$$

where we put $m = n - r + n_1 - n_s$. The following lemma can be proved by the same argument as that in the proof of [10, Lemma 5.8][2].

LEMMA 5. *The product $x_2 x_1^{n_1 - n_2} x_2 \cdot \cdots \cdot x_1^{n_{s-1} - n_s} x_2$ is equal to bS or $a^2 b a^2 b S$, where $S = w_1 \cdot \cdots \cdot w_k$ and $(w_1, \ldots, w_k)$ is a reduced sequence with $w_1 \in \{a, a^2\}$.*

[2] In Lemma 5.8 of [10], $n_i - n_{i+1}$ is further assumed to be even, but the proof still works under the assumption $n_i - n_{i+1} \neq 1, 2, 3$.

Since x_1 is equal to aba, the relation (1) contradicts the uniqueness of a factorization by a reduced sequence. This completes the proof of Lemma 4. $\square$

We are now ready to prove our main theorem.

Proof (of Theorem 1). Since SBLFs which are not relatively minimal can be obtained by blowing up relatively minimal SBLFs (see [8, Exercises 8.1.8. (a)], for example), we can assume f is relatively minimal without loss of generality. We will prove Theorem 1 by induction on $l = \#\mathscr{C}_f$. The statement for the case $l = 0$ was proved in [6] and also in [9]. Thus, we assume l is greater than 0. By Lemma 1, we can further assume that W_f is equal to $(\alpha; S_r T(n_1, \dots, n_s))$.

We first consider the case X is not simply connected. In this case, it is easy to verify that s is equal to 0. By Lemma 2, there exists a genus-1 SBLF $f : X' \to S^2$ without Lefschetz critical points such that X is diffeomorphic to $X' \# r\overline{\mathbf{CP}^2}$. Thus, we can deduce the conclusion from the induction hypothesis.

We next consider the case X is simply connected. If r is not equal to 0, we can reduce the number of Lefschetz critical points using Lemma 2 and the conclusion holds by the induction hypothesis. Assume that r is equal to 0 and $s = l$. By Lemma 4, there exists a number $i \in \{1, \dots, l-1\}$ such that $n_i - n_{i+1}$ is equal to $1, 2$ or 3.

If $n_i - n_{i+1}$ is equal to 1, then W_f is Hurwitz equivalent to the following sequence:

$$(\alpha; S_1 T(n_1 + 1, \dots, n_{i-1} + 1, n_i, n_{i+2} \dots, n_l))$$

since the sequence $T(n_i, n_{i+1})$ is Hurwitz equivalent to $S_1 T(n_i)$. Thus, the conclusion holds by the induction hypothesis.

If $n_i - n_{i+1}$ is equal to 2, then W_f is Hurwitz equivalent to the following sequence:

$$(\alpha; T(n_i, n_{i+1}) T(n_1 - 4, \dots, n_{i-1} - 4, n_{i+2} \dots, n_l))$$

since the composition $t_{\beta - n_{i+1}\alpha} \circ t_{\beta - n_i \alpha}$ is equal to $(t_\alpha t_\beta)^3 t_\alpha^{-4}$. By Lemma 2, there exists a genus-1 SBLF $f' : X' \to S^2$ with $l - 2$ Lefschetz critical points such that X is diffeomorphic to $X' \# S$, where S is an S^2-bundle over S^2. Thus, the conclusion holds by the induction hypothesis.

If $n_i - n_{i+1}$ is equal to 3, then W_f is Hurwitz equivalent to $(\alpha; T(0, -3) W)$, where W is some sequence which consists of $l - 2$ simple closed curves. We take a regular value $q_0 \in S^2$ of f and paths γ_1 and γ_2 from q_0 to the images of Lefschetz critical points so that the corresponding vanishing cycles are equal to β and $\beta - 3\alpha$, respectively (see the far left of Figure 4). It is not hard to

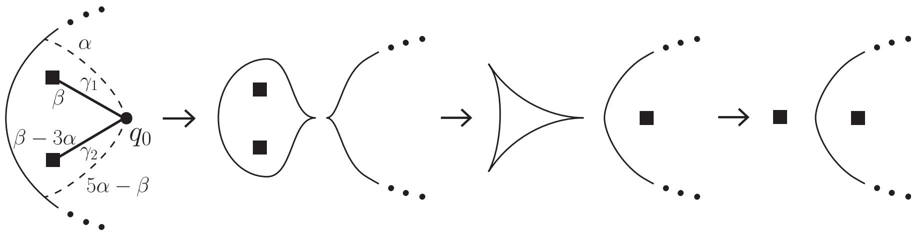

Fig. 4. Configurations of critical value sets. The squares represent the images of Lefschetz critical points.

see that two vanishing cycles of indefinite folds determined by the dashed arcs in the far left of Figure 4 are α and $5\alpha - \beta$, respectively. In particular, these vanishing cycles intersect at one point transversely. Thus, we can apply a fold merge to f along the dashed arcs. The critical value set is changed as described in the second figure in Figure 4. We can further apply an unsink to the cusp in the right side of the figure and sinks twice to the two Lefschetz critical points, resulting the critical value set described in the third figure in Figure 4. Since the genus of a regular fiber in the preimage of the triangle is 1, we can further apply a wrinkle so that the critical point set on the triangle is changed into an achiral Lefschetz critical points with null-homotopic vanishing cycle. Eventually, we can obtain a genus-1 SBLF $f' : X' \to S^2$ with $l - 1$ Lefschetz critical points so that X is diffeomorphic to $X' \# \overline{\mathbf{CP}}^2$. Thus, the conclusion holds by the induction hypothesis. This completes the proof of Theorem 1. $\square$

Acknowledgement

This work is part of the author's PhD thesis at Osaka University. The author is grateful to his supervisor Ryushi Goto for careful reading of the first draft and for valuable comments. The author would like to thank Hisaaki Endo for introducing him to this problem and for fruitful discussions.

References

[1] S. Akbulut and Ç. Karakurt, Every 4-manifold is BLF, J. Gökova Geom. Topol. GGT **2** (2008), 83–106.
[2] D. Auroux, S. K. Donaldson, and L. Katzarkov, Singular Lefschetz pencils, Geom. Topol. **9** (2005), 1043–1114.
[3] R. İ. Baykur, Existence of broken Lefschetz fibrations, Int. Math. Res. Not. IMRN (2008), Art. ID rnn 101, 15.

[4] R. İ. Baykur, Topology of broken Lefschetz fibrations and near-symplectic four-manifolds, Pacific J. Math. **240** (2009), no. 2, 201–230.
[5] R. İ. Baykur, Broken Lefschetz fibrations and smooth structures on 4-manifolds, Geom. Topol. Monogr. **12** (2012), 9–34.
[6] R. İ. Baykur and S. Kamada, Classification of broken Lefschetz fibrations with small fiber genera, to appear in J. Math. Soc. Japan, available at http://arxiv.org/abs/1010.5814.
[7] D. T. Gay and R. Kirby, Constructing Lefschetz-type fibrations on four-manifolds, Geom. Topol. **11** (2007), 2075–2115.
[8] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.
[9] K. Hayano, On genus-1 simplified broken Lefschetz fibrations, Algebr. Geom. Topol. **11** (2011), no. 3, 1267–1322.
[10] K. Hayano, A note on sections of broken Lefschetz fibrations, Bull. Lond. Math. Soc. **44** (2012), no. 4, 823–836.
[11] A. Kas, On the deformation types of regular elliptic surfaces, Complex analysis and algebraic geometry, 1977, pp. 107–111.
[12] Y. Lekili, Wrinkled fibrations on near-symplectic manifolds, Geom. Topol. **13** (2009), no. 1, 277–318. Appendix B by R. İnanç Baykur.
[13] W. Magnus, A. Karrass, and D. Solitar, Combinatorial group theory: Presentations of groups in terms of generators and relations, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966.
[14] B. Moishezon, Complex surfaces and connected sums of complex projective planes, Lecture Notes in Mathematics, Vol. 603, Springer-Verlag, Berlin, 1977. With an appendix by R. Livne.
[15] P. S. Pao, The topological structure of 4-manifolds with effective torus actions. I, Trans. Amer. Math. Soc. **227** (1977), 279–317.
[16] T. Perutz, Lagrangian matching invariants for fibred four-manifolds. I, Geom. Topol. **11** (2007), 759–828.
[17] T. Perutz, Lagrangian matching invariants for fibred four-manifolds. II, Geom. Topol. **12** (2008), no. 3, 1461–1542.
[18] J. Williams, The *h*-principle for broken Lefschetz fibrations, Geom. Topol. **14** (2010), no. 2, 1015–1061.

Kenta Hayano
Department of Mathematics
Graduate School of Science
Hokkaido University
Sapporo, Hokkaido 060-0810, *Japan*
E-mail: k-hayano@math.sci.hokudai.ac.jp
URL: http://www.math.sci.hokudai.ac.jp/k-hayano/index_en.html