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Liouville-type theorems of p-harmonic maps with free boundary values

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Abstract. In this paper, we study free boundary-value problems for p-harmonic maps on half simple spaces of Euclidean space, and obtain some Liouville-type theorems.

1. Introduction and main results

Let (M,g) be a Riemannian manifold of dimension $m \geq 3$ with boundary $\partial M \neq \emptyset$. (N,h) be another Riemannian manifold of dimension $n \geq 2$. Denote S a given closed submanifold of N of dimension d, $1 \leq d \leq n-1$. For a map $u : M \to N$ such that $u(\partial M) \subset S$, we call ∂M the free boundary of map u and S the supporting manifold for the free boundary values.

If u is a critical point of p-energy functional $E_p(u) = \frac{1}{p}\int_M |\mathrm{d}u|^p v_g$ amongst maps satisfying a free boundary condition $u(\partial M) \subset S$, then we call u a p-harmonic map with free boundary. We refer to [1], [2], [3], [5] for the existence, regularity and minimizing properties of p-harmonic maps with boundary-value.

In this paper, we will prove some new type of Liouville theorems for p-harmonic maps with free boundary. Our results concern the asymptotic behavior of p-harmonic maps at infinity. For $p = 2$, we refer to [7] and [9] for this type of Liouville theorems.

Denote by $\mathbf{R}_+^m$ $(m \geq 3)$ the half simple space of Euclidean space $\mathbf{R}^m$ and g_0 the standard Euclidean metric on $\mathbf{R}_+^m$. We can state our main results as follows:

Theorem A. *For $p \in [2,m)$, let $u : (\mathbf{R}_+^m, g_0) \to (N,h)$ be a C^1 p-harmonic map with free boundary condition: $u(\partial \mathbf{R}_+^m) \subset S \subset N$, $\frac{\partial u}{\partial \nu}(x) \perp T_{u(x)}S$ for any $x \in \partial \mathbf{R}_+^m$, where ν is the unit normal to $\partial \mathbf{R}_+^m$. If the p-energy $E_p(u) < \infty$, then u must be a constant map.*

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Theorem B. *For* $p \in [2, m)$*, let* $u : (\mathbf{R}_+^m, g_0) \to (N, h)$ *be a* C^1 *p-harmonic map with free boundary condition:* $u(\partial \mathbf{R}_+^m) \subset S \subset N$, $\frac{\partial u}{\partial v}(x) \perp T_{u(x)}S$ *for any* $x \in \partial \mathbf{R}_+^m$*. If* $u(x) \to Q_0 \in S$ *as* $|x| \to \infty$*, then u must be a constant map.*

By the way, using similar method as in the proof of Theorem B, we have the following Liouville-type theorem for p-harmonic maps which is the generalization of Jin's result for harmonic maps in [7].

Theorem C. *For* $p \in [2, m)$*, let* $u : (\mathbf{R}^m, g_0) \to (N, h)$ *be a* C^1 *p-harmonic map,* $m \geq 3$*. If* $u(x) \to Q_0 \in N$ *as* $|x| \to \infty$*, then u must be a constant map.*

2. Proof of Theorem A

In this section, we will prove the following Theorem A$'$ which is slightly more general than Theorem A while taking $f \equiv 1$ there.

Theorem A$'$. *For* $p \in [2, m)$*, let* $u : (\mathbf{R}_+^m, fg_0) \to (N, h)$ *be a* C^1 *p-harmonic map with free boundary condition:* $u(\partial \mathbf{R}_+^m) \subset S \subset N$, $\frac{\partial u}{\partial v}(x) \perp T_{u(x)}S$ *for any* $x \in \partial \mathbf{R}_+^m$*, where f is some positive function on* $\mathbf{R}_+^m$ *which satisfy*

$$(\varepsilon - (m-p))f(x) \leq \frac{m-p}{2} \frac{\partial f}{\partial x_i} \cdot x_i, \qquad \text{for some constant } \varepsilon > 0. \tag{2.1}$$

If the p-energy $E_p(u) < \infty$*, then u must be a constant map.*

Proof. For the case of $\mathbf{R}_+^m$ with the Riemannian metric $g = fg_0$ for some positive function f on $\mathbf{R}_+^m$, the p-energy density can be written as

$$|\mathrm{d}u|^p = \left(f^{-1}(x) h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i} \right)^{p/2}, \tag{2.2}$$

and

$$|\mathrm{d}u|^p v_g = f^{(m-p)/2}(x) \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i} \right)^{p/2} \mathrm{d}x. \tag{2.3}$$

For $t \geq 0$, we define a family $\{V_t\} : \mathbf{R}_+^m \to N$ of maps by $V_t(x) := u(tx)$ for $x \in \mathbf{R}_+^m$, and set

$$\Phi(R, t) := \frac{1}{p} \int_{B(R)} |\mathrm{d}V_t|^p v_g, \tag{2.4}$$

where $B(R) = \mathbf{R}_+^m \cap \{x : |x| \leq R\}$. Then, applying Green's theorem, we calculate

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} &= \int_{B(R)}\left\langle \mathrm{d}^*(|\mathrm{d}u|^{p-2}\mathrm{d}u), \mathrm{d}u\left(r\frac{\partial}{\partial r}\right)\right\rangle \mathrm{d}x \\ &\quad + R\int_{\partial B(R)\cap\{x_m>0\}} |\mathrm{d}u|^{p-2}\left\langle \mathrm{d}u\left(\frac{\partial}{\partial \nu}\right), \mathrm{d}u\left(\frac{\partial}{\partial r}\right)\right\rangle \sigma_R \\ &\quad + \int_{\partial B(R)\cap\{x_m=0\}} |\mathrm{d}u|^{p-2}\left\langle \mathrm{d}^*u(\nu), \frac{\mathrm{d}V_t}{\mathrm{d}t}\bigg|_{t=1}\right\rangle \mathrm{d}x'.\end{aligned}$$

where $\frac{\partial}{\partial \nu} = f^{-1}\frac{\partial}{\partial r}$ is the unit normal, σ_R denotes the volume element of the induced Riemannian metric on $\partial B(R)$. By virtue of the p-harmonic condition $\mathrm{d}^*(|\mathrm{d}u|^{p-2}\mathrm{d}u) = 0$, the free boundary condition and $\mathrm{d}u(\frac{\partial}{\partial \nu}) = f^{-1}\,\mathrm{d}u(\frac{\partial}{\partial r})$, it follows that

$$\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} \geq 0. \tag{2.5}$$

On the other hand, re-parameterizing the integral (2.4), we get

$$\begin{aligned}\Phi(R,t) &= \frac{1}{p}\int_{B(R)} f^{(m-p)/2}(x)\left(h_{\alpha\beta}(u(tx))\frac{\partial u^\alpha(tx)}{\partial x_i}\frac{\partial u^\beta(tx)}{\partial x_i}\right)^{p/2}\mathrm{d}x \\ &= \frac{t^{p-m}}{p}\int_{B(tR)} f^{(m-p)/2}\left(\frac{x}{t}\right)\left(h_{\alpha\beta}(u(x))\frac{\partial u^\alpha(x)}{\partial x_i}\frac{\partial u^\beta(x)}{\partial x_i}\right)^{p/2}\mathrm{d}x.\end{aligned} \tag{2.6}$$

Set $\tilde{e}_p(u) = \left(h_{\alpha\beta}(u(x))\frac{\partial u^\alpha(x)}{\partial x_i}\frac{\partial u^\beta(x)}{\partial x_i}\right)^{p/2}$, by a direct calculation, we obtain from (2.1) that

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} &= \frac{p-m}{p}\int_{B(R)} f^{(m-p)/2}(x)\tilde{e}_p(u)\mathrm{d}x \\ &\quad - \frac{m-p}{2p}\int_{B(R)} f^{(m-p-2)/2}(x)\tilde{e}_p(u)\cdot\left(\frac{\partial f}{\partial x_i}\cdot x_i\right)\mathrm{d}x \\ &\quad + \frac{1}{p}\int_{\partial B(R)\cap\{x_m\geq 0\}} R^{m-1}f^{(m-p)/2}(x)\tilde{e}_p(u)\sigma_R \\ &\leq -\varepsilon\Phi(R,1) + R\frac{\mathrm{d}}{\mathrm{d}R}\Phi(R,1).\end{aligned} \tag{2.7}$$

Combining (2.5) and (2.7), we have $-\varepsilon\Phi(R,1) + R\frac{\mathrm{d}}{\mathrm{d}R}\Phi(R,1) \geq 0$. Therefore, for all $R > 0$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}R}\{R^{-\varepsilon}\Phi(R,1)\} \geq 0. \tag{2.8}$$

Now, suppose that u is a nonconstant p-harmonic map, by the C^1 regularity, $|du|^p$ cannot vanish identically on some open set in $\mathbf{R}_+^m$. Thus there exists $R_0 > 0$ and $C > 0$ such that $\int_{B(R_0)} |du|^p v_g \geq C$. Meanwhile, for all $R \geq R_0$, we have from (2.8) that

$$\int_{B(R)} |du|^p v_g \geq \left(\frac{R}{R_0}\right)^{\varepsilon} \int_{B(R_0)} |du|^p v_g \geq C\left(\frac{R}{R_0}\right)^{\varepsilon}.$$

Since $\varepsilon > 0$, hence

$$E_p(u) = \frac{1}{p} \lim_{R\to\infty} \int_{B(R)} |du|^p v_g \geq \infty,$$

which gives a contradiction to the finiteness condition of $E_p(u)$. We complete the proof of Theorem A′ and Theorem A as a corollary of Theorem A′. □

3. Proofs of Theorems B and C

It is obvious that Theorem B is the special case of the following theorem while taking $f \equiv 1$ there.

THEOREM B′. *For $p \in [2, m)$, let $u : (\mathbf{R}_+^m, fg_0) \to (N, h)$ be a C^1 p-harmonic map with free boundary condition: $u(\partial\mathbf{R}_+^m) \subset S \subset N$, $\frac{\partial u}{\partial \nu}(x) \perp T_{u(x)}S$ for any $x \in \partial\mathbf{R}_+^m$, where f is some positive function on $\mathbf{R}_+^m$ satisfying the following two conditions:*

(1) *there are constants $\varepsilon > 0$, $R_0 > 0$, such that*

$$(\varepsilon - (m-p))f(x) \leq \frac{m-p}{2}\frac{\partial f}{\partial x_i}\cdot x_i, \qquad \text{for } |x| \geq R_0; \tag{3.1}$$

(2) *with the same constants ε, R_0 as in* (1), *there is a constant $C > 0$, such that*

$$f^{(m-p)/2}(x) \leq C|x|^{\varepsilon-(m-p)}, \qquad \text{for } |x| \geq R_0. \tag{3.2}$$

If $u(x) \to Q_0 \in S$ as $|x| \to \infty$, then u must be a constant map.

PROOF. We will prove Theorem B′ by contradiction. Denote by $B(R)$ the geodesic ball centered at origin with radius R in $\mathbf{R}^m$. Set

$$E_p(B(R)) = \frac{1}{p}\int_{\mathbf{R}_+^m \cap B(R)} |du|^p v_g. \tag{3.3}$$

Suppose that p-harmonic map u is not a constant map, then the assumption (3.1) on f and Theorem A′ imply that the p-energy $E_p(u)$ of u must be infinite.

That's to say $E_p(B(R)) \to \infty$ as $R \to \infty$, from which, we would derive an upper and a lower bound for the growth rate of $E_p(B(R))$ as $R \to \infty$, the two bounds will contradict to each other, at that time, we will complete the proof.

Step I Modification of the p-harmonic map u at boundary $\partial \mathbf{R}^m_+$.

Since $\lim\limits_{|x| \to \infty} u(x) = Q_0$, there exists a neighborhood $U_{r_0} = \{(x_1, \ldots, x_m) : |x_m| < r_0\}$ of $\partial \mathbf{R}^m_+$ such that the image $U_{r_0} \cap \mathbf{R}^m_+$ of u lies on the standard neighborhood $\mathcal{N}(S)$ of S, that means, for every $y \in \mathcal{N}(S)$, there exists only one point $y' \in S$ such that y' is a projection of y along the unique geodesic minimizing the distance between two points y and y'. Let $\bar{x} = (x_1, \ldots, x_{m-1}, -x_m)$ and $x = (x_1, \ldots, x_{m-1}, x_m)$, if $\bar{x} \in U_{r_0} \backslash \mathbf{R}^m_+$ is the reflection point of $x \in \mathbf{R}^m_+$, we project $u(x)$ onto S along the minimal geodesic γ, denote by $\tilde{u}(x) \in S$, extending γ to some point Q such that $\operatorname{dist}(u(x), \tilde{u}(x)) = \operatorname{dist}(Q, \tilde{u}(x))$, then we define the reflection $\tilde{u}(x)$ as follows

$$\begin{cases} \tilde{u}(x) = u(x), & x \in \mathbf{R}^m_+, \\ \tilde{u}(x) = Q = u(\bar{x}), & x \in U_{r_0} \backslash \mathbf{R}^m_+. \end{cases} \tag{3.4}$$

According to the arguments in part 4 of [4], we know that $\tilde{u} : U_{r_0} \cup \mathbf{R}^m_+ \to N$ is a smooth map.

Step II The upper bound for the growth rate of $E_p(B(R))$.

According to theorem 5.1 in [6] (see also [7]), we can choose a local coordinate neighborhood U of Q_0 in N such that $Q_0 = 0$ and, for any $y \in U$, the metric tensor $h = h_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$ satisfies (for two matrices A, B, by $A \geq B$, we mean that $A = B + D$ for a positive semi-definite matrix D)

$$\left(\frac{\partial h_{\alpha\beta}(y)}{\partial y^\gamma} y^\gamma + 2 h_{\alpha\beta}(y) \right) \geq (h_{\alpha\beta}(y)). \tag{3.5}$$

Now, since $u(x) \to Q_0 = 0$ as $|x| \to \infty$, there exists $R_1 > 0$ such that for $|x| > R_1$, $u(x) \in U$, and

$$\left(\frac{\partial h_{\alpha\beta}(u)}{\partial u^\gamma} u^\gamma + 2 h_{\alpha\beta}(u) \right) \geq (h_{\alpha\beta}(u)). \tag{3.6}$$

Since $u : (\mathbf{R}^m_+, f g_0) \to (N, h)$ is a p-harmonic map, it follows that $\tilde{u}(x) : (\mathbf{R}^m_+ \cap U_{r_0}, f g_0) \to (N, h)$ is also a p-harmonic map and then, for $\omega(x) \in C^2_0(\mathbf{R}^m_+ \cap U_{r_0} \backslash B(R_1), \exp^{-1}_{Q_0}(U))$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(\tilde{u}(x) + t\omega(x))|_{t=0} = 0, \tag{3.7}$$

which jointly with (2.3) leads to

$$\int_{\mathbf{R}^m_{r_0}\setminus B(R_1)} A(f,u,Du)\left(2h_{\sigma\delta}(\tilde{u})\frac{\partial\tilde{u}^\sigma}{\partial x_j}\frac{\partial\omega^\delta}{\partial x_j}+\frac{\partial h_{\sigma\delta}}{\partial y^\gamma}\omega^\gamma\frac{\partial\tilde{u}^\sigma}{\partial x_j}\frac{\partial\tilde{u}^\delta}{\partial x_j}\right)\mathrm{d}x=0, \tag{3.8}$$

where $A(f,u,Du):=f^{(m-p)/2}\big(h_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_i}\big)^{(p-2)/2}$ and $\mathbf{R}^m_{r_0}=\{x\,|\,x\in\mathbf{R}^m,x_m>-r_0\}$.

For $0<\tilde{\varepsilon},s\le r_0$, we define Lipschitz functions $\phi(t)$ and $\Phi(t)$ with compact supports:

$$\varphi_{\tilde{\varepsilon}}(t):=\begin{cases}1, & \text{for } t\le 1,\\ 1+\frac{1-t}{\tilde{\varepsilon}}, & \text{for } 1<t<1+\tilde{\varepsilon},\\ 0, & \text{for } t\ge 1+\tilde{\varepsilon},\end{cases} \tag{3.9}$$

$$\Phi(x_m):=\begin{cases}1, & 0\le x_m,\\ 1+\frac{x_m}{s}, & -s<x_m<0,\\ 0, & -r_0<x_m\le -s,\end{cases} \tag{3.10}$$

and choose

$$\phi(|x|)=\varphi_{\tilde{\varepsilon}}\left(\frac{(|x|)}{R}\right)\left(1-\varphi_{r_0}\left(\frac{|x|}{R_1}\right)\right),\qquad \text{for } R>2R_1. \tag{3.11}$$

Notice that, for $R<|x|<R(1+\tilde{\varepsilon})$, $\frac{\partial\varphi_{\tilde{\varepsilon}}(|x|/R)}{\partial x_i}=-\frac{1}{R\tilde{\varepsilon}}\frac{x_i}{|x|}$. Substituting $\omega=\phi(|x|)\Phi(x_m)\tilde{u}(x)$ into (3.8), and taking the limit as $\tilde{\varepsilon}\to 0$, then we obtain

$$\begin{aligned}\int_{\mathbf{R}^m_{r_0}\cap(B(R)\setminus B(R_2))} & A(f,u,Du)\left(2h_{\sigma\delta}(\tilde{u})+\frac{\partial h_{\sigma\delta}(\tilde{u})}{\partial y^\gamma}\tilde{u}^\gamma\right)\frac{\partial\tilde{u}^\sigma}{\partial x_j}\frac{\partial\tilde{u}^\delta}{\partial x_j}\,\mathrm{d}x+D(R_1)\\ &=\int_{\partial B(R)\cap\mathbf{R}^m_{r_0}} A(f,u,Du)\left(2h_{\sigma\delta}(\tilde{u})\frac{\partial\tilde{u}^\sigma}{\partial x_j}\nu^j\tilde{u}^\delta\Phi(x_m)\right)\sigma_R\\ &\quad-\int_{\mathbf{R}^m_{r_0}\cap(B(R)\setminus B(R_1))} 2A(f,u,Du)h_{\sigma\delta}(\tilde{u})\frac{\partial\tilde{u}^\sigma}{\partial x_m}\tilde{u}^\delta(x)\frac{\mathrm{d}\Phi(x_m)}{\mathrm{d}x_m}\,\mathrm{d}x,\end{aligned} \tag{3.12}$$

where $R_2=2R_1$, $\nu=(\nu^1,\nu^2,\ldots,\nu^m)$ is the outer normal on $\partial B(R)$, σ_R denotes the volume element of the induced Riemannian metric on $\partial B(R)$, and letting $s\to 0$, it follows that

$$\begin{aligned}\int_{\mathbf{R}^m_{+}\cap(B(R)\setminus B(R_2))} & A(f,u,Du)\left(2h_{\sigma\delta}(u)+\frac{\partial h_{\sigma\delta}(u)}{\partial y^\gamma}u^\gamma\right)\frac{\partial u^\sigma}{\partial x_j}\frac{\partial u^\delta}{\partial x_j}\,\mathrm{d}x+D(R_1)\\ &=\int_{\partial B(R)\cap\mathbf{R}^m_{+}} A(f,u,Du)\left(2h_{\sigma\delta}(u)\frac{\partial u^\sigma}{\partial x_j}\nu^j u^\delta\Phi(x_m)\right)\sigma_R\end{aligned} \tag{3.13}$$

and

$$D(R_1) = \int_{\mathbf{R}_+^m \cap (B(R_2) \setminus B(R_1))} A(f, u, Du) \Bigg\{ -2h_{\sigma\delta}(u) \frac{\partial u^\sigma}{\partial x_j} u^\delta \frac{\partial \varphi_{r_0}\left(\frac{|x|}{R_1}\right)}{\partial x_j} + \left(2h_{\sigma\delta}(u) + \frac{\partial h_{\sigma\delta}}{\partial y^\gamma} u^\gamma\right) \frac{\partial u^\sigma}{\partial x_j} \frac{\partial u^\delta}{\partial x_j} \left(1 - \varphi_{r_0}\left(\frac{|x|}{R_1}\right)\right) \Bigg\} \mathrm{d}x. \tag{3.14}$$

By means of (3.6), we obtain from (3.13) that

$$\int_{\mathbf{R}_+^m \cap (B(R) \setminus B(R_2))} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x + D(R_1) \le 2 \int_{\partial B(R) \cap \mathbf{R}_+^m} A(f, u, Du) \left(h_{\sigma\delta}(u) \frac{\partial u^\sigma}{\partial x_j} \nu^j u^\delta\right) \sigma_R. \tag{3.15}$$

For $R > R_2$, set

$$Z(R) = \int_{\mathbf{R}_+^m \cap (B(R) \setminus B(R_2))} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x + D(R_1). \tag{3.16}$$

Following the arguments similar to Jin [7], we can derive

$$Z(R) \le C\eta(R) \cdot R^\varepsilon, \qquad \text{for } R \ge R_3, \tag{3.17}$$

where $\eta(R)$ is a non-increasing function on (R_3, ∞) such that $\eta(R) \to 0$ as $R \to \infty$ and $\eta(R) \ge \max_{\partial B(R)} (h_{\alpha\beta} u^\alpha u^\beta)^{p/2}$. Therefore, we obtain an upper bound for the growth rate of $E_p(B(R))$:

$$\begin{aligned} E_p(B(R)) &= \frac{1}{p} \int_{\mathbf{R}_+^m \cap (B(R) \setminus B(R_2))} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x \\ &\quad + \frac{1}{p} \int_{B(R_2) \cap \mathbf{R}_+^m} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x \\ &= \frac{1}{p} [Z(R) - D(R_1)] + \frac{1}{p} \int_{B(R_2) \cap \mathbf{R}_+^m} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x \\ &\le C\left[\eta(R) + \frac{c(u)}{R^\varepsilon}\right] \cdot R^\varepsilon, \end{aligned} \tag{3.18}$$

where, $c(u) = \frac{1}{p} \int_{B(R_2) \cap \mathbf{R}_+^m} f^{(m-p)/2} \left(h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x - \frac{1}{p} D(R_1)$ is a constant depending only on the p-harmonic map u.

Step III The lower bound for the growth rate of $E_p(B(R))$.

Proceeding the similar argument as in the proof of Theorem A′, we can easily get a lower bound for the growth rate of $E_p(B(R))$ as follows:

$$E_p(B(R)) \geq c_1 + c_2(u)R^{\varepsilon}, \qquad \text{for } R > R_5, \tag{3.19}$$

where $c_1, c_2(u)$ are some constants.

Now, a contradiction appears as $R \to \infty$ from (3.18) and (3.19), which implies Theorem B′. □

We can prove Theorem C in the following more general frame, i.e., we have

Theorem C′. *For $p \in [2, m)$, let $u : (\mathbf{R}^m, fg_0) \to (N, h)$ be a C^1 p-harmonic map, where f is some positive function on $\mathbf{R}^m$ satisfying* (3.1), (3.2) *and in additional*

$$\frac{\partial}{\partial r}(r \cdot f(x)) \geq 0, \qquad \text{on } \mathbf{R}^m,\ r = |x|. \tag{3.20}$$

If $u(x) \to Q_0$ as $|x| \to \infty$, then u must be a constant map.

Before starting with the proof of Theorem C′, we quote the result which concerns the finiteness of the p-energy of the p-harmonic map.

Lemma 1 ([8, Theorem 9]). *Suppose that $m > p$, and $\frac{\partial}{\partial r}(r \cdot f(x)) \geq 0$, $r = |x|$. Let $u : (\mathbf{R}^m, fg_0) \to (N, h)$ be a p-harmonic map of $(\mathbf{R}^m, fg_0)$ into an n-dimensional Riemannian manifold N. If the p-energy $E_p(u)$ of u is finite, then u is a constant map.*

Proof (of theorem C′). Suppose that the p-harmonic map u is not a constant map, then Lemma 1 (with the assumption (3.20) on f) implies that the p-energy of u must be infinite. Then, similar to the proof of Theorem B′, we can obtain an upper bound for the growth rate of $E_p(B(R))$:

$$E_p(B(R)) := \frac{1}{p}\int_{B(R)} |\mathrm{d}u|^p v_g \leq C\left[(\eta(R)) + \frac{c(u)}{R^{\varepsilon}}\right] \cdot R^{\varepsilon}. \tag{3.21}$$

Now, define a family $V_t : (\mathbf{R}^m, fg_0) \to (N, h)$ of maps as $V_t(x) := u(tx)$, for $x \in \mathbf{R}^m$, $t > 0$ and set

$$\Phi(R, t) = \frac{1}{p}\int_{B(R)\backslash B(R_1)} |\mathrm{d}V_t|^p \mathrm{d}x, \qquad \text{for } R > R_1. \tag{3.22}$$

Then we know from (2.5) that

$$\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} \geq 0. \tag{3.23}$$

On the other hand, re-parameterizing the integral (3.22) and calculating $\frac{\partial}{\partial t}\Phi(R,t)|_{t=1}$ directly, we get

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} &= \frac{R}{p}\int_{\partial B(R)} B(f,u,Du)\sigma_R - \frac{R_1}{p}\int_{\partial B(R_1)} B(f,u,Du)\sigma_R \\ &\quad - \int_{B(R)\setminus B(R_1)} f^{(m-p-2)/2}\left(\frac{m-p}{2}\right) \\ &\quad \times \frac{\partial f}{\partial x_i} x_i \left(h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_i}\right)^{p/2} \mathrm{d}x,\end{aligned} \tag{3.24}$$

where, we denote $B(f,u,Du) = f^{(m-p)/2}(x)\big(h_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_i}\big)^{p/2}$. (3.24) together with (3.1) implies that

$$\frac{\partial}{\partial t}\Phi(R,t)\bigg|_{t=1} \leq -\varepsilon\Phi(R,1) + R\frac{d}{dR}\Phi(R,1) - R_1\int_{\partial B(R_1)} B(f,u,Du)\sigma_R. \tag{3.25}$$

Set $H_1 = R_1\int_{\partial B(R_1)} B(f,u,Du)\sigma_R$, then (3.23) and (3.25) yield

$$R\frac{\mathrm{d}}{\mathrm{d}R}\Phi(R,1) - \varepsilon\Phi(R,1) - H_1 \geq 0.$$

By setting $H_0 = -\varepsilon\int_{B(R_1)} e_p(u)v_g + \varepsilon H_1$, the last inequality is rewritten as

$$R\left\{E_p(B(R)) + \frac{1}{\varepsilon}H_0\right\}' - \varepsilon\left\{E_p(B(R)) + \frac{1}{\varepsilon}H_0\right\} \geq 0,$$

and then, for all $R > R_1$, we have

$$\left\{R^{-\varepsilon}\left(E_p(B(R)) + \frac{1}{\varepsilon}H_0\right)\right\}' \geq 0.$$

Since $E_p(B(R)) \to \infty$ as $R \to \infty$, there exists $R_5 > R_1$ such that

$$\left\{R^{-\varepsilon}\left(E_p(B(R)) + \frac{1}{\varepsilon}H_0\right)\right\} \geq \left\{R_5^{-\varepsilon}\left(E_p(B(R_5)) + \frac{1}{\varepsilon}H_0\right)\right\} > 0$$

holds for $R > R_5$. Therefore

$$E_p(B(R)) + \frac{1}{\varepsilon}H_0 \geq c_1(u)R^\varepsilon \qquad \text{for } R > R_5, \tag{3.26}$$

which contradicts to (3.21). This contradiction implies Theorem C′ and then Theorem C. □

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