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The mapping class group of a punctured surface is generated by three elements

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Abstract. Let $\mathrm{Mod}(\Sigma_{g,p})$ be the mapping class group of a closed oriented surface $\Sigma_{g,p}$ of genus $g \geq 1$ with p punctures. Wajnryb proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by two elements. Korkmaz proved that one of these generators may be taken to be a Dehn twist. Korkmaz also proved the same result in the case of $\mathrm{Mod}(\Sigma_{g,1})$. For $p \geq 2$, we prove that $\mathrm{Mod}(\Sigma_{g,p})$ is generated by three elements.

1. Introduction

Let $\Sigma_{g,p}$ be a closed oriented surface of genus $g \geq 1$ with arbitrarily chosen p points (which we call punctures). Let $\mathrm{Mod}(\Sigma_{g,p})$ be the *mapping class group* of $\Sigma_{g,p}$, i.e., the group of homotopy classes of orientation-preserving homeomorphisms which preserve the set of punctures. Let $\mathrm{Mod}^{\pm}(\Sigma_{g,p})$ be the *extended mapping class group* of $\Sigma_{g,p}$, i.e., the group of homotopy class of all (including orientation-reversing) homeomorphisms which preserve the set of punctures. By $\mathrm{Mod}^0(\Sigma_{g,p})$ we will denote the subgroup of $\mathrm{Mod}(\Sigma_{g,p})$ which fixes the punctures pointwise. It is clear that we have the exact sequence:

$$1 \to \mathrm{Mod}^0(\Sigma_{g,p}) \to \mathrm{Mod}(\Sigma_{g,p}) \to \mathrm{Sym}_p \to 1,$$

where the last projection is given by the restriction of a homeomorphism to its action on the punctures.

The problem of finding a set of generators for the mapping class group of a closed surface was first considered by Dehn. He proved in [De] that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by a finite set of Dehn twists. Thirty years later, Lickorish [Li] showed that $3g-1$ Dehn twists generate $\mathrm{Mod}(\Sigma_{g,0})$. This number was improved to $2g+1$ by Humphries [Hu]. Humphries proved, moreover, that in fact the number $2g+1$ is minimal; i.e. $\mathrm{Mod}(\Sigma_{g,0})$ cannot be generated by $2g$ (or less) Dehn twists. Johnson [Jo] proved that the $2g+1$ Dehn twists also generate $\mathrm{Mod}(\Sigma_{g,1})$. In the case of multiple punctures the mapping class group can be generated by $2g+p$ twists for $p \geq 1$ (see [Ge]).

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It is possible to obtain smaller generating sets of $\mathrm{Mod}(\Sigma_{g,p})$ by using elements other than Dehn twists. N. Lu (see [Lu]) constructed a generating set of $\mathrm{Mod}(\Sigma_{g,0})$ consisting of 3 elements. This result was improved by Wajnryb who found the smallest possible generating set of $\mathrm{Mod}(\Sigma_{g,0})$ consisting of 2 elements (see [Wa]). Korkmaz proved in [Ko] that one of these generators may be taken to be a Dehn twist. Moreover, he proved that $\mathrm{Mod}^{\pm}(\Sigma_{g,1})$ can be generated by two elements.

In this paper we show the following two results.

(a) For $g \geq 1$, $p \geq 2$, $\mathrm{Mod}(\Sigma_{g,p})$ is generated by 3 elements one of which is a Dehn twist.

(b) For $g \geq 1$, $p \geq 2$, $\mathrm{Mod}^{\pm}(\Sigma_{g,p})$ is generated by 3 elements one of which is a Dehn twist.

2. Preliminaries

Let c be a simple closed curve on $\Sigma_{g,p}$. Then the (right handed) Dehn twist C about c is the homotopy class of the homeomorphism obtained by cutting $\Sigma_{g,p}$ along c, twisting one of the side by 360° to the right and gluing two sides of c back to each other. We denote curves on $\Sigma_{g,p}$ by letters a, b, c, d and corresponding Dehn twists about them by capital letters A, B, C, D.

A small regular neighborhood of an arc $s_{i,j}$ joining two punctures x_i and x_j of $\Sigma_{g,p}$ is denoted by $N(s_{i,j})$. The (right hand) half twist along $s_{i,j}$ is denoted by $H_{i,j}$. To be precise, $H_{i,j}$ is a self-homeomorphism of $\Sigma_{g,p}$, supported in $N(s_{i,j} \cup x_i \cup x_j)$, which leaves $s_{i,j}$ invariant and interchanges x_i, x_j, such that $H_{i,j}^2$ is the right handed Dehn twist along $\partial N(s_{i,j} \cup x_i \cup x_j)$.

We define the curves a_i, b, c_i and d_i on $\Sigma_{g,p}$ as shown in Figure 1.

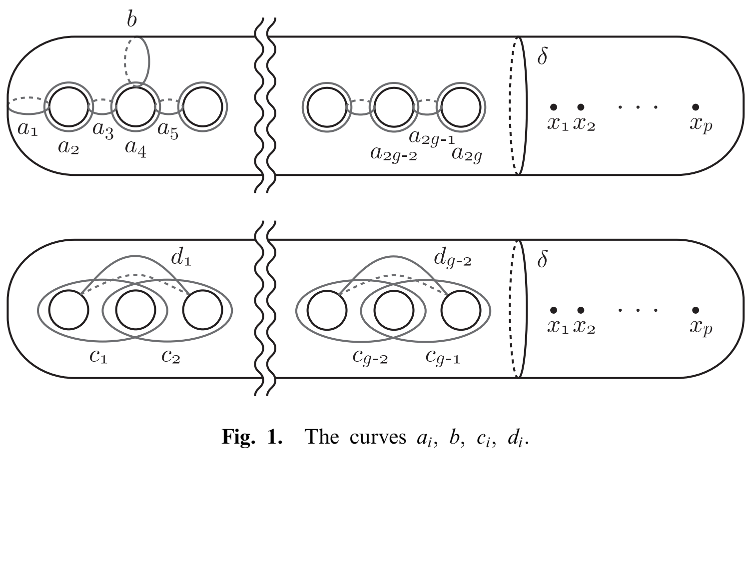

Fig. 1. The curves a_i, b, c_i, d_i.

If F and G are two homeomorphisms, then the composition FG means that G is applied first.

We recall the following basic facts.

LEMMA 1. *Let c be a simple closed curve on $\Sigma_{g,p}$, let F be a self-homeomorphism of $\Sigma_{g,p}$ and let $F(c) = d$. Then $FCF^{-1} = D^r$, where $r = \pm 1$ depending on whether F is orientation-preserving or orientation-reversing.*

LEMMA 2. *Let c and d be two simple closed curves on $\Sigma_{g,p}$. If c is disjoint from d, then $CD = DC$.*

Let S denote the product $A_{2g}A_{2g-1} \dots A_2A_1$ of $2g$ Dehn twists in $\mathrm{Mod}(\Sigma_{g,p})$ and let G be the subgroup of $\mathrm{Mod}(\Sigma_{g,p})$ generated by B and $SH_{1,p}$, where $H_{1,p}$ is the half twist about an arc $s_{1,p}$ joining x_1 and x_p which is disjoint from the punctures x_j $(j = 2, \dots, p-1)$ and the loop δ in Figure 1. The following lemmas are obtained by the arguments in Section 3 of [Ko].

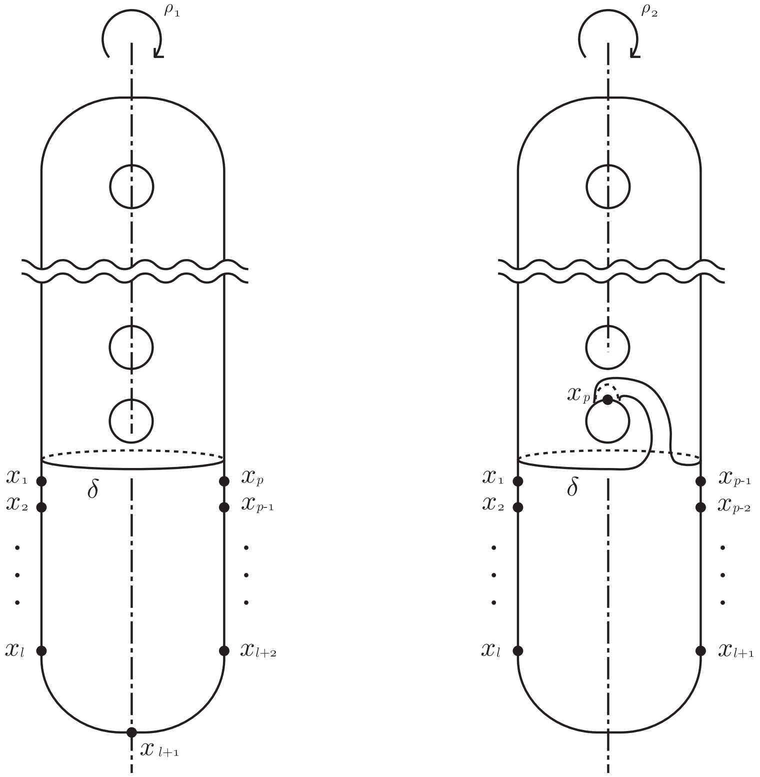

Fig. 2. Involutions ρ_1 and ρ_2, when p is odd.

LEMMA 3. $C_1, \dots, C_{g-1}, D_1, \dots, D_{g-2} \in G$.

LEMMA 4. $A_1, \dots, A_{2g} \in G$.

3. The mapping class group

In this section we prove that the mapping class group $\mathrm{Mod}(\Sigma_{g,p})$ is generated by three elements. Throughout this section, G' denotes the subgroup of $\mathrm{Mod}(\Sigma_{g,p})$ generated by B, $SH_{1,p}$ and a certain element, T, of $\mathrm{Mod}(\Sigma_{g,p})$.

Let us embed $\Sigma_{g,p}$ in Euclidean space in two different ways as shown in Figure 2 or Figure 3 according as the number p of the punctures is odd or even. Each embedding gives a natural involution of the surface—the half turn rotation around its axis of symmetry. Let us call these involutions ρ_1 and ρ_2,

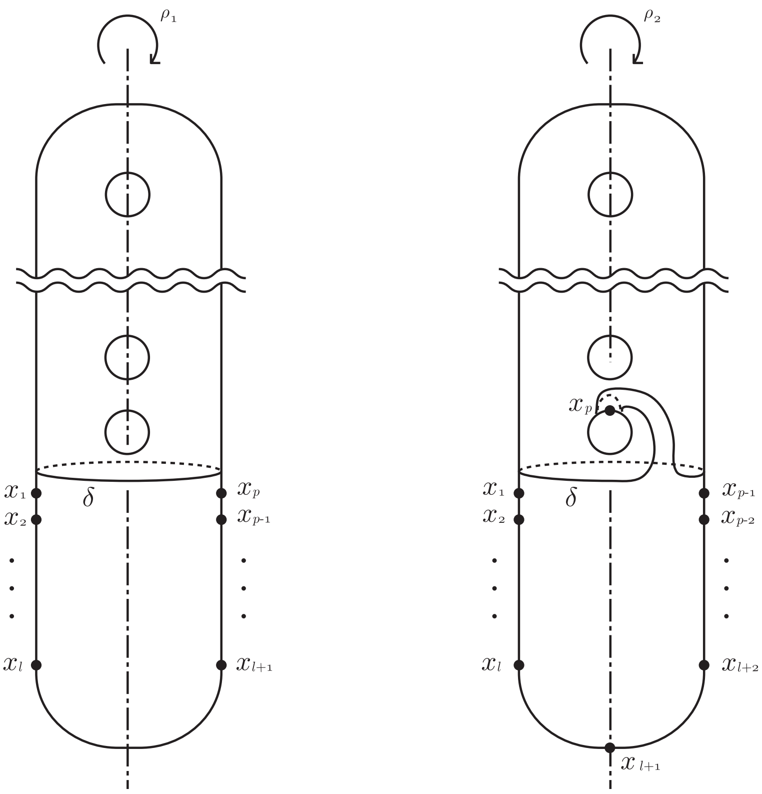

Fig. 3. Involutions ρ_1 and ρ_2, when p is even.

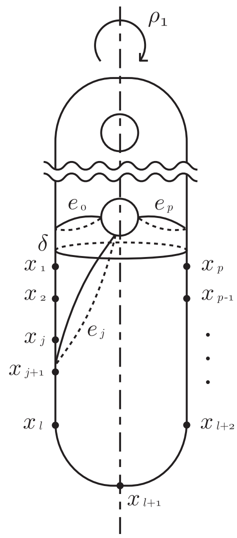

Fig. 4. The curves e_j.

and set $T = \rho_1\rho_2$. These involutions ρ_1 and ρ_2 are constructed by modifying the ones introduced by Kassabov [Ka].

On the set of punctures, T acts as a long cycle

$$T(x_p) = x_1 \qquad \text{and} \qquad T(x_j) = x_{j+1} \qquad \text{for } 1 \le j \le p-1.$$

Let G' be the subgroup of $\mathrm{Mod}(\Sigma_{g,p})$ generated by B, SH_{1p} and T. We prove that G' includes $\mathrm{Mod}^0(\Sigma_{g,p})$. In [Ge] it is shown that $\mathrm{Mod}^0(\Sigma_{g,p})$ is generated by the Dehn twists about the curves b, a_i $(i = 1, \dots, 2g)$, and e_j $(j = 1, \dots, p-1)$, where the curve e_j are as shown in Figure 4.

Lemma 5. *The homeomorphism T acts on the set of curves, $\{e_0, \dots, e_{p-1}\}$, as follows:*

$$T(e_j) = e_{j+1} \qquad (j = 0, \dots, p-1).$$

Proof. Figure 5 shows the ρ_2-orbit of e_j and the ρ_1-orbit of $\rho_2(e_j)$. It is clear from the picture that $e_{j+1} = \rho_1\rho_2(e_j) = T(e_j)$. □

The curve δ separates $\Sigma_{g,p}$ into two components: the first one, denoted by Σ, is a surface of genus g with one boundary component and no punctures. The second one, denoted by D, is a disk with p puncture points.

Lemma 6. $E_0, \dots, E_{p-1} \in G'$.

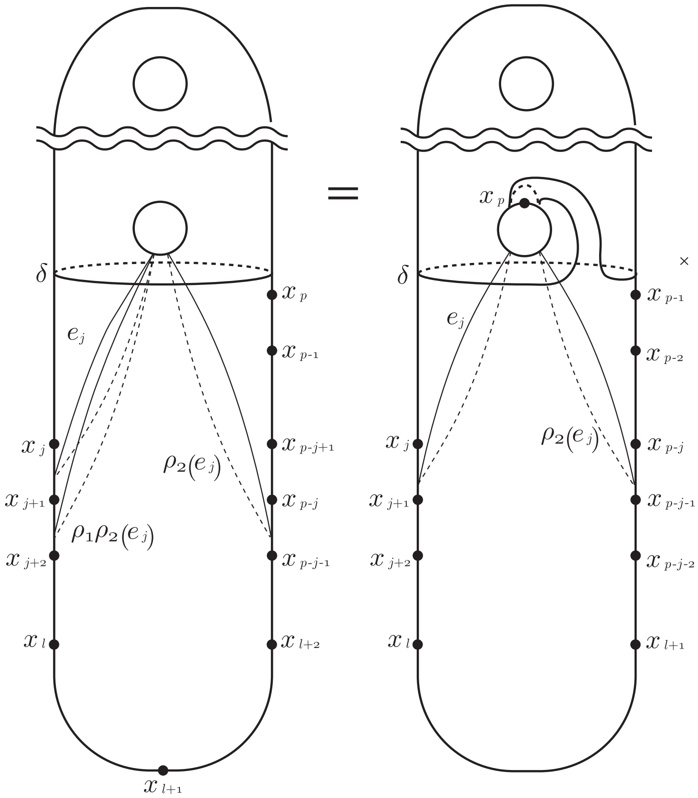

Fig. 5. The ρ_2-orbit of e_j and the ρ_1-orbit of $\rho_2(e_j)$.

PROOF. Let $\mathrm{Mod}(\Sigma)$ be the *mapping class group* of Σ, i.e., the group of homotopy classes of orientation-preserving homeomorphisms which restrict to the identity on the boundary. Let $\imath$ be the inclusion $\imath : \Sigma \to \Sigma_{g,p}$. If F is any homeomorphism of Σ *representing an element of* $\mathrm{Mod}(\Sigma)$, then we may extend it by the identity on D to get a well-defined homeomorphism of $\Sigma_{g,p}$. In this way we get an induced homomorphism

$$\imath_* : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(\Sigma_{g,p}).$$

In [Jo], Johnson proved that $\mathrm{Mod}(\Sigma)$ is generated by $B, A_1, \dots, A_{2g}$. Since $B, A_1, \dots, A_{2g}$ are in $G \subset G'$, G' contains $\imath_*(\mathrm{Mod}(\Sigma))$. Therefore E_0 is in $\imath_*(\mathrm{Mod}(\Sigma)) \subset G'$. Using Lemma 5 we can prove that all $E_j = T^j E_0 T^{-j}$ are in G'. □

COROLLARY 7. $\mathrm{Mod}^0(\Sigma_{g,p}) \subset G'$.

We now recall a simple but useful fact.

LEMMA 8. *Let G be a group which is an extension of a group Q by a group N, i.e., there is a short exact sequence,*

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1.$$

Then a subgroup H of G is equal to G if (and only if) H contains $i(N)$ and $\pi|_H$ is a surjection to Q.

We now prove the first main result of the paper:

THEOREM 9. *Suppose that $g \geq 1$ and $p \geq 2$. Then the mapping class group $\mathrm{Mod}(\Sigma_{g,p})$ is generated by B, $SH_{1,p}$ and T.*

PROOF. It is clear that we have the exact sequence:

$$1 \to \mathrm{Mod}^0(\Sigma_{g,p}) \to \mathrm{Mod}(\Sigma_{g,p}) \xrightarrow{\pi'} \mathrm{Sym}_p \to 1.$$

Since $\mathrm{Mod}^0(\Sigma_{g,p}) \subset G'$ by Corollary 7, Lemma 8 tells us that we have only to show $\pi'(G') = \mathrm{Sym}_p$. Since $A_1, \ldots, A_{2g}$ and $SH_{1,p}$ are in G', $H_{1,p}$ is in G'. Therefore, we can find that $H_{j,j+1} = T^j H_{1,p} T^{-j} \in G'$ $(j = 1, \ldots, p-1)$. It is clear that the image of $H_{j,j+1}$ is $(j, j+1)$. Since $(1,2), \ldots, (p-1, p)$ generate Sym_p, we see $\pi'(G') = \mathrm{Sym}_p$. This completes the proof of Theorem 9. □

4. The extended mapping class group

In this section we prove that the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma_{g,p})$ is also generated by three elements.

Let us embed $\Sigma_{g,p}$ in $\mathbf{R}^3$ as shown in Figure 6 or Figure 7 according as the number of p of the punctures is odd or even. Let R denote the reflection across the xz-plane and let T' denote the product $R\rho_2$.

We can find that

$$R(x_j) = x_{p-j+1} = \rho_1(x_j) \tag{1}$$

$$R(e_j) = e_{p-j+2} = \rho_1(e_j). \tag{2}$$

THEOREM 10. *Suppose that $g \geq 1$ and $p \geq 2$. Then the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma_{g,p})$ is generated by B, $SH_{1,p}$ and T'.*

PROOF. Let H' be the subgroup of $\mathrm{Mod}^{\pm}(\Sigma_{g,p})$ generated by B, $SH_{1,p}$ and T'. By (1) and (2), we find that the action of $T' = R\rho_2$ on the set of punctures and the curve e_j is identical with that of $T = \rho_1\rho_2$. From the proof of Lemma 6 and Theorem 9, we find that B, A_i $(i = 1, \ldots, 2g)$, E_j and $H_{j,j+1}$

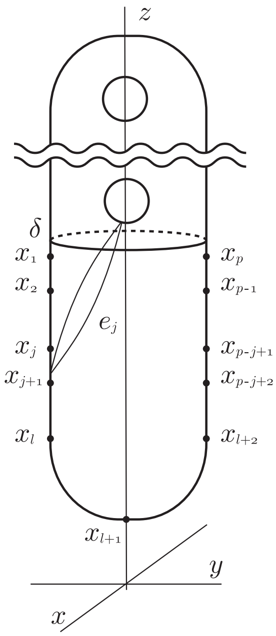

Fig. 6. Involution R, when p is odd.

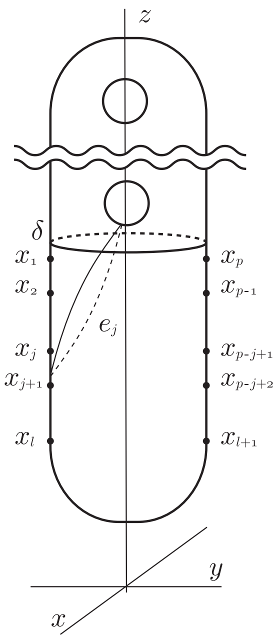

Fig. 7. Involution R, when p is even.

$(j = 1, \dots, p-1)$ are in H'. Therefore, $\mathrm{Mod}(\Sigma_{g,p})$ is the subgroup of H'. It is clear that we have the exact sequence:

$$1 \to \mathrm{Mod}(\Sigma_{g,p}) \to \mathrm{Mod}^{\pm}(\Sigma_{g,p}) \xrightarrow{\pi''} \mathbf{Z}/2\mathbf{Z} \to 1.$$

Thus, we have only to show $\pi''(H') = \mathbf{Z}/2\mathbf{Z}$ by virtue of Lemma 8. Since T' is the homotopy class of an orientation reversing homeomorphism, $\pi''(H') = \mathbf{Z}/2\mathbf{Z}$. This completes the proof of Theorem 10. □

References

[BF] T. E. Brendle and B. Farb. Every mapping class group is generated by 3 torsion elements and by 6 involutions. J. Algebra **278** (2004), 187–198.

[De] M. Dehn. Papers on group theory and topology. Springer-Verlag, New York, 1987 (Die Gruppe der Abbildungsklassen, Acta Math. Vol. **69** (1938), 135–206).

[Ge] S. Gervais. A finite presentation of the mapping class group of a punctured surface. Topology **40** (2001), No. 4, 703–725.

[Hu] S. P. Humphries. Generators for the mapping class group. Topology of low-dimensional manifolds Proc. Second Sussex Conf. Chelwood Gate 1977, Lecture Notes in Math. **722** (1979), Springer, 44–47.

[Jo] D. Johnson. The structure of Torelli group I: A finite set of generators for $\mathscr{I}$. Ann. of Math. **118** (1983), 423–442.

[Ka] M. Kassabov. Generating Mapping Class Groups by Involutions. arXiv:math.GT/0311455 v1 25 Nov 2003.

[Ko] M. Korkmaz. Generating the surface mapping class group by two elements. Trans. Amer. Math. Soc. **357** (2005), 3299–3310.

[Li] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. Proc. Cambridge Philos. Soc. **60** (1964), 769–778.

[Lu] N. Lu. On the mapping class groups of the closed orientable surfaces. Topology Proc. **13** (1988), 293–324.

[Wa] B. Wajnryb. Mapping class group of a surface is generated by two elements. Topology. **35** (1996), 377–383.

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