Hiroshima Math. J.
43 (2013), 305–333

Self-similar radial solutions to a class of strongly coupled reaction-diffusion systems with cross-diffusion

Dedicated to Professor Bernd Kawohl on the occasion of his sixtieth birthday

Dirk Horstmann

(Received March 16, 2012)
(Revised September 26, 2012)

Abstract. This paper establishes some existence and nonexistence results of self-similar radial symmetric solutions to some class of strongly coupled reaction-diffusion systems with cross-diffusion. The considered class of systems allows to reduce the problem to a single equation with exponential source terms. Using the famous Mountain Pass Theorem and some smallness conditions on the system parameters it is possible to generalize wellknown results on self-similar radial solutions for a related problem that have been established by Y. Mizutani, N. Muramoto and K. Yoshida in 1999. As an application of the results derived in the present paper it is possible to conclude the existence and nonexistence of self-similar radial solutions for multi-species chemotaxis-model in the conflict-free setting and in the presence of a conflict of interests.

1. Introduction

The purpose of the present paper is to establish some existence and nonexistence results of positive self-similar radial solutions to a class of strongly coupled reaction-diffusion systems with cross-diffusion of the following type:

$$\left.\begin{aligned} (u_i)_t &= \nabla(\nabla u_i - \Theta_i u_i \nabla v), && \text{in } \mathbf{R}^2 \times \{t > 0\} \\ (w_j)_t &= \nabla(\nabla w_j + \Xi_j w_j \nabla v), && \text{in } \mathbf{R}^2 \times \{t > 0\} \\ \varepsilon v_t &= \Delta v + \sum_{k=1}^{N} \alpha_k e^{\xi_k v} \prod_{i=1}^{n} (u_i)^{\beta_{i,k}} \prod_{j=1}^{m} (w_j)^{\delta_{j,k}}, && \text{in } \mathbf{R}^2 \times \{t > 0\} \end{aligned}\right\} \tag{1.1}$$

where $i \in \{1, \dots n\}$, $j \in \{1, \dots, m\}$, $n \in \mathbf{N}$, $m \in \mathbf{N}_0$ and $N \geq 1$. (By $m = 0$ we mean that there is no equation of the type given in the second line of the previous system.) Furthermore, we assume that the constants Θ_i, $\Xi_j > 0$,

2010 *Mathematics Subject Classification*. Primary 35J20, 35K57, 35K40; Secondary 35Q60, 92C17.
Key words and phrases. Self-similarity, radial solutions, mountain pass solution, cross-diffusion systems.

$\alpha_k > 0$ and $\xi_k, \beta_{i,k}, \delta_{j,k} \in \mathbf{R}$. We see that the mass for all u_i and w_j is conserved and therefore,

$$\int_{\mathbf{R}^2} u_i(x,t)dx = \int_{\mathbf{R}^2} u_i(x,0)dx < \infty$$

and

$$\int_{\mathbf{R}^2} w_j(x,t)dx = \int_{\mathbf{R}^2} w_j(x,0)dx < \infty \qquad \text{for all } t > 0$$

hold. As a restriction we are seeking for solutions satisfying that

$$v(x,t) > 0 \qquad \text{and} \qquad \int_{\mathbf{R}^2} v(x,t) < \infty \qquad \text{for all } t > 0.$$

Problems of this kind have several applications in the sciences. For example, multi-species chemotaxis models as they have been studied and analyzed in [6, 10, 11, 20] also belong to this class.

Assuming that

$$u_i(x,t) = \frac{1}{t}\varphi_i\left(\frac{|x|}{\sqrt{t}}\right), \qquad w_j(x,t) = \frac{1}{t}\omega_j\left(\frac{|x|}{\sqrt{t}}\right) \qquad \text{and} \qquad v(x,t) = \psi\left(\frac{|x|}{\sqrt{t}}\right)$$

and introducing the new variable $r = |x|/\sqrt{t}$, we see that our system is transformed into

$$\left.\begin{array}{r} (\varphi_i' - \Theta_i\varphi_i\psi')' + \frac{1}{r}(\varphi_i' - \Theta_i\varphi_i\psi') + \frac{r}{2}\varphi_i' + \varphi_i = 0 \\ (\omega_j' + \Xi_j\omega_j\psi')' + \frac{1}{r}(\omega_j' + \Xi_j\omega_j\psi') + \frac{r}{2}\omega_j' + \omega_j = 0 \\ \psi'' + \frac{1}{r}\psi' + \frac{\varepsilon r}{2}\psi' + \sum_{k=1}^{N}\alpha_k e^{\xi_k\psi}\prod_{i=1}^{n}(\varphi_i)^{\beta_{i,k}}\prod_{j=1}^{m}(\omega_j)^{\delta_{j,k}} = 0 \\ \varphi_i'(0) = \omega_j'(0) = \psi'(0) = 0 \end{array}\right\}. \tag{1.2}$$

However, for each i and j one can solve the equations for φ_i and ω_j in dependence of ψ and, therefore, we see that

$$\varphi_i(r) = c_i e^{-r^2/4} e^{\Theta_i\psi(r)} \qquad \text{and} \qquad \omega_j(r) = d_j e^{-r^2/4} e^{-\Xi_j\psi(r)}$$

with

$$c_i = \varphi_i(0)e^{-\Theta_i\psi(0)} > 0 \qquad \text{and} \qquad d_j = \omega_j(0)e^{\Xi_j\psi(0)} > 0.$$

This allows us to reduce the problem to a single equation for the function ψ. The new problem for the unknown function ψ is now given by:

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \sum_{l=1}^{z_1} \mu_l e^{-\zeta_l r^2/4} e^{\chi_l \psi} + \sum_{h=1}^{z_2} \lambda_h e^{-\Lambda_h r^2/4} e^{-\nu_h \psi} = 0 \tag{1.3}$$

$$\psi'(0) = 0 \tag{1.4}$$

$$\int_0^\infty r\psi(r)dr < \infty, \tag{1.5}$$

under the assumption that

$$\xi_k + \sum_{i=1}^{n} \beta_{i,k}\Theta_i - \sum_{j=1}^{m} \delta_{i,k}\Xi_i \neq 0 \qquad \text{for all } k \in \{1, \dots, N\}$$

and where we used for $z_1 + z_2 = N$ the notations,

$$\zeta_k := \sum_{i=1}^{n} \beta_{i,k} + \sum_{j=1}^{m} \delta_{i,k} \qquad \text{and}$$

$$\mu_k := \alpha_k \prod_{i=1}^{n} (c_i)^{\beta_{i,k}} \prod_{j=1}^{m} (d_j)^{\delta_{j,k}}, \qquad \text{for those } z_1 \text{ indices } k, \text{ for that}$$

$$\chi_k := \xi_k + \sum_{i=1}^{n} \beta_{i,k}\Theta_i - \sum_{j=1}^{m} \delta_{i,k}\Xi_i > 0$$

and

$$\Lambda_k := \sum_{i=1}^{n} \beta_{i,k} + \sum_{j=1}^{m} \delta_{i,k} \qquad \text{and}$$

$$\lambda_k := \alpha_k \prod_{i=1}^{n} (c_i)^{\beta_{i,k}} \prod_{j=1}^{m} (d_j)^{\delta_{j,k}} \qquad \text{for those } z_2 \text{ indices } k, \text{ for that}$$

$$-\nu_k := \xi_k + \sum_{i=1}^{n} \beta_{i,k}\Theta_i - \sum_{j=1}^{m} \delta_{i,k}\Xi_i < 0.$$

Hypothesis 1.1. *Throughout the present paper we will make the following two assumptions:*

(1) *There is at least one k such that $\mu_k \neq 0$.*

(2) *For simplicity, let us assume that $\zeta_k = 1 = \Lambda_k$ for all indices k.*

Remark 1. *It seems that Assumption* (2) *of Hyphesis* 1.1 *can easily be relaxed to some more general conditions on ζ_k and Λ_k. There will be some additional remarks on this point in the closing section of the present paper.*

To establish our existence results we use some previous results by Y. Mizutani, N. Muramoto and K. Yoshida [13]. In their paper [13] they proved similar results on the existence and nonexistence of self-similar radial symmetric solutions for the problem

$$\left.\begin{aligned} u_t &= \nabla(\nabla u - u\nabla v), && \text{in } \mathbf{R}^2 \times \{t > 0\} \\ \varepsilon v_t &= \Delta v + \alpha u, && \text{in } \mathbf{R}^2 \times \{t > 0\} \end{aligned}\right\}. \tag{1.6}$$

Looking for a self-similar radially symmetric solution $u(x,t) = \varphi(|x|/\sqrt{t})/t = \varphi(r)/t$ and $v(x,t) = \psi(|x|/\sqrt{t}) = \psi(r)$ this leads them to the following system

$$\left.\begin{aligned} (\varphi' - \chi\varphi\psi')' + \tfrac{1}{r}(\varphi' - \psi\varphi\psi') + \tfrac{r}{2}\varphi' + \varphi = 0 \\ \psi'' + \tfrac{1}{r}\psi' + \tfrac{\varepsilon r}{2}\psi' + \alpha\varphi = 0 \\ \varphi'(0) = \psi'(0) = 0, \end{aligned}\right\} \tag{1.7}$$

resp. to the single equation

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \mu e^{-r^2/4} e^{\psi} = 0 \tag{1.8}$$

$$\psi'(0) = 0 \tag{1.9}$$

$$\int_0^\infty r\psi(r)dr < \infty. \tag{1.10}$$

Therefore, the results and proofs in the present paper will naturally and closely follow the lines of argumentation in [13] and we generalize the results given there to a more general class of systems. Furthermore, we will use the notations introduced and used in [13] to make the comparison as simple and transparent as possible. Additionally, throughout the text we will always allude to the corresponding results in [13].

However, at several points the arguments by Y. Mizutani, N. Muramoto and K. Yoshida cannot be adopted and, therefore, some modifications are needed at some places. This is true especially in the case when $m \geq 1$ and when there exist at least one $\lambda_k \neq 0$. This case may correspond to cross-diffusion models describing chemotactic motion in the presence of a conflict of interests. For those kind of problems the author of the present paper is not aware of any existence or nonexistence results like those presented here. Therefore, the presented existence results seem to be completely new.

2. Existence of a mountain pass solution for a related Dirichlet problem

Before we formulate our results for the whole space we will look for positive radial solutions of a related Dirichlet problem on a disk at first. These solutions will then be used to construct solutions of (1.3)–(1.5) on the whole space $\mathbf{R}^2$. To simplify notation we will set $B_R := \{x \in \mathbf{R}^2 \mid |x| < R\}$ and look at the following Dirichlet problem:

$$\left.\begin{aligned} \nabla(e^{\varepsilon|x|^2/4}\nabla v) + \left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l v} + \sum_{h=1}^{z_2}\lambda_h e^{-v_h v}\right)e^{(\varepsilon-1)|x|^2/4} = 0 \quad \text{in } B_R \\ v = 0 \quad \text{on } \partial B_R. \end{aligned}\right\} \tag{2.1}$$

To establish the existence of a positive radially symmetric solution of this problem, we will (as it has also been done in [13]) make use of the famous Mountain Pass Theorem by A. Ambrosetti and P. Rabinowitz [2]. For the reader's convenience we recall the definition of a Palais-Smale sequence, the Palais-Smale condition and the formulation of the Mountain Pass Theorem at first.

Definition 1 (Palais-Smale sequence (compare for example [19])). Let $\mathscr{X}$ be a Banach space and $\mathscr{J} \in C^1(\mathscr{X}, \mathbf{R})$. We say that $(x_n)_{n\in\mathbf{N}} \subset \mathscr{X}$ is a Palais-Smale sequence for $\mathscr{J}$ if

$$|\mathscr{J}(x_n)| \leq c \qquad \text{for some } c, \tag{2.2}$$

uniformly in n, while

$$\mathscr{J}'(x_n) \to 0 \qquad \text{in } \mathscr{X}' \text{ as } n \to \infty. \tag{2.3}$$

Definition 2 (Palais-Smale condition (compare for example [19])). Let $\mathscr{X}$ be a Banach space and $\mathscr{J} \in C^1(\mathscr{X}, \mathbf{R})$. Then we say that $\mathscr{J}$ satisfies the Palais-Smale condition, if any Palais-Smale sequence $(x_n)_{n\in\mathbf{N}} \subset \mathscr{X}$ has a (strongly) convergent subsequence.

Theorem 1 (Mountain Pass Theorem (compare for example [19])). *Let $\mathscr{X}$ be a Banach space, $\mathscr{U}_\rho = \{x \in \mathscr{X} \mid \|x\|_{\mathscr{X}} < \rho\}$ and $\mathscr{J} \in C^1(\mathscr{X}, \mathbf{R})$ satisfying the Palais-Smale condition. Suppose that*

(1) $\mathscr{J}(0) = 0$.

(2) *There exist a $\rho > 0$ and an $\alpha > 0$ such that*

$$\inf_{\|x\|=\rho} \mathscr{J}(x) \geq \alpha. \tag{2.4}$$

(3) *There exists $y_0 \in \mathscr{X} \backslash \overline{\mathscr{U}}_\rho$ such that*

$$\mathscr{J}(y_0) < 0. \tag{2.5}$$

We now define $\mathfrak{P} = \{\wp \in C([0,1], \mathscr{X}) \mid \wp(0) = 0, \wp(1) = y_0\}$. Then

$$\varsigma = \inf_{\wp\in\mathfrak{P}} \sup_{x\in\wp([0,1])} \mathscr{J}(x)$$

is a critical value of $\mathscr{J}$.

Seeking for radially symmetric solutions we will also use the following Theorem which is due to R. S. Palais [18].

Theorem 2 (Principle of symmetric criticality [18]). *Let G be a topological group which continuously acts on a Hilbert space $\mathcal{X}$, that is,*

$$G \times \mathcal{X} \to \mathcal{X} : [g, x] \to gx$$

is a continuous map such that

$$1 \cdot x = x,$$

$$(gh)x = g(hx),$$

$$x \mapsto gx \text{ is linear.}$$

Furthermore assume that $\|gx\| = \|x\|$. Let $\mathcal{J} \in C^1(\mathcal{X}, \mathbf{R})$ satisfy $\mathcal{J} \circ g = \mathcal{J}$ for every $g \in G$. If x is a critical point of $\mathcal{J}$ restricted to $\{x \in \mathcal{X} \mid gx = x$ for all $g \in G\}$, x is a critical point of $\mathcal{J}$.

With the help of these theorems we will establish the following existence result (its pendant in [13] is given by Proposition 1) for (2.1).

Theorem 3. *For sufficiently small R there exists a radially symmetric positive solution $v(x)$ of the Dirichlet problem* (2.1).

We define the Hilbert space $H := \{v \in W_0^{1,2}(B_R) \mid v(v) = v(|x|)\}$ with the inner product

$$(u, v)_H = \int_{B_R} e^{\varepsilon|x|^2/4} \nabla u \nabla v \, dx$$

and its corresponding norm

$$\|v\|_H^2 = \left(\int_{B_R} e^{\varepsilon|x|^2/4} \nabla v \nabla v \, dx \right).$$

We see that

$$\|\nabla v\|_{L^2(B_R)} \le \|v\|_H.$$

Now, let us set

$$\mathcal{J}(v) = \frac{1}{2}\|v\|_H^2 - \int_{B_R} \left(\sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}(e^{\chi_l v} - 1) - \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h}(e^{-v_h v} - 1) \right) e^{(\varepsilon-1)|x|^2/4} \, dx. \quad (2.6)$$

Remark 2. *We observe that $\mathscr{J}$ satisfies the assumptions from Theorem 2 since H can be written as $H = \{v \in W_0^{1,2}(B_R) \mid v(gx) = v(x) \text{ for all } g \in O(2)\}$, where $O(2)$ denotes the orthogonal transformation group and obviously $H \subset W_0^{1,2}(B_R)$.*

Proof (Proof of Theorem 3). We will show the proof for $\varepsilon \geq 1$. The case $0 < \varepsilon < 1$ is done in the analogous way. Therefore, we will phrase the corresponding statements in some explicit remarks within the proof. To establish our result we have to proceed several steps for showing that $\mathscr{J}$ satisfies the conditions of Theorem 1.

To show this claim we choose $\rho > 0$ arbitrarily but fixed and set

$$\mathscr{U} := \{v \in H \mid \|v\|_H < \rho\}.$$

A useful tool will be the so-called Moser-Trudinger inequality and one of its consequences:

Theorem 4 (Moser-Trudinger inequality [14]). *Let Ω be a domain in $\mathbf{R}^2$ such that*

$$|\Omega| = \int_\Omega dx < \infty.$$

Let $u \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dx \leq 1.$$

Then, if $\alpha \leq 8\pi$, there exists a positive constant c such that

$$\int_\Omega e^{\alpha|u|^2}\, dx \leq c|\Omega|.$$

Corollary 1 (See for example [16]). *Let Ω be a domain in $\mathbf{R}^2$ such that*

$$|\Omega| = \int_\Omega dx < \infty.$$

Let $u \in W_0^{1,2}(\Omega)$. Then there exists a positive constant c such that

$$\int_\Omega e^{|u|}\, dx \leq c|\Omega| \exp\left(\frac{1}{16\pi} \|\nabla u\|^2_{L^2(\Omega)}\right).$$

Remark 3. *New exponential Sobolev inequalities like the statement of the previous Corollary for different classes of function are established in* [12, Theorem 2.3 and equations (2.10) and (2.11)].

Since

$$\|\nabla v\|_{L^2(B_R)} \le \|v\|_H$$

we see by an application of the Moser-Trudinger inequality that for $v \in \partial\mathscr{U}$ (i.e. $\|v\|_H = \rho$):

$$\begin{aligned}
\mathscr{J}(v) &\geq \frac{1}{2}\|v\|_H^2 - \int_{B_R}\left(\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}(e^{\chi_l|v|}-1) + \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\right)e^{(\varepsilon-1)|x|^2/4}\,dx \\
&\geq \frac{\rho^2}{2} - \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}e^{(\varepsilon-1)R^2/4}\int_{B_R}(e^{\chi_l|v|}-1)dx - \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}e^{(\varepsilon-1)R^2/4}\pi R^2 \\
&\geq \frac{\rho^2}{2} + \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}e^{(\varepsilon-1)R^2/4}\pi R^2(1 - c_{B_R}e^{\chi_l^2\rho^2/16\pi}) - \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}e^{(\varepsilon-1)R^2/4}\pi R^2 \\
&\geq \frac{\rho^2}{2} + \left(\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}(1 - c_{B_R}e^{\chi_l^2\rho^2/16\pi}) - \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\right)e^{(\varepsilon-1)R^2/4}\pi R^2.
\end{aligned}$$

REMARK 4. *For $0 < \varepsilon < 1$ we get:*

$$\begin{aligned}
\mathscr{J}(v) &\geq \frac{1}{2}\|v\|_H^2 - \int_{B_R}\left(\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}(e^{\chi_l|v|}-1) + \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\right)e^{(\varepsilon-1)|x|^2/4}\,dx \\
&\geq \frac{\rho^2}{2} + \left(\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}(1 - c_{B_R}e^{\chi_l^2\rho^2/16\pi}) - \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\right)\pi R^2.
\end{aligned}$$

Choosing R sufficiently small we can always guarantee that

$$\mathscr{J}(v) > \frac{\rho^2}{4}.$$

To apply the Mountain Pass Theorem we have also to check its second assumption. As in [13] we, therefore, look at the (on B_R) positive function:

$$v^* = b - \frac{b}{R}|x|,$$

where $b > 0$ is fixed and will be determined later. Obviously this function belongs to H and we easily see that:

$$\|v^*\|_H = \frac{2b}{R}\sqrt{\pi(e^{\varepsilon R^2/4}-1)/\varepsilon}$$

and

$$
\begin{aligned}
\mathscr{J}(v^*) &= \frac{1}{2}\|v^*\|_H^2 - \int_{B_R} \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}(e^{\chi_l v^*} - 1)e^{(\varepsilon-1)|x|^2/4}\, dx \\
&\quad + \int_{B_R} \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h}(e^{-v_h v^*} - 1)e^{(\varepsilon-1)|x|^2/4}\, dx \\
&= \frac{1}{2}\left(\frac{4b^2\pi}{\varepsilon R^2}e^{\varepsilon R^2/4} - \frac{4b^2\pi}{\varepsilon R^2}\right) - \int_{B_R} \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}(e^{\chi_l v^*} - 1)e^{(\varepsilon-1)|x|^2/4}\, dx \\
&\quad + \int_{B_R} \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h}(e^{-v_h v^*} - 1)e^{(\varepsilon-1)|x|^2/4}\, dx \\
&\leq \frac{1}{2}e^{\varepsilon R^2/4}\frac{b^2}{R^2}\pi R^2 + \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}\left(\pi R^2 - e^{\chi_l b}\int_{B_R} e^{-\chi_l b|x|/R}\, dx\right) \\
&\quad + \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h} e^{-v_h b} e^{(\varepsilon-1)R^2/4} \int_{B_R} e^{v_h b|x|/R}\, dx \\
&= \frac{e^{\varepsilon R^2/4} b^2 \pi}{2} + \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}\left(\pi R^2 - 2\pi e^{\chi_l b}\int_0^R r e^{-\chi_l br/R}\, dr\right) \\
&\quad + \sum_{h=1}^{z_2} 2\pi \frac{\lambda_h}{v_h} e^{-v_h b} e^{(\varepsilon-1)R^2/4} \int_0^R r e^{v_h br/R}\, dr.
\end{aligned}
$$

Thus, we see that:

$$
\begin{aligned}
\mathscr{J}(v^*) \leq \frac{e^{\varepsilon R^2/4}b^2\pi}{2} &+ \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}\left(\pi R^2 - \frac{2\pi R^2 e^{\chi_l b}}{\chi_l^2 b^2} + \frac{2\pi R^2}{\chi_l b} + \frac{2\pi R^2}{\chi_l^2 b^2}\right) \\
&+ \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h}\left(\frac{2\pi R^2 e^{-v_h b}}{v_h^2 b^2} + \frac{2\pi R^2}{v_h b} - \frac{2\pi R^2}{v_h^2 b^2}\right)e^{(\varepsilon-1)R^2/4}.
\end{aligned}
$$

Remark 5. *For $0 < \varepsilon < 1$ one gets*

$$
\begin{aligned}
\mathscr{J}(v^*) \leq \frac{e^{\varepsilon R^2/4}b^2\pi}{2} &+ \sum_{l=1}^{z_1} \frac{\mu_l}{\chi_l}\left(\pi R^2 - \frac{2\pi R^2 e^{\chi_l b}}{\chi_l^2 b^2} + \frac{2\pi R^2}{\chi_l b} + \frac{2\pi R^2}{\chi_l^2 b^2}\right) \\
&+ \sum_{h=1}^{z_2} \frac{\lambda_h}{v_h}\left(\frac{2\pi R^2 e^{-v_h b}}{v_h^2 b^2} + \frac{2\pi R^2}{v_h b} - \frac{2\pi R^2}{v_h^2 b^2}\right).
\end{aligned}
$$

Now, choosing b sufficiently large we can guarantee that the inequalities

$$
\mathscr{J}(v^*) < 0 \qquad \text{and} \qquad \|v^*\|_H > \rho
$$

are fulfilled. (Remember that there is at least one $\mu_l \neq 0$ according to Hypothesis 1.1(1).)

Finally we have to check whether $\mathscr{J}$ satisfies the Palais-Smale condition or not. To show this property of $\mathscr{J}$ let $(v_n)_{n\in\mathbf{N}} \subset H$ satisfy

- $\mathscr{J}(v_n)$ is bounded.
- $\mathscr{J}'(v_n) \to 0$ in H' as $n \to \infty$.

It is left to show that $(v_n)_{n\in\mathbf{N}}$ has a strongly convergent subsequence. To do so, we first prove that the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded in $W_0^{1,2}(B_R)$. Straightforward calculation give us that

$$\mathscr{J}'(v)\phi = \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v \cdot \nabla\phi \, dx - \int_{B_R}\left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l v}\phi + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h v}\phi\right)e^{(\varepsilon-1)|x|^2/4}\, dx$$

for all $\phi \in H$ and, therefore,

$$\mathscr{J}'(v)\frac{ve^{bv}}{1+e^{bv}} = \int_{B_R} e^{\varepsilon|x|^2/4}\frac{e^{2bv}+e^{bv}+bve^{bv}}{(1+e^{bv})^2}|\nabla v|^2 dx$$
$$- \int_{B_R}\left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l v}\frac{ve^{bv}}{1+e^{bv}} + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h v}\frac{ve^{bv}}{1+e^{bv}}\right)e^{(\varepsilon-1)|x|^2/4}\, dx$$

and

$$\mathscr{J}'(v)\frac{ve^{bv}}{1+e^{bv}} \le \frac{3}{2}\|v\|_H^2 - \int_{B_R}\sum_{l=1}^{z_1}\mu_l e^{\chi_l v}\frac{ve^{bv}}{1+e^{bv}}e^{(\varepsilon-1)|x|^2/4}\, dx$$
$$- \int_{B_R}\sum_{h=1}^{z_2}\lambda_h e^{-\nu_h v}\frac{ve^{bv}}{1+e^{bv}}e^{(\varepsilon-1)|x|^2/4}\, dx$$

for $\frac{ve^{bv}}{1+e^{bv}} \in H$ with some $b > 0$, since

$$\left|\frac{e^{2bv}+e^{bv}+bve^{bv}}{(1+e^{bv})^2}\right| \le \frac{3}{2} \qquad \text{for all } v \in \mathbf{R}.$$

Remark 6. *If all $\lambda_h = 0$, then we can proceed exactly as in* [13] *and one changes the proof by looking at $\mathscr{J}'(v)\cdot v$ instead of $\mathscr{J}'(v)\frac{ve^{bv}}{1+e^{bv}}$.*

Additionally we see that for $0 < a < b$ and for all $v \in \mathbf{R}$ the following lower estimates hold true:

$$\frac{a}{4}\frac{ve^{(b-a)v}}{1+e^{bv}} + e^{-av} - 1 \ge -1,$$

and

$$\frac{a}{4}\frac{ve^{(a+b)v}}{1+e^{bv}} - e^{av} + 1 = \frac{a}{4}ve^{av} - e^{av} + 1 - \frac{1}{4}\frac{ave^{av}}{1+e^{bv}}$$
$$\geq -\left(\frac{e^3}{4} - 1\right) - K_{a,b},$$

where $K_{a,b}$ is a positive constant depending on a and b.

Thus, one can conclude that for $M := \left\{\max_h\ v_h,\ \max_l\ \chi_l\right\}$

$$\mathscr{I}(v) - \frac{1}{4}\mathscr{I}'(v)\frac{ve^{(M+1)v}}{1+e^{(M+1)v}}$$
$$\geq \frac{1}{8}\|v\|_H^2 + \int_{B_R}\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{1}{4}\frac{\chi_l ve^{(M+1)v}}{1+e^{(M+1)v}}e^{\chi_l v} - e^{\chi_l v} + 1\right)e^{(\varepsilon-1)|x|^2/4}\,dx$$
$$+ \int_{B_R}\sum_{h=1}^{z_2}\frac{\lambda_h}{v_h}\left(\frac{1}{4}\frac{v_h ve^{(M+1)v}}{1+e^{(M+1)v}}e^{-v_h v} + e^{-v_h v} - 1\right)e^{(\varepsilon-1)|x|^2/4}\,dx$$
$$\geq \frac{1}{8}\|v\|_H^2 - \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l,M}\right)\int_{B_R} e^{(\varepsilon-1)|x|^2/4}\,dx$$
$$- \int_{B_R}\sum_{h=1}^{z_2}\frac{\lambda_h}{v_h}e^{(\varepsilon-1)|x|^2/4}\,dx.$$

$$\mathscr{I}(v) - \frac{1}{4}\mathscr{I}'(v)\frac{ve^{(M+1)v}}{1+e^{(M+1)v}} \geq \frac{1}{8}\|v\|_H^2 - \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l,M}\right)\int_{B_R} e^{(\varepsilon-1)|x|^2/4}\,dx$$
$$- \sum_{h=1}^{z_2}\frac{\lambda_h}{v_h}\int_{B_R} e^{(\varepsilon-1)|x|^2/4}\,dx$$
$$\geq \frac{1}{8}\|v\|_H^2 - \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l,M}\right)\pi R^2 e^{(\varepsilon-1)R^2/4}$$
$$- \sum_{h=1}^{z_2}\frac{\lambda_h}{v_h}\pi R^2 e^{(\varepsilon-1)R^2/4}.$$

Remark 7. *For $0 < \varepsilon < 1$ one gets*

$$\mathscr{I}(v) - \frac{1}{4}\mathscr{I}'(v)\frac{ve^{(M+1)v}}{1+e^{(M+1)v}} \geq \frac{1}{8}\|v\|_H^2 - \sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l,M}\right)\pi R^2$$
$$- \sum_{h=1}^{z_2}\frac{\lambda_h}{v_h}\pi R^2.$$

As a consequence we have:

$$\begin{aligned}\|v_n\|_H^2 &\le 8|\mathcal{J}(v_n)| + 2\|\mathcal{J}'(v_n)\|_{H'} \cdot \left\|\frac{v_n e^{(M+1)v_n}}{1+e^{(M+1)v_n}}\right\|_H \\ &\quad + 8\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l, M}\right)\pi R^2 e^{(\varepsilon-1)R^2/4} \\ &\quad + 8\sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\pi R^2 e^{(\varepsilon-1)R^2/4} \\ &\le 8|\mathcal{J}(v_n)| + \frac{9}{2}\|\mathcal{J}'(v_n)\|_{H'}^2 + \frac{2}{9}\left\|\frac{v_n e^{(M+1)v_n}}{1+e^{(M+1)v_n}}\right\|_H^2 \\ &\quad + 8\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l, M}\right)\pi R^2 e^{(\varepsilon-1)R^2/4} \\ &\quad + 8\sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\pi R^2 e^{(\varepsilon-1)R^2/4}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|v_n\|_H^2 &\le 8|\mathcal{J}(v_n)| + \frac{9}{2}\|\mathcal{J}'(v_n)\|_{H'}^2 + \frac{1}{2}\|v_n\|_H^2 \\ &\quad + 8\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l, M}\right)\pi R^2 e^{(\varepsilon-1)R^2/4} \\ &\quad + 8\sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\pi R^2 e^{(\varepsilon-1)R^2/4},\end{aligned}$$

resp.

$$\begin{aligned}\frac{1}{2}\|v_n\|_H^2 &\le 8|\mathcal{J}(v_n)| + \frac{9}{2}\|\mathcal{J}'(v_n)\|_{H'}^2 \\ &\quad + 8\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l, M}\right)\pi R^2 e^{(\varepsilon-1)R^2/4} \\ &\quad + 8\sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\pi R^2 e^{(\varepsilon-1)R^2/4}.\end{aligned}$$

REMARK 8. *Consequently, for* $0 < \varepsilon < 1$ *we have:*

$$\frac{1}{2}\|v_n\|_H^2 \le 8|\mathscr{J}(v_n)| + \frac{9}{2}\|\mathscr{J}'(v_n)\|_{H'}^2 + 8\pi R^2\left(\sum_{l=1}^{z_1}\frac{\mu_l}{\chi_l}\left(\frac{e^3}{4} - 1 + K_{\chi_l, M}\right) + \sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}\right).$$

Now, since we assumed that $\mathscr{J}(v_n)$ is bounded and $\mathscr{J}'(v_n) \to 0$ in H' as $n \to \infty$ we can conclude that the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded in $W_0^{1,2}(B_R)$. According to the wellknown Sobolev's embedding theorems (see for example [1, Theorem 4.12, page 85]) we know that $(v_n)_{n\in\mathbf{N}}$ has a subsequence $(v_{n_k})_{n_k\in\mathbf{N}}$ that converges weakly in $W_0^{1,2}(B_R)$ and strongly in $L^2(B_R)$. For this subsequence we now take a closer look at $\|v_{n_k} - v_{n_i}\|_H$.

We see that:

$$\begin{aligned}\|v_{n_k} - v_{n_i}\|_H^2 &= \int_{B_R} e^{\varepsilon|x|^2/4}|\nabla v_{n_k} - \nabla v_{n_i}|^2 dx \\ &= \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v_{n_k}(\nabla v_{n_k} - \nabla v_{n_i})dx \\ &\quad - \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v_{n_i}(\nabla v_{n_k} - \nabla v_{n_i})dx.\end{aligned}$$

$$\begin{aligned}\|v_{n_k} - v_{n_i}\|_H^2 &= \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v_{n_k}(\nabla v_{n_k} - \nabla v_{n_i})dx \\ &\quad - \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v_{n_i}(\nabla v_{n_k} - \nabla v_{n_i})dx \\ &\quad - \int_{B_R}\sum_{l=1}^{z_1}\mu_l e^{\chi_l v_{n_k}}(v_{n_k} - v_{n_i})e^{(\varepsilon-1)|x|^2/4}\, dx \\ &\quad - \int_{B_R}\sum_{h=1}^{z_2}\lambda_h e^{-\nu_h v_{n_k}}(v_{n_k} - v_{n_i})e^{(\varepsilon-1)|x|^2/4}\, dx \\ &\quad + \int_{B_R}\left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l v_{n_i}}(v_{n_k} - v_{n_i})\right)e^{(\varepsilon-1)|x|^2/4}\, dx \\ &\quad + \int_{B_R}\left(\sum_{h=1}^{z_2}\lambda_h e^{-\nu_h v_{n_i}}(v_{n_k} - v_{n_i})\right)e^{(\varepsilon-1)|x|^2/4}\, dx \\ &\quad + \int_{B_R}\left(\sum_{l=1}^{z_1}\mu_l(e^{\chi_l v_{n_k}} - e^{\chi_l v_{n_i}})(v_{n_k} - v_{n_i})\right)e^{(\varepsilon-1)|x|^2/4}\, dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_R}\left(\sum_{h=1}^{z_2}\lambda_h(e^{-\nu_h v_{n_k}}-e^{-\nu_h v_{n_i}})(v_{n_k}-v_{n_i})\right)e^{(\varepsilon-1)|x|^2/4}\,dx\\
\leq{}& |\mathscr{J}'(v_{n_k})(v_{n_k}-v_{n_i})|+|\mathscr{J}'(v_{n_i})(v_{n_k}-v_{n_i})|\\
&+\int_{B_R}\left(\sum_{l=1}^{z_1}\mu_l(e^{\chi_l|v_{n_k}|}+e^{\chi_l|v_{n_i}|})|v_{n_k}-v_{n_i}|\right)e^{(\varepsilon-1)|x|^2/4}\,dx\\
&+\int_{B_R}\left(\sum_{h=1}^{z_2}\lambda_h(e^{\nu_h|v_{n_k}|}+e^{\nu_h|v_{n_i}|})|v_{n_k}-v_{n_i}|\right)e^{(\varepsilon-1)|x|^2/4}\,dx.
\end{aligned}
$$

Let us now recall that (from the assumptions made) we know that for any ϵ there exists $n(\epsilon)$ such that

$$|\mathscr{J}'(v_n)\phi|\leq\epsilon\|\phi\|_H\leq\frac{1}{4}\|\phi\|_H^2+\epsilon^2\qquad\text{for all }\ \phi\in H,$$

if $n\geq n(\epsilon)$. Thus one can show by some easy calculations and an application of Hölder's inequality and Moser-Trudinger's inequality that there exists a positive constant c such that

$$\|v_{n_k}-v_{n_i}\|_H^2\leq 4\epsilon^2+c\|v_{n_k}-v_{n_i}\|_{L^2(B_R)}.$$

Since $(v_{n_k})_{n_k\in\mathbf{N}}$ is strongly convergent in $L^2(B_R)$ we conclude that

$$\lim_{k,i\to\infty}\|v_{n_k}-v_{n_i}\|_H^2=0,$$

i.e. $(v_{n_k})_{n_k\in\mathbf{N}}$ is a strongly convergent sequence in H.

Summing up our results shown so far, we have proven that $\mathscr{J}$ satisfies the needed Palais-Smale condition.

Since $\mathscr{J}$ satisfies all assumptions needed, we now conclude from Theorem 1 that there exists a solution of our Dirichlet problem. As a critical point of our functional $\mathscr{J}$ this solution solves our problem (2.1) in the weak sense. However, it is not only a weak solution of (2.1) but in fact a classical one. Using standard arguments from elliptic regularity theory one can easily derive higher regularity results for the mountain pass solution of (2.1). Its positivity follows from the strong maximum principle for elliptic equation (see for instance [7, Theorem 4, page 333]. Since the critical point belongs (by construction) to the Hilbert space H we see immediately that it is a radially symmetric function. □

3. Some a priori estimates on the solutions

Before we prove the existence of a solution to

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \sum_{l=1}^{z_1} \mu_l e^{-r^2/4} e^{\chi_l \psi} + \sum_{h=1}^{z_2} \lambda_h e^{-r^2/4} e^{-v_h \psi} = 0 \tag{3.1}$$

$$\psi'(0) = 0 \tag{3.2}$$

$$\int_0^\infty r\psi(r)dr < \infty, \tag{3.3}$$

we will show some of its properties. Exactly as in [13] we set

$$I(\varepsilon) = \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds.$$

From [13] we know the following Lemma:

Lemma 1 (compare Lemma 1 in [13]). *For $\varepsilon > 0$ the expression $I(\varepsilon)$ is represented as*

$$I(\varepsilon) = \begin{cases} \frac{\log(\varepsilon)}{(\varepsilon-1)} & \text{if } \varepsilon \neq 1 \\ 1 & \text{if } \varepsilon = 1. \end{cases}$$

For a proof of this lemma see [13, Proof of Lemma 1, page 147]. In the following we will denote a solution ψ of (3.1)–(3.3) with $\psi(0) = a$ by $\psi(r; a)$. The next lemma (that corresponds to Lemma 2 in [13]) shows that the solution of (3.1)–(3.3) is monotone decreasing and bounded from below.

Lemma 2 (Monotonicity and boundedness). *Let $\mu_l(\varepsilon) = \mu_l I(\varepsilon)$, $\lambda_h(\varepsilon) = \lambda_h I(\varepsilon)$, $M_{\mu_l(\varepsilon)} := \max_l \mu_l(\varepsilon)$, $M_{\lambda_h(\varepsilon)} := \max_h \lambda_h(\varepsilon)$, $M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} := \max\{M_{\mu_l(\varepsilon)}, M_{\lambda_h(\varepsilon)}\}$, $\chi := \max_l \chi_l$ and $v := \max_h v_h$. Then, for $r > 0$, the following estimates hold:*

1. $\psi'(r; a) < 0$.
2. $\psi'(r; a) > -\sum_{l=1}^{z_1} \frac{\mu_l r e^{\chi_l a}}{2} - \sum_{h=1}^{z_2} \frac{\lambda_h r e^{-v_h \psi(r;a)}}{2}$.
3. *If there is at least one $\lambda_h \neq 0$ then,*

$$\psi(r, a) > \frac{1}{v} \ln(e^{-vNM_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a}} + e^{va - vNM_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a}} - 1),$$

and if all $\lambda_h = 0$, then $\psi(r, a) > a - NM_{\mu_l(\varepsilon)} e^{\chi a}$.

Proof. 1. Since ψ solves (3.1) we see that,

$$(p(r)\psi')' + \sum_{l=1}^{z_1} \mu_l p(r) e^{-r^2/4} e^{\chi_l \psi} + \sum_{h=1}^{z_2} \lambda_h p(r) e^{-r^2/4} e^{-v_h \psi} = 0 \tag{3.4}$$

where $p(r) = re^{\varepsilon r^2/4} > 0$. Integrating this equation leads to:

$$p(r)\psi' = -\sum_{l=1}^{z_1} \mu_l \int_0^r se^{(\varepsilon-1)s^2/4} e^{\chi_l \psi}\, ds - \sum_{h=1}^{z_2} \lambda_h \int_0^r se^{(\varepsilon-1)s^2/4} e^{-v_h \psi}\, ds < 0. \quad (3.5)$$

2. From the first statement of the lemma we conclude that $\psi(0; a) > \psi(r; a)$. From (3.5) and the monotonicity statement we see that:

$$\begin{aligned}
p(r)\psi' &= -\sum_{l=1}^{z_1} \mu_l \int_0^r se^{(\varepsilon-1)s^2/4} e^{\chi_l \psi}\, ds - \sum_{h=1}^{z_2} \lambda_h \int_0^r se^{(\varepsilon-1)s^2/4} e^{-v_h \psi}\, ds \\
&> -\sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_0^r se^{(\varepsilon-1)s^2/4}\, ds - \sum_{h=1}^{z_2} \lambda_h e^{-v_h \psi(r;a)} \int_0^r se^{(\varepsilon-1)s^2/4}\, ds \\
&> -\sum_{l=1}^{z_1} \frac{\mu_l r^2 e^{\chi_l a} e^{\varepsilon r^2/4}}{2} - \sum_{h=1}^{z_2} \frac{\lambda_h r^2 e^{\varepsilon r^2/4} e^{-v_h \psi(r;a)}}{2}.
\end{aligned}$$

3. If there is at least one $\lambda_h \neq 0$, then we see that:

$$\begin{aligned}
\int_0^r \frac{\psi'(s;a) e^{v\psi(s;a)}}{1 + e^{v\psi(s;a)}}\, ds &= \frac{1}{v} \ln(1 + e^{v\psi(r;a)}) - \frac{1}{v} \ln(1 + e^{v\psi(0;a)}) \\
&= \frac{1}{v} \ln(1 + e^{v\psi(r;a)}) - \frac{1}{v} \ln(1 + e^{va})
\end{aligned}$$

and therefore

$$\begin{aligned}
\int_0^r \frac{\psi'(s;a) e^{v\psi(s;a)}}{1 + e^{v\psi(s;a)}}\, ds &= -\sum_{l=1}^{z_1} \mu_l \int_0^r \frac{1}{s} \frac{e^{-\varepsilon s^2/4} e^{v\psi(s;a)}}{1 + e^{v\psi(s;a)}} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{\chi_l \psi(\tau;a)}\, d\tau ds \\
&\quad - \sum_{h=1}^{z_2} \lambda_h \int_0^r \frac{1}{s} \frac{e^{-\varepsilon s^2/4} e^{v\psi(s;a)}}{1 + e^{v\psi(s;a)}} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{-v_h \psi(\tau;a)}\, d\tau ds \\
&> -\sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds \\
&\quad - \sum_{h=1}^{z_2} \lambda_h \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} \frac{e^{(v-v_h)\psi(s;a)}}{1 + e^{v\psi(s;a)}} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds \\
&> -\sum_{l=1}^{z_1} \mu_l(\varepsilon) e^{\chi_l a} - \sum_{h=1}^{z_2} \lambda_h(\varepsilon) \\
&> -N \cdot M_{\mu_l(\varepsilon), \lambda_h(\varepsilon)} e^{\chi a}.
\end{aligned}$$

Thus, we conclude that:

$$1 + e^{\nu\psi(r;a)} > (1 + e^{\nu a})e^{-\nu N M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a}}$$

and, therefore,

$$\psi(r;a) > \frac{1}{\nu} \ln(e^{-\nu N M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a}} + e^{\nu a - \nu N M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a}} - 1).$$

If all $\lambda_h = 0$, then we can easily derive that

$$\psi(r,a) > a - N M_{\mu_l(\varepsilon)} e^{\chi a}$$

from equation (3.5). □

The previous lemma implies that ψ is monotone decreasing and strictly bounded away from zero, i.e.

$$\lim_{r\to\infty} \psi(r;a) > 0,$$

if

$$a - N \cdot M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a} > 0.$$

Lemma 3 (Positivity). *We set* $\psi(\infty;a) = \lim_{r\to\infty} \psi(r;a)$ *and assume that*

$$0 < N \cdot M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} < \frac{1}{\chi}\frac{1}{e}.$$

Furthermore, we denote the intersection points of the line $y = N \cdot M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}$ *and the curve* $y = ae^{-\chi a}$, *by* $\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$ *and* $\Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$, *where*

$$\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} < \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}.$$

If

$$\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} < a < \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}},$$

then $\psi(\infty;a) > 0$ *holds.*

This lemma follows immediately from the third statement of Lemma 2.

Lemma 4 (Upper bounds for positive solutions). *Let* $\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} < a < \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$, $c_\varepsilon = \max\{1, 1/\varepsilon\}$ *and* $\kappa_\varepsilon = \min\{1, \varepsilon\}$, *then*

$$\psi(r;a) < \psi(\infty;a) + \left(\sum_{l=1}^{z_1} \mu_l e^{\chi_l a} + \sum_{h=1}^{z_2} \lambda_h\right) c_\varepsilon e^{-\kappa_\varepsilon r^2/4}.$$

(A condition for the positivity of the solution of problem (1.8)–(1.10) is given by Lemma 3 in [13] and an upper bound is presented by Lemma 4 in [13].)

Proof. Since $\psi(r;a) \geq 0$ we see that:

$$\begin{aligned}\psi(r;a) &= \psi(\infty;a) + \sum_{l=1}^{z_1} \mu_l \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{\chi_l \psi} \, d\tau ds \\ &\quad + \sum_{h=1}^{z_2} \lambda_h \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{-\nu_h \psi} \, d\tau ds \\ &< \psi(\infty;a) + \sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} \, d\tau ds \\ &\quad + \sum_{h=1}^{z_2} \lambda_h \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} \, d\tau ds.\end{aligned}$$

If $\varepsilon \geq 1$, then we derive:

$$\begin{aligned}\psi(r;a) &< \psi(\infty;a) + \sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} \, d\tau ds \\ &\quad + \sum_{h=1}^{z_2} \lambda_h \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} \, d\tau ds \\ &< \psi(\infty;a) + \sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} e^{(\varepsilon-1)s^2/4} \int_0^s \tau \, d\tau ds \\ &\quad + \sum_{h=1}^{z_2} \lambda_h \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} e^{(\varepsilon-1)s^2/4} \int_0^s \tau \, d\tau ds.\end{aligned}$$

And thus:

$$\begin{aligned}\psi(r;a) &< \psi(\infty;a) + \sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a}}{2} \int_r^\infty s e^{-\varepsilon s^2/4} e^{(\varepsilon-1)s^2/4} \, ds \\ &\quad + \sum_{h=1}^{z_2} \frac{\lambda_h}{2} \int_r^\infty s e^{-\varepsilon s^2/4} e^{(\varepsilon-1)s^2/4} \, ds \\ &= \psi(\infty;a) + \left(\sum_{l=1}^{z_1} \mu_l e^{\chi_l a} + \sum_{h=1}^{z_2} \lambda_h \right) e^{-r^2/4}.\end{aligned}$$

On the other hand, if $0 < \varepsilon < 1$, then we can show that:

$$\psi(r;a) < \psi(\infty;a) + \sum_{l=1}^{z_1} \mu_l e^{\chi_l a} \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau \, d\tau ds$$

$$+ \sum_{h=1}^{z_2} \lambda_h \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau \, d\tau ds$$

$$= \psi(\infty;a) + \sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a}}{2} \int_r^\infty s e^{-\varepsilon s^2/4} \, ds$$

$$+ \sum_{h=1}^{z_2} \frac{\lambda_h}{2} \int_r^\infty s e^{-\varepsilon s^2/4} \, ds$$

$$= \psi(\infty;a) + \left(\sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a}}{\varepsilon} + \sum_{h=1}^{z_2} \frac{\lambda_h}{\varepsilon} \right) e^{-\varepsilon r^2/4}. \qquad \square$$

The following lemma is similar to Lemma 5 in [13].

Lemma 5. *We set*

$$h(t) = t e^{(\varepsilon-1)t^2/4} \int_t^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \, ds, \qquad c_1 := \max\{a,b\} \qquad \text{and} \qquad c_2 := \min\{a,b\}.$$

Then

1. $\int_0^\infty h(r)dr = I(\varepsilon)$, *(compare* [13, Lemma 5 (i), page 150]*)*
2. $|\psi(r;a) - \psi(r;b)| \le |a-b| \cdot \exp\left(\left(\sum_{l=1}^{z_1} \mu_l e^{\chi_l c_1} + \sum_{h=1}^{z_2} \lambda_h e^{-\nu_h c_2} \right) \int_0^r h(\tau) d\tau \right)$,
3. $|\psi(\infty;a) - \psi(\infty;b)| \le |a-b| \cdot \exp\left(\sum_{l=1}^{z_1} \mu_l(\varepsilon) e^{\chi_l c_1} + \sum_{h=1}^{z_2} \lambda_h(\varepsilon) e^{-\nu_h c_2} \right)$.

Proof. 1. The proof is exactly the same as in [13, Proof of Lemma 5 (i), page 150 f.].

2. We see that:

$$\psi(r;a) = a - \sum_{l=1}^{z_1} \mu_l \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{\chi_l \psi(\tau;a)} \, d\tau ds$$

$$- \sum_{h=1}^{z_2} \lambda_h \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{-\nu_h \psi(\tau;a)} \, d\tau ds$$

$$= a - \sum_{l=1}^{z_1} \mu_l \int_0^r \tau e^{(\varepsilon-1)\tau^2/4} e^{\chi_l \psi(\tau;a)} \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4} \, ds d\tau$$

$$- \sum_{h=1}^{z_2} \lambda_h \int_0^r \tau e^{(\varepsilon-1)\tau^2/4} e^{-\nu_h \psi(\tau;a)} \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4} \, ds d\tau.$$

As a consequence we have:

$$
\begin{aligned}
&|\psi(r;a)-\psi(r;b)| \\
&\quad \le |a-b| + \sum_{l=1}^{z_1}\mu_l \int_0^r \tau e^{(\varepsilon-1)\tau^2/4}|e^{\chi_l\psi(\tau;a)} - e^{\chi_l\psi(\tau;b)}| \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4}\, ds d\tau \\
&\qquad + \sum_{h=1}^{z_2}\lambda_h \int_0^r \tau e^{(\varepsilon-1)\tau^2/4}|e^{-\nu_h\psi(\tau;a)} - e^{-\nu_h\psi(\tau;b)}| \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4}\, ds d\tau.
\end{aligned}
$$

Since

$$|e^{\chi_l\psi(\tau;a)} - e^{\chi_l\psi(\tau;b)}| \le e^{\chi_l c_1}|\psi(\tau;a)-\psi(\tau;b)|$$

where $c_1 := \max\{a,b\}$ and

$$|e^{-\nu_h\psi(\tau;a)} - e^{-\nu_h\psi(\tau;b)}| \le e^{-\nu_h c_2}|\psi(\tau;a)-\psi(\tau;b)|$$

where $c_2 := \min\{a,b\}$, we conclude that:

$$
\begin{aligned}
&|\psi(r;a)-\psi(r;b)| \\
&\quad \le |a-b| + \sum_{l=1}^{z_1}\mu_l e^{\chi_l c_1} \int_0^r \tau e^{(\varepsilon-1)\tau^2/4}|\psi(\tau;a)-\psi(\tau;b)| \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4}\, ds d\tau \\
&\qquad + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h c_2} \int_0^r \tau e^{(\varepsilon-1)\tau^2/4}|\psi(\tau;a)-\psi(\tau;b)| \int_\tau^r \frac{1}{s} e^{-\varepsilon s^2/4}\, ds d\tau \\
&\quad \le |a-b| + \sum_{l=1}^{z_1}\mu_l e^{\chi_l c_1} \int_0^r |\psi(\tau;a)-\psi(\tau;b)| h(\tau) d\tau \\
&\qquad + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h c_2} \int_0^r |\psi(\tau;a)-\psi(\tau;b)| h(\tau) d\tau \\
&\quad = |a-b| + \left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l c_1} + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h c_2}\right) \int_0^r |\psi(\tau;a)-\psi(\tau;b)| h(\tau) d\tau.
\end{aligned}
$$

Thus, we get from Gronwall's inequality the claim of statement 2. that:

$$|\psi(r;a)-\psi(r;b)| \le |a-b| \cdot \exp\left(\left(\sum_{l=1}^{z_1}\mu_l e^{\chi_l c_1} + \sum_{h=1}^{z_2}\lambda_h e^{-\nu_h c_2}\right) \int_0^r h(\tau) d\tau\right). \quad (3.6)$$

3. Sending $r \to \infty$ on both sides of inequality (3.6), we get the third claim of the lemma. $\square$

4. Existence and nonexistence results for positive solutions

Now we have all tools at hand to establish our existence and nonexistence results for problem (1.3)–(1.5), resp. (3.1)–(3.3). Let us denote by v_c the classical solution of the Dirichlet problem (2.1). Since v_c is radial we see that it satisfies the following conditions:

$$\left.\begin{aligned} (re^{\varepsilon r^2/4}v_c')' + \left(\sum_{l=1}^{z_1}\mu_l re^{\chi_l v_c} + \sum_{h=1}^{z_2}\lambda_h re^{-\nu_h v_c}\right)e^{(\varepsilon-1)r^2/4} = 0 \quad \text{in } B_R \\ v'(0) = 0 \text{ and } v_c(R) = 0. \end{aligned}\right\} \tag{4.1}$$

Lemma 6. *Let v_c be the classical (positive) solution of* (2.1). *For sufficiently small R this solution satisfies $v_c(0) > \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$, where $\Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$ is the largest intersection point of the line*

$$y = N \cdot M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} \qquad \text{and the curve } y = ae^{-\chi a}.$$

(This result corresponds to Lemma 6 in [13].)

Proof. Suppose that $0 < v_c(0) < \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$. Since $v_c(r) > 0$ for $r \le R$ we have

$$-\sum_{h=1}^{z_2}\frac{\lambda_h}{\nu_h}(e^{-\nu_h v_c} - 1) > 0$$

and, therefore, we see from equation (2.6) that

$$\begin{aligned} \mathscr{I}(v_c) &< \frac{1}{2}\int_{B_R} e^{\varepsilon|x|^2/4}|\nabla v_c|^2 dx \\ &\le \pi\int_0^R re^{\varepsilon r^2/4}(v_c')^2 dr. \end{aligned}$$

From the second statement of Lemma 2 we conclude that

$$\begin{aligned} \mathscr{I}(v_c) &< \pi\int_0^R re^{\varepsilon r^2/4}\left(\sum_{l=1}^{z_1}\frac{\mu_l re^{\chi_l v_c(0)}}{2} + \sum_{h=1}^{z_2}\frac{\lambda_h r}{2}\right)^2 dr \\ &\le \pi e^{\varepsilon R^2/4}\left(\sum_{l=1}^{z_1}\frac{\mu_l e^{\chi_l v_c(0)}}{2} + \sum_{h=1}^{z_2}\frac{\lambda_h}{2}\right)^2\int_0^R r^3\, dr. \end{aligned}$$

Since $v_c(0)$ may depend on R we conclude that

$$\mathscr{I}(v_c) \le \frac{1}{16}\pi R^4 e^{\varepsilon R^2/4}\left(\sum_{l=1}^{z_1}\frac{\mu_l e^{\chi_l \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}}}{2} + \sum_{h=1}^{z_2}\frac{\lambda_h}{2}\right)^2.$$

Choosing R sufficiently small one can always ensure that $\mathscr{J}(v_c) < \frac{\rho^2}{4}$. However this is a contradiction to what we have shown in our existence proof. Therefore, we see that $v_c(0) > \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$. □

As in [13, Lemma 7] for problem (1.8)–(1.10) we have the following statement about the properties of the set of possible initial data for ψ in our problem (1.3)–(4).

LEMMA 7. *We set*

$$\mathfrak{O}_+ := \{a \in (1/\chi, \infty) \mid \psi(\infty; a) > 0\} \quad \text{and} \quad \mathfrak{O}_- := \{a \in (1/\chi, \infty) \mid \psi(\infty; a) < 0\}.$$

then $\mathfrak{O}_+$ *and* $\mathfrak{O}_-$ *are open sets.*

This lemma follows directly from Lemma 5.3.

Now, in the following we can formulate our main theorem.

THEOREM 5. *Let*

$$0 < N \cdot M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} < \frac{1}{\chi} \cdot \frac{1}{e}.$$

1. *There exists an* $a^* > 1/\chi$ *such that problem* (3.1) *with* $\psi(0) = a^*$ *admits a positive solution satisfying* (3.3). *Furthermore* $\psi(0)$ *tends to infinity as*

$$M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} \to 0.$$

2. *There exists a positive* $a_* < 1/\chi$ *such that problem* (3.1) *with* $\psi(0) = a_*$ *admits a positive solution satisfying* (3.3). *Furthermore there exists a* μ^* *such that if*

$$\sum_{l=1}^{z_1} \mu_l(\varepsilon) > \mu^*$$

there are no positive solutions to our problem (3.1).

PROOF (Proof of Theorem 5). 1. At first we show the first statement of Theorem 5.

As we have seen $v_c(r)$ is a monotone decreasing function and $v_c(R) = 0$. Thus we conclude that $v_c(r) < 0$ for $r > R$. This implies that $v_c(0)$ belongs to $\mathfrak{O}_-$. However, if

$$\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} < a < \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}},$$

we know from Lemma 3, that $a \in \mathfrak{O}_+$. As a consequence we see that $\mathfrak{O}_+ \neq \emptyset$ and $\mathfrak{O}_- \neq \emptyset$. Choosing $a^* = \inf \mathfrak{O}_-$ it follows that

$$a^* \in [\Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}, \infty).$$

Lemma 7 implies that a^* is neither in $\mathfrak{O}_+$ nor in $\mathfrak{O}_-$ and therefore, $\psi(\infty; a^*) = 0$. Furthermore, we know from Lemma 4 that

$$\begin{aligned}
\int_0^\infty r\psi(r; a^*)dr &< \left(\sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a^*}}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) \int_0^\infty r c_\varepsilon e^{-\kappa_\varepsilon r^2/4}\, dr \\
&= \left(\sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a^*}}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) 2c_\varepsilon/\kappa_\varepsilon \\
&\le \left(\sum_{l=1}^{z_1} \frac{\mu_l}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) 2e^{\chi a^*} c_\varepsilon/\kappa_\varepsilon \\
&\le N M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a^*} c_\varepsilon/\kappa_\varepsilon.
\end{aligned}$$

Since

$$a^* \ge \Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}}$$

and

$$\Gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} \to \infty$$

as

$$M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} \to 0,$$

we have that

$$\psi(0) \to \infty \qquad \text{as } M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} \to 0.$$

So the first statement of our Theorem has been shown.

2. Now we prove the second statement of our Theorem.

Equation (3.5) gives us

$$\begin{aligned}
\psi'(r) = &-\sum_{l=1}^{z_1} \frac{\mu_l}{r} e^{\varepsilon r^2/4} \int_0^r s e^{(\varepsilon-1)s^2/4} e^{\chi_l \psi}\, ds \\
&-\sum_{h=1}^{z_2} \frac{\lambda_h}{r} e^{\varepsilon r^2/4} \int_0^r s e^{(\varepsilon-1)s^2/4} e^{-\nu_h \psi}\, ds.
\end{aligned}$$

From the fact that $\psi(r, a)$ is monotone decreasing we know that

$$\psi'(r) < -\sum_{l=1}^{z_1} \frac{\mu_l}{r} e^{\varepsilon r^2/4} e^{\chi_l \psi(r)} \int_0^r s e^{(\varepsilon-1)s^2/4}\, ds$$

$$-\sum_{h=1}^{z_2} \frac{\lambda_h}{r} e^{\varepsilon r^2/4} e^{-\nu_h \psi(0)} \int_0^r s e^{(\varepsilon-1)s^2/4}\, ds$$

$$< -\left(\sum_{l=1}^{z_1} \frac{\mu_l}{r} e^{\varepsilon r^2/4} \int_0^r s e^{(\varepsilon-1)s^2/4}\, ds\right) e^{\chi \psi(r)}.$$

This inequality implies:

$$(-e^{-\chi\psi(r)})' < -\chi \sum_{l=1}^{z_1} \frac{\mu_l}{r} e^{\varepsilon r^2/4} \int_0^r s e^{(\varepsilon-1)s^2/4}\, ds.$$

Integrating this inequality from 0 to ∞ gives us:

$$-e^{-\chi\psi(\infty)} + e^{-\chi\psi(0)} < -\chi \sum_{l=1}^{z_1} \mu_l(\varepsilon)$$

and as a consequence

$$\psi(\infty) < -\frac{1}{\chi} \ln\left(\chi \sum_{l=1}^{z_1} \mu_l(\varepsilon) + e^{-\chi\psi(0)}\right).$$

As a consequence we see that

$$\psi(\infty) < 0, \qquad \text{if } \chi \sum_{l=1}^{z_1} \mu_l(\varepsilon) \geq 1.$$

Furthermore we know that:

$$\psi(\infty; a) = a - \sum_{l=1}^{z_1} \mu_l \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{\chi_l \psi}\, d\tau ds$$

$$-\sum_{h=1}^{z_2} \lambda_h \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{-\nu_h \psi}\, d\tau ds$$

$$< a - \sum_{l=1}^{z_1} \mu_l e^{\chi_l \psi(\infty; a)} \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds$$

$$= a - \sum_{l=1}^{z_1} \mu_l(\varepsilon) e^{\chi_l \psi(\infty; a)}.$$

Similar as in Lemma 7 we now set

$$\mathfrak{U}_{+} = \{a \in (0, 1/\chi) \mid \psi(\infty; a) > 0\}$$

and

$$\mathfrak{U}_{-} = \{a \in (0, 1/\chi) \mid \psi(\infty; a) < 0\}.$$

These set are open what can be shown exactly in the same way as the claim of Lemma 7.

Thus, if we have

$$\gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}} < a < 1/\chi,$$

then we see that $a \in \mathfrak{U}_{+}$. However, our calculations from above show that:

$$\psi(\infty; 0) < -\sum_{l=1}^{z_1} \mu_l(\varepsilon) e^{\chi_l \psi(\infty; 0)} < 0.$$

Therefore, we see that an $a \in (0, \gamma_{M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)}})$ exists such that $\psi(\infty; a) < 0$ and as a consequence $\mathfrak{U}_{+} \neq \emptyset$ and $\mathfrak{U}_{-} \neq \emptyset$. Now we set

$$a_* = \sup \mathfrak{U}_{-} < 1/\chi.$$

Exactly as in the arguments to establish the first claim of our theorem we see that $a_* \notin \mathfrak{U}_{+}$ and $a_* \notin \mathfrak{U}_{-}$. So $\psi(\infty; a_*) = 0$. The boundedness of the solution is once again guaranteed by Lemma 4 since we see that:

$$\begin{aligned}
\int_0^\infty r\psi(r; a_*)dr &< \left(\sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a_*}}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) \int_0^\infty r c_\varepsilon e^{-\kappa_\varepsilon r^2/4}\, dr \\
&= \left(\sum_{l=1}^{z_1} \frac{\mu_l e^{\chi_l a_*}}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) 2c_\varepsilon/\kappa_\varepsilon \\
&\le \left(\sum_{l=1}^{z_1} \frac{\mu_l}{2} + \sum_{h=1}^{z_2} \frac{\lambda_h}{2}\right) 2e^{\chi a_*} c_\varepsilon/\kappa_\varepsilon \\
&\le N M_{\mu_l(\varepsilon),\lambda_h(\varepsilon)} e^{\chi a_*} c_\varepsilon/\kappa_\varepsilon.
\end{aligned}$$

This gives us the second statement of the Theorem and completes its proof. □

5. Comments and some application of the existence and nonexistence results

It has already been mentioned in the introduction of the present paper that some multi-species chemotaxis models belong to such kind of systems studied here. As a concrete example one can look at the following chemotaxis model in the presence of a conflict of interest (see [20, Definition 1, page 646] and [10, Remark 1.1, page 233, and Definition 5.2, page 249] for a definition of a system describing chemotactic motion in the presence of a conflict of interests):

$$\left.\begin{aligned} u_t &= \nabla(\nabla u - \chi u \nabla v), && \text{in } \mathbf{R}^2 \times \{t > 0\} \\ w_t &= \nabla(\nabla w + \nu w \nabla v), && \text{in } \mathbf{R}^2 \times \{t > 0\} \\ \varepsilon v_t &= \Delta v + \alpha_1 u + \alpha_2 w, && \text{in } \mathbf{R}^2 \times \{t > 0\}. \end{aligned}\right\} \tag{5.1}$$

Looking for self-similar radial symmetric solution of (5.1) leads to

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \mu e^{-r^2/4} e^{\chi\psi} + \lambda e^{-r^2/4} e^{-\nu\psi} = 0 \tag{5.2}$$

$$\psi'(0) = 0 \tag{5.3}$$

$$\int_0^\infty r\psi(r)\,dr < \infty, \tag{5.4}$$

where

$$\mu = \alpha_1 \varphi(0) e^{-\chi\psi(0)} > 0 \qquad \text{and} \qquad \lambda = \alpha_2 \omega(0) e^{\nu\psi(0)} > 0.$$

Therefore, the results of the present paper lead to the following statements for this concrete example that can be viewed as an application of Theorem 5:

Theorem 6. *Let*

$$0 < \max\{\mu(\varepsilon), \lambda(\varepsilon)\} < \frac{1}{\chi} \cdot \frac{1}{e},$$

where

$$\mu(\varepsilon) := \mu \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds$$

and

$$\lambda(\varepsilon) := \lambda \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} \int_0^s \tau e^{(\varepsilon-1)\tau^2/4}\, d\tau ds.$$

1. *There exists an $a^* > 1/\chi$ such that problem* (5.2) *with $\psi(0) = a^*$ admits a positive solution satisfying* (5.4). *Furthermore $\psi(0)$ tends to infinity as*

$$\max\{\mu(\varepsilon), \lambda(\varepsilon)\} \to 0.$$

2. *There exists a positive* $a_* < 1/\chi$ *such that problem* (5.2) *with* $\psi(0) = a_*$ *admits a positive solution satisfying* (5.4). *Furthermore there exists a* μ^* *such that if*

$$\mu(\varepsilon) > \mu^*$$

there are no positive solutions to our problem (5.2).

Besides [13] and the present paper there are also other results on self-similar solutions for problem (1.6) resp. (1.8)–(1.10). To list some of these results let us refer for example to [3, 4, 5, 8, 9, 15] and [17].

Different from the ansatz followed in the present paper and in [13] Y. Naito, T. Suzuki and K. Yoshida looked for self-similar solutions of system (1.6) and showed the existence of such solutions in [17] by assuming that $u(x,t) = \varphi(x/\sqrt{t})/t$ and $v(x,t) = \psi(x/\sqrt{t})$. This leads them to the system:

$$\left.\begin{array}{ll} 0 = \nabla(\nabla\varphi - \varphi\nabla\psi) + \frac{x}{2}\nabla\varphi + \varphi, & x \in \mathbf{R}^2 \\ 0 = \Delta\psi + \frac{\varepsilon x}{2}\nabla\psi + \varphi, & x \in \mathbf{R}^2 \\ 0 \le \varphi, \psi \text{ in } \mathbf{R}^2 \text{ and } \varphi(x), \psi(x) \to 0 \text{ as } |x| \to \infty. & \end{array}\right\} \tag{5.5}$$

The existence of a solution for this system has been shown first in [15, Theorem 1.1 and Theorem 1.2, page 429] while it is shown in [17] that any classical solution of (5.5) has to be radially symmetric about the origin and satisfies $\varphi, \psi \in L^1(\mathbf{R}^2)$. In addition it was shown that the solution set of (5.5) can be expressed as a one-parameter family $\mathscr{S} = \{(\varphi(s), \psi(s)) : s \in \mathbf{R}\}$. If $\lambda(s) := \|\varphi(s)\|_{L^1(\mathbf{R}^2)}$, then the solution $(\varphi(s), \psi(s))$ and $\lambda(s)$ satisfy the following properties:

1. $s \mapsto (\varphi(s), \psi(s)) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ and $s \mapsto \lambda(s) \in \mathbf{R}$ are continuous.
2. $(\varphi(s), \psi(s)) \to (0,0)$ in $C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ and $\lambda(s) \to 0$ as $s \to -\infty$.
3. $\|\psi(s)\|_{L^\infty(\mathbf{R}^2)} \to \infty$, $\lambda(s) \to 8\pi$, and $\varphi(s)dx \rightharpoonup 8\pi\delta_0(dx)$ in the sense of measure as $s \to \infty$, where $\delta_0(dx)$ denotes Dirac's δ-function with support in the origin.
4. $0 < \lambda(s) < 8\pi$ for $s \in \mathbf{R}$, if $0 < \varepsilon \le 1/2$, and

$$0 < \lambda(s) \le \max\{4\pi^3/3, 4\pi^3\varepsilon^2/3\}$$

for $s \in \mathbf{R}$, if $\varepsilon > 1/2$.

These properties result in the existence of a critical value

$$8\pi \le \lambda^* \le \max\{4\pi^3/3, 4\pi^3\varepsilon^2/3\}$$

such that for $\lambda \in (0, \lambda^*)$ there exists a solution in $\mathscr{S}$ such that $\|\varphi\|_{L^1(\mathbf{R}^2)} = \lambda$ and for $\lambda > \lambda^*$, there exists no solution in $\mathscr{S}$ satisfying $\|\varphi\|_{L^1(\mathbf{R}^2)} = \lambda$.

The more recent paper [3] follows a different approach resp. another goal. It is concerned with self-similar blow-up solutions (as it has also been done in [4, 5, 8, 9]) in two spatial dimensions.

Biler et al. show in [3] that for the parabolic-parabolic case of (1.6) the threshold value deciding whether blow-up can take place or not is not as clear as in the parabolic-elliptic case, in which solutions with mass above the threshold value M_c always blow up. In [3] the author study forward self-similar solutions of the parabolic-parabolic system (1.6) and prove that (in some cases) such solutions globally exist even if their total mass is above the threshold value M_c of the parabolic-elliptic case.

Let us finally mention that it seems to be possible to establish results similar to those presented in the present paper without the simplifying assumption on ζ_k and Λ_k in Hypothesis 1.1. The results of the present paper seem to remain true if one assumes that $\zeta_k = C_{\text{const}} = \Lambda_k$ for all indices k with an arbitrary positive constant C_{const} (that is not necessarily equal to 1) and if one, therefore, replaces expressions like $(\varepsilon - 1)$ by $(\varepsilon - C_{\text{const}})$. Of course this will also lead to a necessary change in $I(\varepsilon)$ that will then be represented by

$$I(\varepsilon) = \begin{cases} \frac{\log(\varepsilon) - \log(C_{\text{const}})}{(\varepsilon - C_{\text{const}})} & \text{if } \varepsilon \neq C_{\text{const}} \\ \frac{1}{C_{\text{const}}} & \text{if } \varepsilon = C_{\text{const}} \end{cases}$$

instead of the given representation in Lemma 1. At some other parts of the present paper some additional technicalities will also be needed but these changes seem to be (only) marginal and not so complicated.

Acknowledgement

The author thanks the anonymous referees for reading the submitted manuscript carefully and for giving hints that allowed to shorten the proof of Lemma 6.

References

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, Pure and Applied Mathematcis, Volume 140, second edition, Academic Press, 2003.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), pp. 349–381.

[3] P. Biler, L. Corrias and J. Dolbeault, Large mass self-similar solutions of the parabolic-parabolic Keller-Segel model of chemotaxis, J. Math. Biol. **63** (2011), pp. 1–32.

[4] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel and S. C. Venkataramani, Attraction, Diffusion and Collapse, Nonlinearity **12** (1999), pp. 1071–1098.

[5] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel and S. C. Venkataramani, Blowup in the Chemotaxis equation, Technical Report University of Chicago.

[6] E. E. Espejo Arenas, A. Stevens, J. J. L. Velázquez, Simultaneous finite time blow-up in a two-species model for chemotaxis, Analysis **29** (2009), pp. 317–338.

[7] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Volume 19, AMS 1998.

[8] M. A. Herrero, E. Medina and J. J. L. Velázquez, Finite-time aggregation into a single point in a reaction diffusion system, Nonlinearity **10** (1997), pp. 1739–1754.

[9] M. A. Herrero, E. Medina and J. J. L. Velázquez, Self-similar blow-up for a reaction diffusion system, Journal of Computational and Applied Mathematics **97** (1998), pp. 99–119.

[10] D. Horstmann, Generalizing Keller-Segel: Lyapunov functionals, steady state analysis and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species, J. Non. Science **21** (2011), pp. 231–270.

[11] D. Horstmann and M. Lucia, Nonlocal elliptic boundary value problems related to chemotactic movement of mobile species, RIMS Kokuroku Bessatsu B15 (2009), pp. 39–72.

[12] D. Horstmann and M. Lucia, Uniqueness and symmetry of equilibria in a chemotaxis model, Journal für die reine und angewandte Mathematik (Crelles Journal) **654** (2011), pp. 83–124.

[13] Y. Mizutani, N. Muramoto and K. Yoshida, Self-similar radial solutions to a parabolic system modelling chemotaxis via variational method, Hiroshima Math. J. **29** (1999), pp. 145–160.

[14] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Math. J. **20** (1971), pp. 1077–1092.

[15] N. Muramoto, Y. Naito and K. Yoshida, Existence of self-similar solutions to a parabolic system modelling chemotaxis, Japan J. Indust. Appl. Math. **17** (2000), pp. 427–451.

[16] T. Nagai, T. Senba and K. Yoshida, Application of the Moser-Trudinger inequality to a parabolic system of chemotaxis, Funkc. Ekvacioj, Ser. Int. **40** (1997), pp. 411–433.

[17] Y. Naito, T. Suzuki and K. Yoshida, Self-similar solutions to a parabolic system modeling chemotaxis, J. Differential Equations **184** (2002), pp. 386–421.

[18] R. S. Palais, The principle of symmetric criticality, Comm. Math. Phys. **69** (1979), pp. 19–30.

[19] M. Struwe, Variational Methods, Second, revised and substantially expanded edition, Springer 1996.

[20] G. Wolansky, Multi-components chemotactic system in absence of conflicts, Euro. Jnl. of Applied Mathematics **13** (2002), pp. 641–661.

Dirk Horstmann
Mathematisches Institut
Universität zu Köln
Weyertal 86-90, *D*-50931 *Köln*
Germany
URL: http://www.mi.uni-koeln.de/~dhorst
E-mail: dhorst@math.uni-koeln.de