Hiroshima Math. J.
40 (2010), 229–239

Criteria for singularities of smooth maps from the plane into the plane and their applications

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(Received April 2, 2009)
(Revised December 11, 2009)

Abstract. We give useful criteria for lips, beaks and swallowtail singularities of a smooth map from the plane into the plane. As an application of the criteria, we discuss the singularities for a Cauchy problem of single conservation law.

1. Introduction

Singularities of map germs have long been studied, especially up to the equivalence under coordinate changes in both source and target ($\mathscr{A}$-equivalence). According to [2], "classification" for map germs with $\mathscr{A}$-equivalence means finding lists of germs, and showing that all germs satisfying certain conditions are equivalent to a germ on the list. Classification is well understood, with many good references in the literature. "Recognition" means finding criteria which will describe which germ on the list a given germ is equivalent to (see [2]). The classification problem and recognition problem for map germs from the plane into the plane up to $\mathscr{A}$-equivalence was studied by J. H. Rieger [9]. He classified map germs $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ with corank one and $\mathscr{A}_e$-codimension ≤ 6. Table 1 shows the list of the $\mathscr{A}_e$-codimension ≤ 3 local singularities obtained in [9]. Some of these singularities are also called as follows: $4_{2,+}$ (*lips*), $4_{2,-}$ (*beaks*), 5 (*swallowtail*). These singularities are depicted in Figure 1. Rieger also discussed the recognition of these map germs after normalizing the coordinate system as $(u, v) \mapsto (u, f_2(u, v))$. However, for applications, criteria of recognition without using normalization are not only more convenient but also indispensable in some cases.

In this paper, we give criteria for the lips, the beaks and the swallowtails of a map germ $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ without using the normalizations (Theorem 3). Since they only use the information of the Taylor coefficients of the germ, Theorem 3 can be applied directly for the recognition of the lips, the beaks and the swallowtail on explicitly parameterized maps. Using the criteria, we study

The author is supported by Grant-in-aid for Young scientists (B) No. 20740028.
2000 *Mathematics Subject Classification*. 57R45.
Key words and phrases. criteria, singularity, recognition.

Table 1. Classification of $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$

Name	Normal form	$\mathscr{A}_e$-codimension
Immersion	(u, v)	0
Fold	(u, v^2)	0
Cusp	$(u, v^3 + uv)$	0
$4_{k,\pm}$	$(u, v^3 \pm u^k v),\ k = 2, 3$	$k-1$
5	$(u, uv + v^4)$	1
$6_\pm$	$(u, uv + v^5 \pm v^7)$	2
11_5	$(u, uv^2 + v^4 + v^5)$	2

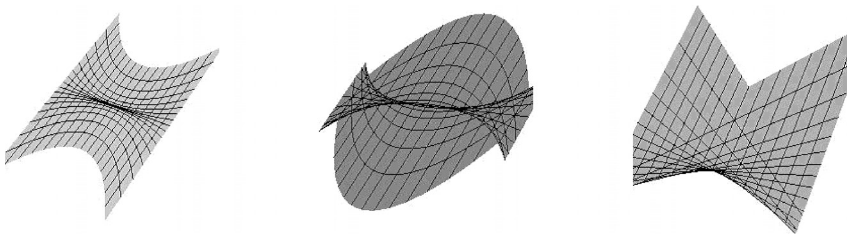

Fig. 1. Lips, beaks and swallowtail

singularities of a conservation law about a time variable. We study singularities of geometric solutions of the equation and show the singularities that appear for the first time are generically the lips (Section 3).

The case of wave front surfaces in 3-space, criteria for the cuspidal edge and the swallowtail were given by M. Kokubu et al. [7]. By using them, we studied local and global behaviors of flat fronts in hyperbolic 3-space. Using them, K. Saji et al. [11] introduced the singular curvature on the cuspidal edge and investigated its properties.

Criteria for other singularities of fronts and their applications were given in [1, 5, 12]. Recently, several applications of these criteria were considered in various situations [3, 5, 6, 8, 10]. Throughout this paper, we work in the C^∞-category.

2. Preliminaries and statements of criteria

Let $U \subset \boldsymbol{R}^2$ be an open set and $f : (U, p) \to (\boldsymbol{R}^2, 0)$ a map germ. We call $q \in U$ a *singular point* of f if $\operatorname{rank}(df)_q \leq 1$. We denote by $S(f) \subset U$ the set of singular points of f. Two map germs $f_i : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ $(i = 1, 2)$ are *$\mathscr{A}$-equivalent* if there exist diffeomorphism map germs $\Phi_i : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ $(i = 1, 2)$ such that $f_1 \circ \Phi_1 = \Phi_2 \circ f_2$ holds. For a positive integer k, a map germ $f : (U, p) \to (\boldsymbol{R}^2, 0)$ is *k-determined* if any $g : (U, p) \to (\boldsymbol{R}^2, 0)$ satisfying the condition that the k-jet $j^k g(p)$ of g is equal to $j^k f(p)$, is $\mathscr{A}$-equivalent to f. The following fact is well-known.

Fact 1. ([9, Lemma 3.2.2 and 3.1.3]) *The lips and the beaks* $(x, y) \mapsto (x, y^3 \pm xy)$ *are three-determined. The swallowtail* $(x, y) \mapsto (x, xy + y^4)$ *is four-determined.*

Let $f : (U, p) \to (\boldsymbol{R}^2, 0)$ be a map germ. A singular point q is of *corank one* if $\operatorname{rank}(df)_q = 1$. If p is a corank one singular point of f, then there exists a neighborhood V of p and a never vanishing vector field $\eta \in \mathfrak{X}(V)$ such that $df_q(\eta) = 0$ holds for any $q \in S(f) \cap V$. We call η the *null vector field*. We define a function which plays a crucial role in our criteria. Let (u_1, u_2) be coordinates of U. Define the *discriminant function* λ *of* f by

$$\lambda(u_1, u_2) = \det\left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right)(u_1, u_2). \tag{1}$$

Then $S(f) = \lambda^{-1}(0)$ holds. We call $p \in S(f)$ a *non-degenerate singular point* if $d\lambda(p) \neq 0$ and a *degenerate singular point* if $d\lambda(p) = 0$. Note that a non-degenerate singular point is of corank one. The terminologies "discriminant function", "null vector field" and "non-degeneracy" are defined in [7] in order to state criteria for fronts in the 3-space. Our definitions of these three terminologies are similar. These notions also play a key role to identify singularities for our case. This seems to be related to the correspondence between singularities of front and its projection to the limiting tangent plane. This correspondence is discussed in [12].

We review the criteria for the fold and the cusp, due to Whitney [13] (see also [12]).

Fact 2. ([13, Proposition 2.1]) *For a map germ* $f : (U, p) \to (\boldsymbol{R}^2, 0)$, f *at* p *is* $\mathscr{A}$*-equivalent to the fold if and only if* $\eta\lambda(p) \neq 0$.

Furthermore, f *at* p *is* $\mathscr{A}$*-equivalent to the cusp if and only if* p *is non-degenerate,* $\eta\lambda(p) = 0$ *and* $\eta\eta\lambda(p) \neq 0$.

Here, $\eta\lambda$ means the directional derivative $D_\eta\lambda$. The main result of this paper is the following.

Theorem 3. *For a map germ* $f : (U, p) \to (\boldsymbol{R}^2, 0)$, *the following hold.*

(1) f *is* $\mathscr{A}$*-equivalent to the lips if and only if* p *is of corank one,* $d\lambda(p) = 0$ *and* λ *has a Morse type critical point of index* 0 *or* 2 *at* p, *namely,* $\det \operatorname{Hess} \lambda(p) > 0$.

(2) f *is* $\mathscr{A}$*-equivalent to the beaks if and only if* p *is of corank one,* $d\lambda(p) = 0$, λ *has a Morse type critical point of index* 1 *at* p (*i.e.,* $\det \operatorname{Hess} \lambda(p) < 0$.) *and* $\eta\eta\lambda(p) \neq 0$.

(3) *f is $\mathscr{A}$-equivalent to the swallowtail if and only if $d\lambda(p) \neq 0$, $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*

Here, for a function $\lambda : (U, u_1, u_2) \to \boldsymbol{R}$, Hess λ is the matrix defined by Hess $\lambda = (\partial^2\lambda/\partial u_i\,\partial u_j)_{i,j=1,2}$. Remark that in Theorem 3 (1), $\eta\eta\lambda(p) \neq 0$ is automatically satisfied because of the symmetricity of Hess λ and the inequality det Hess $\lambda(p) > 0$.

EXAMPLE 4. Let us put

$$f_{\mathrm{l}}(u,v) = (u, v^3 + u^2 v), \quad f_{\mathrm{b}}(u,v) = (u, v^3 - u^2 v) \quad \text{and} \quad f_{\mathrm{s}}(u,v) = (u, v^4 + uv).$$

Since these are nothing but the defining formula for the lips, the beaks and the swallowtail, these maps satisfy the conditions in Theorem 3. The discriminant functions for these maps are

$$\lambda_{\mathrm{l}} = 3v^2 + u^2, \quad \lambda_{\mathrm{b}} = 3v^2 - u^2 \quad \text{and} \quad \lambda_{\mathrm{s}} = 4v^3 + u,$$

respectively. Thus λ_{l} and λ_{b} have a Morse type critical point at the origin. Furthermore, the null vector field can be chosen as $\eta = (0,1)$ for all maps. It holds that $\eta\eta\lambda_{\mathrm{b}} \neq 0$, and that $d\lambda_{\mathrm{s}} \neq 0$, $\eta\lambda_{\mathrm{s}} = \eta\eta\lambda_{\mathrm{s}} = 0$ and $\eta\eta\eta\lambda_{\mathrm{s}} \neq 0$ at the origin. Thus we see that each of the conditions in Theorem 3 is satisfied for each map. These observations together with the following Lemma 1 confirm the only if part of Theorem 3.

EXAMPLE 5. Let $\gamma : I \to \boldsymbol{R}^2$ be a plane curve with $\gamma'(t) \neq 0$ for any $t \in I$. The *tangential ruling map* R_γ of γ is the map $R_\gamma : (t,u) \mapsto \gamma(t) + u\gamma'(t)$. The discriminant function and the null vector field of R_γ are $\lambda = u\kappa$ and $\eta = (-1, 1)$, respectively, where κ is the curvature of γ. Thus we have

$$\text{Hess } \lambda(t,0) = \begin{pmatrix} 0 & \kappa' \\ \kappa' & 0 \end{pmatrix} \quad \text{and} \quad \eta\eta\lambda(t,0) = -2\kappa'.$$

Using Theorem 3, R_γ at $(t_0, 0)$ is $\mathscr{A}$-equivalent to the beaks if and only if $\kappa(t_0) = 0$ and $\kappa'(t_0) \neq 0$ holds (See figure 2).

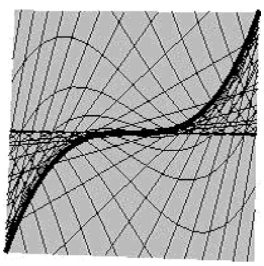

Fig. 2. The beaks on the tangential ruling map of (t, t^3) at $(t,u) = (0,0)$.

To prove Theorem 3, we need the following lemma.

Lemma 1. *For a map germ $f : (U, p) \to (\boldsymbol{R}^2, 0)$, the conditions in Theorem 3 are independent of the choice of coordinates of the source and target. To be precise, the rank of $(df)_p$, the non-degeneracy of p, and the sign of* det Hess $\lambda(p)$, *are independent of the choice of both coordinates on the source and target. Suppose further that p is non-degenerate, and let $\lambda(u_1, u_2)$ and $\tilde{\lambda}(v_1, v_2)$ are area density functions of f, and η and $\tilde{\eta}$ are null vector fields of f, then the following hold:*

- $\eta\lambda(p) = 0$ *if and only if* $\tilde{\eta}\tilde{\lambda}(p) = 0$.
- *If* $\eta\lambda(p) = \tilde{\eta}\tilde{\lambda}(p) = 0$, *then* $\eta\eta\lambda(p) \neq 0$ *if and only if* $\tilde{\eta}\tilde{\eta}\tilde{\lambda}(p) \neq 0$.
- *If* $\eta\lambda(p) = \eta\eta\lambda(p) = \tilde{\eta}\tilde{\lambda}(p) = \tilde{\eta}\tilde{\eta}\tilde{\lambda}(p) = 0$, *then* $\eta\eta\eta\lambda(p) \neq 0$ *if and only if* $\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\lambda}(p) \neq 0$.

Proof. Needless to say, $\operatorname{rank}(df)_p$ is independent of the choice of the coordinate systems. If we change the coordinates, then the function λ is multiplied by a non-zero function. Since the vanishing of $d\lambda$ and the sign of det Hess λ do not change under this multiplication, the first part of the lemma is proved. We now prove the second part. We can write $\tilde{\eta} = a_1\xi + a_2\eta$, where a_1, a_2 are functions near p, satisfies $a_1 = 0$ on $S(f)$, and ξ is a vector field transverse to η at p, and assume that $\tilde{\lambda}$ is a multiplication of λ by a non-zero function. Under this setting, since $\{\lambda = 0\} = \{a_1 = 0\}$ holds, one can prove that the non-degeneracy yields the desired equivalences. □

Now we prove Theorem 3; the method of proof is due to Rieger [9].

Proof of (1) and (2). Since p is of corank one, f can be represented as

$$f(u, v) = (u, vf_2(u, v)), \quad p = (0, 0)$$

by Lemma 1. Since $\lambda(p) = 0$ and $d\lambda(p) = 0$, we have $f_2 = (f_2)_u = (f_2)_v = 0$ at p, where $(f_2)_u = \partial f_2/\partial u$ and $(f_2)_v = \partial f_2/\partial v$. Therefore, f can be written as

$$(u, v(au^2 + 2buv + cv^2) + (\text{higher order term})), \ a, b, c \in \boldsymbol{R}.$$

Here, the "higher order term" consists of the terms whose degrees are greater than 3. Since det Hess $\lambda(p) \neq 0$, it holds that a, b or c does not vanish at p. Moreover, since $\eta = (0, 1)$ and $\eta\eta\lambda(p) \neq 0$, it holds that $c \neq 0$. Now, by the coordinate change

$$U = u, \quad V = v + \frac{2b}{3c}u,$$

f can be written as

$$(u, v(\alpha u^2 + \beta v^2) + \gamma u^3 + (\text{higher order term})), \ \alpha, \beta, \gamma \in \boldsymbol{R}.$$

Here, the "higher order term" consists of the terms whose degrees are greater than 3. We remark that the sign of $\alpha\beta$ coincides with the sign of Hess $\lambda(p)$. Hence, by some scaling change and a coordinate change on the target, f can be written as

$$(u, v(u^2 \pm v^2) + (\text{higher order term})). \tag{2}$$

Since the map germ $(u, v(u^2 \pm v^2))$ is three-determined, the map germ (2) is $\mathscr{A}$-equivalent to the lips(+) or the beaks(−). □

Proof of (3). Since f is of corank one, f can be written as $f(u,v) = (u, vh(u,v))$. Then the null vector field is $(0,1)$. Write

$$\begin{aligned} vh(u,v) = {} & a_{11}uv + a_{02}v^2 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + a_{31}u^3v \\ & + a_{22}u^2v^2 + a_{13}uv^3 + a_{04}v^4 + (\text{higher order term}). \end{aligned}$$

Here, the "higher order term" consists of the terms whose degrees are greater than 4. The non-degeneracy of f yields that $a_{11} \neq 0$. If $a_{02} \neq 0$, by Fact 2, f is $\mathscr{A}$-equivalent to the fold. Moreover, if $a_{02} = 0$ and $a_{03} \neq 0$ then by Fact 2, f is $\mathscr{A}$-equivalent to the cusp. Hence we can assume $a_{02} = a_{03} = 0$. Since $\eta\eta\eta\lambda(p) \neq 0$, we have $a_{04} \neq 0$. By the coordinate change

$$\begin{aligned} \tilde{u} &= u, \\ \tilde{v} &= a_{11}v + a_{21}uv + a_{12}v^2 + a_{31}u^2v + a_{22}uv^2 + a_{13}v^3, \end{aligned}$$

f is written as

$$f(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{u}\tilde{v} + \tilde{v}^4 + (\text{higher order term})).$$

Since $(\tilde{u}, \tilde{u}\tilde{v} + \tilde{v}^4)$ is four-determined, it is $\mathscr{A}$-equivalent to $(u, uv + v^4)$. □

3. Singularities of characteristic surfaces of a single conservation law

In this section, we consider the following Cauchy problem of a single conservation law:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t, \boldsymbol{x}) + \displaystyle\sum_{i=1,2} \dfrac{df_i}{dy}(y(t, \boldsymbol{x})) \dfrac{\partial y}{\partial x_i}(t, \boldsymbol{x}) = 0, \\ y(0, \boldsymbol{x}) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, x_2), \end{cases} \tag{C}$$

where, f_1, f_2 and φ are functions. We consider the characteristic surfaces of (C) following the framework of [4].

Let $\pi : PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}) \to \boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}$ be the projective cotangent bundle. Identify $PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}) = (\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}) \times P(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R})$ and denote the local coordinates of this space by $(t, \boldsymbol{x}, y, [\tau : \boldsymbol{\xi} : \eta])$. We consider the canonical contact form,

$$\alpha = [\tau dt + \xi_1 dx_1 + \xi_2 dx_2 + \eta dy].$$

Then the equation (C) is written in the following form:

$$E(1, f_1', f_2', 0) = \left\{ (t, \boldsymbol{x}, y, [\tau : \boldsymbol{\xi} : \eta]) \in PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}) \,\middle|\, \tau + \sum_{i=1,2} f_i'(y)\xi_i = 0 \right\},$$

where $f_i' = df_i/dy$. If (C) has a classical solution y, then the non-zero normal vector $v = (y_t, y_{x_1}, y_{x_2}, -1)$ of smooth hypersurface $(t, \boldsymbol{x}, y(t, \boldsymbol{x})) \subset \boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}$ exists, where, $y_{x_1} = \partial y/\partial x_1$ for example.

Hence we have a Legendrian immersion : $\boldsymbol{R} \times \boldsymbol{R}^2 \to PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R})$:

$$\tilde{y}(t, \boldsymbol{x}) : (t, \boldsymbol{x}) \mapsto (t, \boldsymbol{x}, y(t, \boldsymbol{x}), [v]) \in E(1, f_1', f_2', 0) \subset PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R}).$$

According to this, we define a *geometric solution* of (C) as a Legendrian immersion $L : (U; u_1, u_2) \to E(1, f_1', f_2', 0) \subset PT^*(\boldsymbol{R} \times \boldsymbol{R}^2 \times \boldsymbol{R})$ of a domain $U \subset \boldsymbol{R}^2$ such that $\pi \circ L$ is an embedding. We apply the method of characteristic equation. The characteristic equation associated with (C) through $(0, \boldsymbol{x}_0)$ is

$$\frac{dx_i}{dt}(t) = \frac{df_i}{dy}(y(t, \boldsymbol{x}(t))), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\frac{dy}{dt}(t, \boldsymbol{x}(t)) = 0, \quad y(0, \boldsymbol{x}(0)) = \varphi(\boldsymbol{x}_0).$$

The solution of the characteristic equation can be expressed by

$$x_i(\boldsymbol{u}, t) = u_i + t\frac{df_i}{dy}(\varphi(\boldsymbol{u})), \quad y(0, \boldsymbol{x}(\boldsymbol{u}, 0)) = y(0, \boldsymbol{u}) = \varphi(\boldsymbol{u}), \quad \boldsymbol{u} = (u_1, u_2) \in U. \tag{3}$$

If a map

$$g_t : \boldsymbol{u} \mapsto (x_1(\boldsymbol{u}, t), x_2(\boldsymbol{u}, t)) \tag{4}$$

is non-singular, $y = \varphi((g_t)^{-1}(x_1, x_2))$ is the classical solution of (C) (See [4, Section 5]). Remark that if $t = 0$, g_t is non-singular. Thus, in order to investigate the singularity of (C), we study the singularities of a family of maps g_t. The discriminant function of $g_t(\boldsymbol{u})$ is

$$\det\begin{pmatrix} 1 + tc_{11} & tc_{12} \\ tc_{21} & 1 + tc_{22} \end{pmatrix}, \qquad c_{ij} = \frac{d^2 f_i}{dy^2}(\varphi(\boldsymbol{u}))\frac{\partial \varphi}{\partial u_j}(\boldsymbol{u}).$$

Needless to say, this matrix is never equal to the zero-matrix. This implies that $(t, \boldsymbol{u})$ is a singular point of (3), if and only if $-t^{-1}$ is an eigen value of the matrix $C = (c_{ij})_{i,j=1,2}$. The eigen equation for an eigen value μ of C can be computed as

$$\begin{aligned}
0 &= \det\left(C - \mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} \dfrac{d^2 f_1}{dy^2}(\varphi(\boldsymbol{u}))\dfrac{\partial\varphi}{\partial u_1}(\boldsymbol{u}) - \mu & \dfrac{d^2 f_1}{dy^2}(\varphi(\boldsymbol{u}))\dfrac{\partial\varphi}{\partial u_2}(\boldsymbol{u}) \\ \dfrac{d^2 f_2}{dy^2}(\varphi(\boldsymbol{u}))\dfrac{\partial\varphi}{\partial u_1}(\boldsymbol{u}) & \dfrac{d^2 f_2}{dy^2}(\varphi(\boldsymbol{u}))\dfrac{\partial\varphi}{\partial u_2}(\boldsymbol{u}) - \mu \end{pmatrix} \\
&= \mu\left(\mu - \frac{d^2 f_1}{dy^2}(\varphi(\boldsymbol{u}))\frac{\partial\varphi}{\partial u_1}(\boldsymbol{u}) - \frac{d^2 f_2}{dy^2}(\varphi(\boldsymbol{u}))\frac{\partial\varphi}{\partial u_2}(\boldsymbol{u})\right) \\
&= \mu(\mu - \mathrm{trace}C).
\end{aligned}$$

Hence $(t, \boldsymbol{u})$ is a singular point of (3), if and only if

$$t = -1/\mathrm{trace}\ C.$$

We call C the *shape operator* of (C). Now we consider the first singular point of (4) with respect to t from the initial time $t = 0$.

For a minimal value of $t(\boldsymbol{u}) = -1/\mathrm{trace}\ C$, if $\det \mathrm{Hess}\ t(\boldsymbol{u}) > 0$ holds, then by Theorem 3, the singular point at $\boldsymbol{u}$ is $\mathscr{A}$-equivalent to the lips. Izumiya and Kossioris [4] have developed an unfolding theory and classified the generic singularities of multi-valued solutions in general dimensions. According to it, the first singular point of (4) is generically the lips, where they did not give a condition for the singular point to be equivalent to the lips. Using our criterion for the lips, we detect the singular point and write down an explicit condition for the singular point to be equivalent to the lips. As a corollary of it, we give a simple proof that the first singular point of (4) is generically the lips.

Since the single conservation law (C) is determined by functions (f_1, f_2) and the initial value φ, we may regard that the space of single conservation laws is the space

$$\{(f_1, f_2, \varphi)\} = C^\infty(\boldsymbol{R}, \boldsymbol{R})^2 \times C^\infty(\boldsymbol{R}^2, \boldsymbol{R})$$

with the Whitney C^∞-topology.

Theorem 6. *There exists a residual subset $\mathcal{O} \subset C^\infty(\boldsymbol{R}, \boldsymbol{R})^2 \times C^\infty(\boldsymbol{R}^2, \boldsymbol{R})$ such that for any $(f_1, f_2, \varphi) \in \mathcal{O}$, the map germ (4) defined by (f_1, f_2, φ) at the first singular point with respect to $t > 0$ is $\mathscr{A}$-equivalent to the lips.*

Here, a subset is *residual* if it is a countable intersection of open and dense subsets.

PROOF. Since, for a function w, the behaviors of dw and Hess w are the same as those of w^{-1}, we may calculate these quantities about trace C. By a direct calculation, we have

$$\begin{aligned}\Xi_1(\boldsymbol{u}) &:= (1/t)_{u_1} = f_1^{(3)}(\varphi_1)^2 + f_2^{(3)}\varphi_1\varphi_2 + f_1''\varphi_{11} + f_2''\varphi_{12},\\ \Xi_2(\boldsymbol{u}) &:= (1/t)_{u_2} = f_1^{(3)}\varphi_1\varphi_2 + f_2^{(3)}(\varphi_2)^2 + f_1''\varphi_{12} + f_2''\varphi_{22}\end{aligned}$$

and

$$\begin{aligned}\Xi_3(\boldsymbol{u}) :=\ & \det \mathrm{Hess}\,(1/t)\\ =\ & f_1^{(4)}[f_1^{(3)}(\varphi_1)^2((\varphi_1)^2\varphi_{22} - 2\varphi_1\varphi_2\varphi_{12} + (\varphi_2)^2\varphi_{11})\\ &\quad + f_1''\varphi_1((\varphi_1)^2\varphi_{122} - 2\varphi_1\varphi_2\varphi_{112} + (\varphi_2)^2\varphi_{111})\\ &\quad + f_2^{(3)}\varphi_1\varphi_2((\varphi_1)^2\varphi_{22} - 2\varphi_1\varphi_2\varphi_{12} + (\varphi_2)^2\varphi_{11})\\ &\quad + f_2''\varphi_1((\varphi_1)^2\varphi_{222} - 2\varphi_1\varphi_2\varphi_{122} + (\varphi_2)^2\varphi_{112})]\\ & + f_2^{(4)}[f_1^{(3)}\varphi_1\varphi_2((\varphi_1)^2\varphi_{22} - 2\varphi_1\varphi_2\varphi_{12} + (\varphi_2)^2\varphi_{11})\\ &\quad + f_1''\varphi_2((\varphi_1)^2\varphi_{122} - 2\varphi_1\varphi_2\varphi_{112} + (\varphi_2)^2\varphi_{111})\\ &\quad + f_2^{(3)}(\varphi_2)^2((\varphi_1)^2\varphi_{22} - 2\varphi_1\varphi_2\varphi_{12} + (\varphi_2)^2\varphi_{11})\\ &\quad + f_2''\varphi_2((\varphi_1)^2\varphi_{222} - 2\varphi_1\varphi_2\varphi_{122} + (\varphi_2)^2\varphi_{112})]\\ & + (f_1^{(3)})^2[\varphi_1\varphi_{11}(3\varphi_1\varphi_{22} + 2\varphi_2\varphi_{12}) - 4(\varphi_1)^2(\varphi_{12})^2 - (\varphi_2)^2(\varphi_{11})^2]\\ & + (f_2^{(3)})^2[\varphi_2\varphi_{22}(2\varphi_1\varphi_{12} + 3\varphi_2\varphi_{11}) - (\varphi_1)^2(\varphi_{22})^2 - 4(\varphi_2)^2(\varphi_{12})^2]\\ & + f_1^{(3)}f_2^{(3)}[-2(\varphi_1)^2\varphi_{12}\varphi_{22} - 4\varphi_1\varphi_2(\varphi_{12})^2\\ &\quad + 8\varphi_1\varphi_2\varphi_{11}\varphi_{22} - 2(\varphi_2)^2\varphi_{11}\varphi_{12}]\\ & + f_1^{(3)}[f_1''(3\varphi_1\varphi_{11}\varphi_{122} - 4\varphi_1\varphi_{12}\varphi_{112}\\ &\quad - 2\varphi_2\varphi_{11}\varphi_{112} + \varphi_1\varphi_{22}\varphi_{111} + 2\varphi_2\varphi_{12}\varphi_{111})\\ &\quad + f_2''(-4\varphi_1\varphi_{12}\varphi_{122} + 3\varphi_1\varphi_{11}\varphi_{222}\\ &\quad - 2\varphi_2\varphi_{11}\varphi_{122} + \varphi_1\varphi_{22}\varphi_{112} + 2\varphi_2\varphi_{12}\varphi_{112})]\\ & + f_2^{(3)}[f_1''(2\varphi_1\varphi_{12}\varphi_{122} + \varphi_2\varphi_{11}\varphi_{122}\\ &\quad - 2\varphi_1\varphi_{22}\varphi_{112} - 4\varphi_2\varphi_{12}\varphi_{112} + 3\varphi_2\varphi_{22}\varphi_{111})\\ &\quad + f_2''(2\varphi_1\varphi_{12}\varphi_{222} - 2\varphi_1\varphi_{22}\varphi_{122}\\ &\quad - 4\varphi_2\varphi_{12}\varphi_{122} + \varphi_2\varphi_{11}\varphi_{222} + 3\varphi_2\varphi_{22}\varphi_{112})]\\ &\quad + (f_1'')^2(\varphi_{111}\varphi_{122} - (\varphi_{112})^2) + (f_2'')^2(\varphi_{112}\varphi_{222} - (\varphi_{122})^2)\\ &\quad + f_1''f_2''(\varphi_{111}\varphi_{222} - \varphi_{112}\varphi_{122}),\end{aligned}$$

where for the sake of simplicity, we set

$$f'_\ell := \frac{df_\ell}{dy},\ f''_\ell := \frac{d^2 f_\ell}{dy^2},\ f_\ell^{(m)} := \frac{d^m f_\ell}{dy^m},\ (\ell = 1, 2,\ m = 3, 4)$$

$$\varphi_i := \frac{\partial \varphi}{\partial u_i},\ \varphi_{ij} := \frac{\partial^2 \varphi}{\partial u_i u_j},\ \text{and}\ \varphi_{ijk} := \frac{\partial^3 \varphi}{\partial u_i u_j u_k},\ (i, j, k = 1, 2).$$

Next we consider a map

$$j^4(f_1, f_2, \varphi) : (y, \boldsymbol{u}) \mapsto (j^4 f_1(y), j^4 f_2(y), j^4 \varphi(\boldsymbol{u})) \in J^4(\boldsymbol{R}, \boldsymbol{R})^2 \times J^4(\boldsymbol{R}^2, \boldsymbol{R})$$

and four subsets of jet spaces $J^4(\boldsymbol{R}, \boldsymbol{R})^2 \times J^4(\boldsymbol{R}^2, \boldsymbol{R})$ as follows:

$$\hat{\Xi}_0 := \{j^4(f_1, f_2, \varphi)(y, \boldsymbol{u}) \mid y - \varphi(\boldsymbol{u}) = 0\}$$
$$\hat{\Xi}_1 := \{j^4(f_1, f_2, \varphi)(y, \boldsymbol{u}) \mid \Xi_1(\boldsymbol{u}) = 0\}$$
$$\hat{\Xi}_2 := \{j^4(f_1, f_2, \varphi)(y, \boldsymbol{u}) \mid \Xi_2(\boldsymbol{u}) = 0\}$$
$$\hat{\Xi}_3 := \{j^4(f_1, f_2, \varphi)(y, \boldsymbol{u}) \mid \Xi_3(\boldsymbol{u}) = 0\}.$$

Since the coordinate system of $J^4(\boldsymbol{R}, \boldsymbol{R})^2 \times J^4(\boldsymbol{R}^2, \boldsymbol{R})$ is defined by each coordinate of source and value of derivatives of functions, $\hat{\Xi}_0$, $\hat{\Xi}_1$, $\hat{\Xi}_2$ and $\hat{\Xi}_3$ are algebraic subsets with respect to the coordinates of $J^4(\boldsymbol{R}, \boldsymbol{R})^2 \times J^4(\boldsymbol{R}^2, \boldsymbol{R})$. Comparing the coefficients of φ_{11} and φ_{22} in Ξ_1 and Ξ_2, we see that Ξ_1 and Ξ_2 do not have a common factor. Moreover, $f_1'' \varphi_1 (\varphi_2)^2$ is the coefficient of $\varphi_{111} f_1^{(4)}$ of Ξ_3, but this does not appear in either Ξ_1 or Ξ_2. Hence $S := \cap_{i=0}^{3} \hat{\Xi}_i$ is a closed algebraic subset with codimension 4 in $J^4(\boldsymbol{R}, \boldsymbol{R})^2 \times J^4(\boldsymbol{R}^2, \boldsymbol{R})$. So this set has a standard stratification. Applying the Thom jet transversality theorem to $j^4(f_1, f_2, \varphi)$ and S, there exists a residual subset $\mathcal{O} \subset C^\infty(\boldsymbol{R}, \boldsymbol{R})^2 \times C^\infty(\boldsymbol{R}^2, \boldsymbol{R})$ such that for any $(f_1, f_2, \varphi) \in \mathcal{O}$, the map $j^4(f_1, f_2, \varphi)$ is transverse to S. Since the codimension of S is 4, transversal condition means having no intersection point. If g_{t_0} at $\boldsymbol{u}_0$ is the beaks, there is a singularity of $g_{t-\varepsilon}$ near $\boldsymbol{u}_0$ for a sufficiently small number ε, the beaks never appear at the minimal value of t which g_t is singular. Thus $(f_1, f_2, \varphi) \in \mathcal{O}$ satisfies the desired condition. $\square$

Acknowledgement

The author would like to thank Professors Goro Akagi, Shyuichi Izumiya and Farid Tari for fruitful discussions and helpful advices. The author is grateful to the referee for careful reading and helpful comments.

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