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Stability and bifurcation of periodic travelling waves in a derivative non-linear Schrödinger equation

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Abstract. We study periodic travelling wave solutions of a derivative non-linear Schrödinger equation and show the existence of infinitely many families of semi-trivial solutions (Theorem 2). Each of the families constitutes a branch of travelling waves corresponding to a non-zero integer called the winding number. A sufficient condition for the orbital stability of travelling waves on the branches with positive winding number is given in terms of the wave speed and winding number of the solution (Theorem 3). Bifurcation points are found on each semi-trivial branch of travelling wave solutions (Theorem 4), and the qualitative, and approximately quantitative, orbit-shapes of the bifurcated solutions are given. The stability of the semi-trivial solutions under subharmonic perturbations is studied in Theorem 6, and subharmonic bifurcations are established in Theorem 7.

1. Introduction and main results

In this paper we study the derivative non-linear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}|u|^2 u_x = 0, \tag{1.1}$$

where $u(t,x)$ is a complex-valued function, $\mathrm{i} = \sqrt{-1}$, and the subscripts stand for partial differentiation. This equation is treated under the 2π-periodic boundary condition for the spatial variable x, namely, with $x \in S^1 = \mathbf{R}/(2\pi\mathbf{Z})$, and $t \in \mathbf{R}$.

The following equation related to (1.1),

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2 u)_x = 0 \tag{1.2}$$

was derived in [12] as a compatibility condition within a reductive perturbation expansion. The well-posedness of the initial value problem for (1.2) on $H^s(\mathbf{R},\mathbf{C})$ was studied by [9, 10, 5, 7, 8] for various values of $s > 1/2$. A family of solitary wave solutions of (1.2), belonging to $H^s(\mathbf{R},\mathbf{C})$ $(s > 0)$, was

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found by [11]. The orbital stability of these solitary waves was studied by [4] and completely settled affirmatively by [1]. The method of proof in [1] is very much interesting. The original equation (1.2) is first transformed into the form of (1.1) by the so called gauge transformation. Then the authors of [1] use ingenious variational techniques to establish the orbital stability of the solitary waves found by [11].

One of the reasons of transforming (1.2) into the form of (1.1) is that the latter has a nice (infinite dimensional) Hamiltonian structure. This structure is usefully exploited to prove stability properties of solitary waves. Although (1.2) also has a Hamiltonian structure, it is not easy to deal with in the sense that the linearlized Hamiltonian is indefinite with infinite dimensional positive and negative subspaces. As we will see below, (1.1) and (1.2) share families of travelling wave solutions (the families shown in Theorem 2, below), but their stability properties could be drastically different.

For more details of the origin and physical significance of (1.2), as well as its basic relations to other types of derivative non-linear Schrödinger equation posed on the spatial domain $\mathbf{R}$, we refer to [12, 1, 7].

The local well-posedness of the initial value problem for (1.1) with periodic boundary condition was obtained by Herr in the following theorem.

Theorem 1 (Herr, [6]). *For each* $\mu > 0$ *there exists* $t_0 = t_0(\mu) > 0$ *such that* (1.1) *has a unique solution* $u \in C([-t_0, t_0], H^1(S^1))$ *satisfying the initial condition* $u(0, x) = u_0(x)$ *for all* $u_0 \in H^1(S^1, \mathbf{C})$ *with* $\|u_0\|_{H^1} < \mu$.

The well-posedness established in [6] is for the initial value problem of (1.2) with periodic boundary condition. However, the method of proof applies to (1.1) as well. In fact, (1.2) is first transformed by a periodic version of gauge transform to an equation similar to (1.1), and then various estimates are obtained for the transformed equation to establish well-posedness. Therefore, we state the results in [6] as cited in Theorem 1 which is suitable for our problem.

We are interested in the travelling wave solutions of (1.1) and their stability properties. A solution of (1.1) of the form $u(t, x) = \phi(x - ct)$ with $c \in \mathbf{R}$ is called a 2π-*periodic travelling wave* with *profile* $\phi(\cdot)$ and *velocity* c, when $\phi(x)$ is 2π-periodic in x. The profile is governed by

$$0 = -\phi_{xx} + \mathrm{i}(c - |\phi|^2)\phi_x, \quad \phi(x + 2\pi) = \phi(x) \qquad \forall x \in \mathbf{R}. \tag{1.3}$$

It will turn out that the set of solutions of (1.3) has a rich structure (cf. Theorems 2, 4 and 7, below). We first state the existence of solutions with a simple profile.

Theorem 2 (Existence). *For each* $\ell \in \mathbf{Z} \backslash \{0\}$, (1.1) *has a family of* 2π-*periodic travelling wave solutions* $u(t, x) = \phi_\ell^c(x - ct)$ *defined for* $c > \ell$ *and given* *explicitly by*

$$\phi_\ell^c(x) = \sqrt{c-\ell}\,\mathrm{e}^{\mathrm{i}\ell x}.$$

The proof of this theorem is easy. Just substitute $u(t,x) = \phi_\ell^c(x-ct)$ into (1.1) to find it is a solution.

Note that for each $\ell \in \mathbf{R}\setminus\{0\}$, ϕ_ℓ^c (with $c > \ell$) remains a travelling wave solution of (1.1). However, it is not 2π-periodic unless ℓ is an integer.

The solutions $\phi(x)$ of (1.3) satisfying $\phi(x) \neq 0$ for all $x \in \mathbf{R}$ are expressed as $\phi(x) = r(x)\mathrm{e}^{\mathrm{i}\theta(x)}$, where $r(x) > 0$ and $\theta(x)$ are real valued functions. The 2π-periodicity of $\phi(x)$ implies that for $x \in \mathbf{R}$ we have

$$r(x) = r(x+2\pi),\ \ \theta_x(x) = \theta_x(x+2\pi)\ \text{ and }\ \theta(x+2\pi) = \theta(x) + 2\pi\ell$$

for some integer ℓ. The number ℓ is called the *winding number* of ϕ. If, moreover, the minimal period of $r(x)$ is $2\pi/k$ with $k \in \mathbf{N}$, or $k = 0$ if $r(x) \equiv$ constant, then the solution $\phi(x)$ is said to have the amplitude *modulation number* k, and in this case ϕ is called an (ℓ,k)-type solution. Theorem 2 asserts the existence of the branches of $(\ell,0)$-type solutions for all $\ell \in \mathbf{Z}\setminus\{0\}$. We call the family $B_\ell := \{\phi_\ell^c \,|\, c > \ell\}$ the ℓ-branch of solutions. Theorem 2 may be interpreted as follows. When c passes $\ell \in \mathbf{Z}\setminus\{0\}$, solutions with winding number ℓ of (1.1) bifurcate from the trivial solution $u \equiv 0$ (cf. Remark 1 at the end of § 3).

Our next interest is the stability of the semi-trivial solutions ϕ_ℓ^c with respect to (1.1). Before we state our results, let us give a formal definition of orbital stability.

Definition 1. A periodic travelling wave solution $u(t,x) = U(x-ct)$ of (1.1) is *orbitally stable* for (1.1), if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(S^1)$ satisfies $\inf_{x\in\mathbf{R}} \|u_0 - U(\cdot - x)\|_{H^1} < \delta$, then the solution $u(t,x)$ of (1.1) with $u(0,x) = u_0(x)$ exists globally in time and satisfies

$$\sup_{t\geq 0}\inf_{x\in\mathbf{R}} \|u(t) - U(\cdot - x)\|_{H^1} < \varepsilon. \tag{OS}$$

When $U(x-ct)$ is not orbitally stable, it is called *orbitally unstable*.

The meaning of this definition is as follows. If a solution of (1.1) starts from a sufficiently small neighborhood of the *orbit*$\{U(\cdot - x) \in H^1 \,|\, x \in \mathbf{R}\}$ of the wave profile U, then the solution exists globally in time and remains in a small neighborhood of the *orbit*. Note that for $U = \phi_\ell^c$ the condition (OS) in the last definition may be replaced by

$$\sup_{t\geq 0}\inf_{(\theta,x)\in\mathbf{R}^2} \|u(t) - \mathrm{e}^{\mathrm{i}\theta}U(\cdot - x)\|_{H^1} < \varepsilon, \tag{OS$'$}$$

because $\mathrm{e}^{\mathrm{i}\theta}U(\cdot - x) = U(\cdot - (x - \theta/\ell))$ in this case.

Our result on the stability is given by

Theorem 3 (Stability). *For each integer* $\ell \geq 1$, *the semi-trivial solution* ϕ_ℓ^c *is orbitally stable if* $c > \ell + \dfrac{\ell^2 - 1}{2\ell}$.

This theorem says that for each positive integer ℓ the solution on the branch B_ℓ is stable if its travelling speed is sufficiently large. In particular, solutions on B_1 are always stable. At present, we do not know whether solutions with slower speed (for $\ell \geq 2$) or solutions on B_ℓ with $\ell \leq -1$ are stable or not. The stability and instability theorems in [2, 3], which are mainly developed for solitary pulse waves, are not powerful enough to deal with other cases omitted in Theorem 3. It is easy to verify that ϕ_ℓ^c is also a travelling wave solution of (1.2). However, we do not have any information on the stability properties of ϕ_ℓ^c under the flow of (1.2).

Theorem 4 (Bifurcation). *For* $\ell \in \mathbf{Z} \setminus \{0, 1\}$, *solutions of* (1.3) *bifurcate from the branch* B_ℓ.

(i) *For* $\ell \geq 2$ *and for each* $k \in \{1, \dots, \ell - 1\}$, (ℓ, k)*-type solutions bifurcate from* B_ℓ *at* $c = \ell + (\ell^2 - k^2)/(2\ell)$. *The bifurcated branch exists locally for* $c > \ell + (\ell^2 - k^2)/(2\ell)$.

(ii) *For* $\ell \leq -1$ *and for each* $k \in \mathbf{N}$ $(k > |\ell|)$, (ℓ, k)*-type solutions bifurcate from* B_ℓ *at* $c = \ell + (\ell^2 - k^2)/(2\ell)$. *The bifurcated branch exists locally*

$$\textit{for } c > \ell + (\ell^2 - k^2)/(2\ell) \textit{ if } |\ell| < k < \sqrt{3 + 2\sqrt{3}}|\ell|, \textit{ and}$$

$$\textit{for } c < \ell + (\ell^2 - k^2)/(2\ell) \textit{ if } k > \sqrt{3 + 2\sqrt{3}}|\ell|.$$

The bifurcated (ℓ, k)-type solution is expressed as

$$\phi(x) = r(x) e^{\mathrm{i}(\ell x + \theta(x))}.$$

For the approximate shapes of $r(x)$ and $\theta(x)$, we refer to §4.

There is no bifurcation point on B_1. On each branch B_ℓ with positive winding number ℓ $(\ell \geq 2)$, there are finitely many bifurcation points and the solutions on these branches eventually become orbitally stable. On the other hand, on each branch with negative winding number, there are infinitely many bifurcation points. It is very much likely that the ℓ-winding number solutions in the cases not covered by Theorem 3 are all unstable. As regard to stability of, and bifurcation from, the branches B_ℓ, we will give additional comments in §5.

2. Abstract formulation of the problem

The equation (1.1) is also written as

$$u_t = \mathrm{i}u_{xx} - |u|^2 u_x = -\mathrm{i}(-u_{xx} - \mathrm{i}|u|^2 u_x),$$

which suggests to express the equation as a Hamiltonian PDE;

$$u_t = JE'(u) = -\mathrm{i}E'(u), \tag{2.1}$$

where $J = -\mathrm{i}$ and E is a C^2-functional defined on a real Hilbert space. To cast (1.1) in this formalism is useful to determine orbital stability of travelling wave solutions of (1.1) within the framework developed by Grillakis-Shatah-Strauss [2, 3]. An appropriate Hilbert space in this context is $H^1 = H^1(S^1, \mathbf{C})$, considered as a *real* Hilbert space with the inner product defined by

$$(u, v) := \mathrm{Re} \int_{S^1} u_x \bar{v}_x + \mathrm{Re} \int_{S^1} u\bar{v}. \tag{2.2}$$

The dual space of H^1 is $H^{-1} = H^{-1}(S^1, \mathbf{C})$, and we denote by $\langle f, u \rangle$ the pairing between $f \in H^{-1}$ and $u \in H^1$. Then, thanks to the Riesz' representation theorem, there is a natural isomorphism $I : H^1 \to H^{-1}$, formally defined by $Iu = -u_{xx} + u$, so that

$$\langle Iu, v \rangle = (u, v) \qquad \forall u, v \in H^1, \tag{2.3}$$

and for each $f \in H^{-1}$ there exists a unique u that satisfies $f = Iu$. For $f \in H^{-1}$ with $f = Iu \in L^2 = L^2(S^1, \mathbf{C})$, we have

$$\langle f, v \rangle = \mathrm{Re} \int_{S^1} (u_x \bar{v}_x + u\bar{v}) = \mathrm{Re} \int_{S^1} (-u_{xx} + u)\bar{v} = \mathrm{Re} \int_{S^1} f\bar{v}. \tag{2.4}$$

Let us now define the C^2-functional $E : H^1 \to \mathbf{R}$ by

$$E(u) = \frac{1}{2} \int_{S^1} |u_x|^2 + \frac{1}{4} \mathrm{Im} \int_{S^1} |u|^2 \bar{u} u_x. \tag{2.5}$$

By the Sobolev embedding theorem the functional $E(u)$ is well-defined for $u \in H^1$, and it is then easy to verify that the derivative $E' : H^1 \to H^{-1}$ is given by

$$E'(u) = Iu - u - \mathrm{i}|u|^2 u_x = -u_{xx} - \mathrm{i}|u|^2 u_x.$$

Note that $J : H^{-1} \to H^1$ defined by $Jf = -\mathrm{i}f$ with $f \in D(J) = H^1$ is skew-symmetric in the following sense;

$$\langle g, Jf \rangle = -\langle f, Jg \rangle \qquad \forall f, g \in H^1 = D(J).$$

This is an easy consequence of (2.4). Therefore, (1.1) is indeed written as the Hamiltonian PDE (2.1).

A travelling wave solution for (1.1) with velocity $c \in \mathbf{R}$ is a solution of the form $u(t,x) = \phi(x - ct)$, in which ϕ is called a wave *profile*. If a one-parameter group of unitary operators $\{T(s); s \in \mathbf{R}\}$ on H^1 is defined by

$$(T(s)u)(x) = u(x - s), \tag{2.6}$$

then a travelling wave solution with profile ϕ corresponds to a group orbit through ϕ, namely, $u(t,x) = (T(ct)\phi)(x)$. We call such a solution a *bound state* of (1.1), according to the terminology in [2, 3]. Then, ϕ is a profile of a bound state if and only if it satisfies $E'(\phi) + cQ'(\phi) = 0$, where $Q : H^1 \to \mathbf{R}$ is defined by

$$Q(u) = \frac{1}{2} \operatorname{Im} \int_{S^1} u\bar{u}_x \quad \text{for } u \in H^1 \qquad \text{and} \qquad Q'(u) = \mathrm{i}u_x. \tag{2.7}$$

That is to say, ϕ is the profile of a bound state, if and only if it is a critical point of the functional $H^1 \ni u \mapsto E(u) + cQ(u)$. The critical points are to satisfy

$$0 = -\phi_{xx} + \mathrm{i}(c - |\phi|^2)\phi_x, \qquad \phi \in H^1(S^1, \mathbf{C}), \tag{2.8}$$

which is equivalent to (1.3). It is easy to verify that ϕ_ℓ^c (given in Theorem 2) satisfies (2.8), or (1.3). The value of the functional $E(u) + cQ(u)$ on the critical point is given, as verified by a direct computation, by

$$d_\ell(c) := E(\phi_\ell^c) + cQ(\phi_\ell^c) = -\frac{\pi}{2}\ell(c - \ell)^2, \tag{2.9}$$

which plays an important role to determine stability or instability within the framework of [2, 3]. Note that if $\ell < 0$ (resp. $\ell > 0$), d_ℓ is convex (resp. concave) in c everywhere it is defined.

We will now confirm that conditions stated in the abstract setup of [2] are valid for our problem. Since the group action $T(s)$ is nothing but a spatial translation by s, we have

$$E(T(s)u) = E(u), \quad Q(T(s)u) = Q(u) \qquad \text{for } s \in \mathbf{R},\, u \in H^1. \tag{2.10}$$

To verify

$$T(s)J = JT(-s)^*, \tag{2.11}$$

we have to show that

$$\langle f, T(s)Jg \rangle = \langle f, JT(-s)^*g \rangle$$

holds for all $f \in H^{-1}$ and $g \in D(J) = H^1$. Thanks to (2.4) and using change of variables in the integration, this is certainly true for $f \in H^{-1} \cap L^2$. The density of L^2 in H^{-1} now establishes (2.11). Note that the linear operator $B : H^1 \to H^{-1}$ defined by $Bu := Q'(u) = \mathrm{i}u_x$ is bounded and self-adjoint ($B^* = B$). Since $T'(0)u = -u_x$ with $D(T'(0)) = H^2$, and our Q corresponds to $-Q$ in [2], we also have that

$$-JB \text{ is an extension of } T'(0). \tag{2.12}$$

This follows from

$$-JBu = -JQ'(u) = -(-\mathrm{i})(\mathrm{i}u_x) = -u_x = T'(0)u \qquad \forall u \in H^2.$$

Thanks to (2.10), (2.11), (2.12) and Theorem 2, all of the standing hypotheses in [2] are fulfilled in our problem for each $\ell \in \mathbf{Z}\backslash\{0\}$.

3. Linearized Hamiltonian

In this section we give a proof of Theorem 3. To study stability of the bound state ϕ_ℓ^c, we linearize $\phi \mapsto E'(\phi) + cQ'(\phi)$, not $J(E'(\phi) + cQ'(\phi))$, around $\phi = \phi_\ell^c$. This linearization, denoted by $\mathscr{H}$, is called the linearized Hamiltonian, given by

$$\mathscr{H} = \mathscr{H}_\ell^c := E''(\phi_\ell^c) + cQ''(\phi_\ell^c) : H^1 \to H^{-1}.$$

This is a self-adjoint operator in the sense that

$$\langle \mathscr{H}u, v\rangle = \langle \mathscr{H}v, u\rangle \qquad \forall u, v \in H^1, \tag{3.1}$$

which directly follows from the fact that $\mathscr{H}$ is the second derivative of the C^2-functional $E(u) + cQ(u)$. Eigenvalues λ of $\mathscr{H}$ are real and satisfy, together with the corresponding eigenfunctions $u \in H^1$ ($u \not\equiv 0$), the following equation

$$\lambda u = -u_{xx} + \mathrm{i}(c - r^2)u_x + \ell r^2 u + \ell r^2 \mathrm{e}^{2\mathrm{i}\ell x}\bar{u}, \tag{3.2}$$

where $r = \sqrt{c - \ell}$. Since the operator $\mathscr{H}$ is self-adjoint, its spectrum consists entirely of real eigenvalues. For more details, we have the following proposition.

Proposition 1. *For each $\ell \in \mathbf{Z}\backslash\{0\}$ and $c > \ell$, the eigenvalues λ of $\mathscr{H}_\ell^c$ are given by*

$$\lambda = \lambda_{\ell,\pm}^{c,n} := (n - \ell)^2 + \ell(c - \ell) \pm \ell\sqrt{(n - \ell)^2 + (c - \ell)^2}, \qquad n \in \mathbf{Z},$$

and the following items are true.

(i) *The eigenvalue $\lambda_{\ell,-}^{c,\ell} = 0$ corresponds to an eigenfunction given by*

$$\phi_{\ell,-}^{c,\ell}(x) = \mathrm{i}\mathrm{e}^{\mathrm{i}\ell x} \text{ which comes from } \mathrm{d}\phi_\ell^c(x)/\mathrm{d}x,$$

while the eigenvalue $\lambda_{\ell,+}^{c,\ell} = 2\ell(c-\ell)$ *corresponds to an eigenfunction given by*

$$\phi_{\ell,+}^{c,\ell}(x) = \mathrm{e}^{\mathrm{i}\ell x} \text{ which comes from } \partial\phi_\ell^c(x)/\partial c.$$

(ii) *For each* $n \neq \ell$, *eigenfunctions* $\psi_{\ell,\pm}^{c,n}$ *corresponding to* $\lambda_{\ell,\pm}^{c,n}$ *are given by*

$$\psi_{\ell,+,\mathrm{r}}^{c,n}(x) = (c-\ell)\mathrm{e}^{\mathrm{i}nx} + \left[\sqrt{(n-\ell)^2 + (c-\ell)^2} + (\ell - n)\right]\mathrm{e}^{\mathrm{i}(2\ell-n)x},$$

$$\psi_{\ell,+,\mathrm{i}}^{c,n}(x) = -\mathrm{i}(c-\ell)\mathrm{e}^{\mathrm{i}nx} + \mathrm{i}\left[\sqrt{(n-\ell)^2 + (c-\ell)^2} + (\ell - n)\right]\mathrm{e}^{\mathrm{i}(2\ell-n)x},$$

$$\psi_{\ell,-,\mathrm{r}}^{c,n}(x) = (c-\ell)\mathrm{e}^{\mathrm{i}nx} + \left[-\sqrt{(n-\ell)^2 + (c-\ell)^2} + (\ell - n)\right]\mathrm{e}^{\mathrm{i}(2\ell-n)x},$$

$$\psi_{\ell,-,\mathrm{i}}^{c,n}(x) = -\mathrm{i}(c-\ell)\mathrm{e}^{\mathrm{i}nx} + \mathrm{i}\left[-\sqrt{(n-\ell)^2 + (c-\ell)^2} + (\ell - n)\right]\mathrm{e}^{\mathrm{i}(2\ell-n)x},$$

and the geometric, as well as algebraic, multiplicity of these eigenvalues is 2.

(iii) *For each* $\ell \geq 1$;
(iii-a) $\lambda_{\ell,+}^{c,n} \geq \lambda_{\ell,+}^{c,\ell} = 2\ell(c-\ell) > 0$ *for all* $n \in \mathbf{Z}$, $c > \ell$.
(iii-b) *If* $c > \ell + \dfrac{\ell^2 - 1}{2\ell}$, *then* $\lambda_{\ell,-}^{c,n} > 0$ *for all* $n \in \mathbf{Z}\backslash\{\ell\}$.

(iv) *For* $\ell \geq 2$ *and* $k \in \{1, \dots, \ell-1\}$, *there are two zero eigenvalues* $\lambda_{\ell,-}^{c,\ell-k} = 0 = \lambda_{\ell,-}^{c,\ell+k}$ *at* $c = \ell + \dfrac{\ell^2 - k^2}{2\ell}$, *except for the trivial one* $\lambda_{\ell,-}^{c,\ell} = 0$. *In this case, the* 0*-eigenvalue has geometric, as well as algebraic, multiplicity* 3, *and the eigenspace is spanned by*

$$\mathrm{i}\mathrm{e}^{\mathrm{i}\ell x},$$
$$(\ell+k)\mathrm{e}^{\mathrm{i}(\ell-k)x} - (\ell-k)\mathrm{e}^{\mathrm{i}(\ell+k)x},$$
$$\mathrm{i}(\ell+k)\mathrm{e}^{\mathrm{i}(\ell-k)x} + \mathrm{i}(\ell-k)\mathrm{e}^{\mathrm{i}(\ell+k)x}.$$

(v) *For each* $\ell \geq 2$ *and* $k \in \{1, \dots, \ell-1\}$, *if the wave speed satisfies*

$$\ell + \frac{\ell^2 - k^2}{2\ell} > c > \ell + \frac{\ell^2 - (k+1)^2}{2\ell},$$

then there is no 0*-eigenvalue except for the trivial one* $\lambda_{\ell,-}^{c,\ell}$ *and there are* $2k$ *negative eigenvalues (counted with multiplicity)* $\lambda_{\ell,-}^{c,\ell\pm m}$ *with* $m = 1, \dots, k$.

(vi) *If* $\ell \leq -1$, *then* $\lambda_{\ell,-}^{c,n} > \lambda_{\ell,-}^{c,\ell} = 0$ *for all* $n \in \mathbf{Z}\backslash\{\ell\}$ *and there are at least* $2|\ell| + 1$ *negative eigenvalues* $\lambda_{\ell,+}^{c,\ell\pm k} < 0$ *with* $k = 0, 1, \dots,$

$-\ell$. The dimension of the sum of the eigenspaces corresponding to these eigenvalues equals $2|\ell|+1$.

(vii) *If* $\ell \le -1$ *and* $\mathbf{N} \ni k > |\ell|$, *then* $\lambda_{\ell,+}^{c,\ell \pm k} = 0$ *at* $c = \ell + \dfrac{\ell^2 - k^2}{2\ell}$. *The dimension of the* 0*-eigenspace is* 3 *and it is spanned by*

$$\begin{gathered} \mathrm{i}e^{\mathrm{i}\ell x}, \\ (\ell+k)e^{\mathrm{i}(\ell-k)x} - (\ell-k)e^{\mathrm{i}(\ell+k)x}, \\ \mathrm{i}(\ell+k)e^{\mathrm{i}(\ell-k)x} + \mathrm{i}(\ell-k)e^{\mathrm{i}(\ell+k)x}. \end{gathered}$$

(viii) *If* $\ell \le -1, \mathbf{N} \ni k > |\ell|$ *and* $\ell + \dfrac{\ell^2 - (k+1)^2}{2\ell} > c > \ell + \dfrac{\ell^2 - k^2}{2\ell}$, *then* $\lambda_{\ell,+}^{c,n} \ne 0$ *for all* $n \in \mathbf{Z}$, *and there are* $2k+1$ *negative eigenvalues* $\lambda_{\ell,+}^{c,\ell \pm m} < 0$ *with* $m = 0, \ldots, k$. *The dimension of the sum of the eigenspaces corresponding to these negative eigenvalues is* $2k+1$.

The proposition will be proven at the end of the present section.

We are now ready to prove Theorem 3 by applying the stability criterion in [2], which is stated as follows.

Theorem 5 (Theorem 1 in [2]). *Assume that the initial value problem for* (1.1) *is locally well-posed on* H^1. *If the operator* $\mathscr{H}_\ell^c$ *has the following properties, then the travelling wave solution* ϕ_ℓ^c *is orbitally stable.*

(i) *The kernel of* $\mathscr{H}_\ell^c$ *is spanned by* $T'(0)\phi_\ell^c$, *and* 0 *is a simple eigenvalue of* $\mathscr{H}_\ell^c$.

(ii) *The spectrum of* $\mathscr{H}_\ell^c$ *other than* 0 *is positive and bounded away from zero.*

We roughly show the outline of proof of this theorem, since it is proven as a special case of a more general stability criterion in [2]. However, the idea is simple and clear. The conditions (i) and (ii) in Theorem 5 say that in the orthogonal complement of the kernel of $\mathscr{H}_\ell^c$ (Ker $\mathscr{H}_\ell^c$ is one-dimensional and parallel to the orbit of ϕ_ℓ^c), the linearized Hamiltonian $\mathscr{H}_\ell^c$ is positive definite. Since ϕ_ℓ^c is a critical point of the functional $\phi \mapsto E(\phi) + cQ(\phi)$, this positivity implies the existence of a constant $C > 0$ such that for a sufficiently small neighborhood $\mathscr{U}$ of the orbit through ϕ_ℓ^c,

$$E(v) + cQ(v) - (E + cQ)|_{\mathscr{O}(\phi_\ell^c)} \ge C[\mathrm{dist}_{H^1}(v, \mathscr{O}(\phi_\ell^c))]^2 \qquad \forall v \in \mathscr{U}, \quad \text{(COE)}$$

where $\mathscr{O}(\phi)$ stands for the orbit of ϕ under the group action of $T(s)$ and dist_{H^1} is the distance measured by H^1-norm. Note also that the value of the functional $E + cQ$ is constant on $\mathscr{O}(\phi_\ell^c)$. Since the functionals E and Q are conserved under the flow of (1.1), if the solution starts from the neighborhood $\mathscr{U}$, then it cannot escape from the ε-neighborhood of $\mathcal{O}(\phi_\ell^c)$ because of the condition (COE), where $\varepsilon > 0$ is defined by

$$\varepsilon^2 := \frac{1}{C} \sup\{E(v) + cQ(v) - [E + cQ]|_{\mathcal{O}(\phi_\ell^c)} \mid v \in \mathscr{U}\}.$$

Probably, THEOREM 5 had been known to many people even before [2]. However, the true merit of [2] is that it gives stability and instability criteria when the linearized Hamiltonian has *exactly one negative eigenvalue*. In this case [2] says that the travelling wave solution with profile ϕ_ℓ^c is stable if $d_\ell''(c) \geq 0$, and that it is unstable if $d_\ell''(c) < 0$, where $d_\ell(c)$ is the quantity defined in (2.9). However, in our case not covered by THEOREM 3 with $\ell \geq 2$, the linearized Hamiltonian has more than one negative eigenvalues (cf. PROPOSITION 1) and $d_\ell''(c) < 0$ (cf. (2.9)). The INSTABILITY THEOREM in [3] is of more general character which states that *if the difference between the number of negative eigenvalues of $\mathscr{H}$ and the number of positive eigenvalue of $d_\ell''(c)$ is odd, then the bound state is unstable.* However, this theorem fails to apply to our situation not covered by THEOREM 3, because the difference between the number of negative eigenvalues of $\mathscr{H}$ and the number of positive eigenvalues of $d_\ell''(c)$ is even. This is foreseen, because the instability criterion in [3] is nothing but a sufficient condition for $-\mathrm{i}\mathscr{H}$ to have eigenvalue with non-zero real part, while $-\mathrm{i}\mathscr{H}$ has only pure imaginary eigenvalues in our case.

Thanks to THEOREMS 1 and 2 in §1, together with the properties (2.10), (2.11), (2.12), and $\phi_\ell^c \in C^\infty(S^1)$, the standing hypotheses of [2] are all satisfied. Now, PROPOSITION 1 (i) and (iii) say that the linearized Hamiltonian has its kernel spanned by $T'(0)\phi_\ell^c = -\ell \mathrm{i} \mathrm{e}^{\mathrm{i}\ell x} \neq 0$ and the rest of its spectrum is positive and bounded away from zero for $c > \ell + \dfrac{\ell^2 - 1}{2\ell}$ with $\ell \geq 1$. THEOREM 5 now applies to prove THEOREM 3.

To prove PROPOSITION 1, we substitute the Fourier expansion $u = \sum a_n \mathrm{e}^{\mathrm{i}nx}$ with $a_n \in \mathbf{C}$ into (3.2). Comparing the coefficients of $\mathrm{e}^{\mathrm{i}nx}$ in the resulting equation, we obtain for $n \in \mathbf{Z}$,

$$(\lambda - (n-\ell)^2 - \ell(c-\ell) + \ell(\ell - n))a_n - \ell(c-\ell)\bar{a}_{2\ell-n} = 0.$$

Taking the complex conjugate of this equation and then replacing n by $2\ell - n$, and noting $\lambda \in \mathbf{R}$, (3.2) is found to be equivalent to

$$\begin{cases} [\lambda - (n-\ell)^2 - \ell(c-\ell) + \ell(\ell-n)]a_n - \ell(c-\ell)\bar{a}_{2\ell-n} = 0, \\ -\ell(c-\ell)a_n + [\lambda - (n-\ell)^2 - \ell(c-\ell) - \ell(\ell-n)]\bar{a}_{2\ell-n} = 0. \end{cases} \tag{3.3}$$

This immediately gives the eigenvalue $\lambda_{\ell,\pm}^{c,n}$ given in PROPOSITION 1. The remaining parts of PROPOSITION 1 follow from this and (3.3) by elementary computations, and hence the details are omitted.

REMARK 1. If we linearlize the Hamiltonian around the trivial solution $u = 0$, then $\mathscr{H} u = -u_{xx} + \mathrm{i}cu_x$. The eigenvalues of this operator are given by $\lambda(n, c) = n^2 - cn$, $n \in \mathbf{Z}$. Therefore, as the wave speed c passes integer points $\ell \in \mathbf{Z}\backslash\{0\}$, the eigenvalue crosses $\lambda = 0$. The crossing of eigenvalue causes the bifurcation of ℓ-branch B_ℓ from the zero solution.

4. Bifurcation analysis

THEOREM 4 is proven by PROPOSITION 1 and the Lyapunov-Schmidt Reduction.

4.1. Lyapunov-Schmidt Reduction. We now apply the Lyapunov-Schmidt decomposition. For each $\ell \in \mathbf{Z}\backslash\{0, 1\}$, we denote by c_0 the possible bifurcation point;

$$c_0 = \ell + \frac{\ell^2 - k^2}{2\ell} \qquad \text{for } \begin{cases} k \in \{1, \dots, \ell - 1\} & \text{if } \ell \geq 2 \\ k \in \mathbf{N},\ k > |\ell| & \text{if } \ell \leq -1 \end{cases}$$

and the corresponding amplitude by $r_0 := \sqrt{c_0 - \ell}$. We then write (1.3) as a perturbation around $(\phi, c) = (\phi_\ell^{c_0}, c_0)$ by introducing the new unknown ψ and the new wave speed parameter $\tilde{c}$;

$$\phi = \phi_\ell^{c_0} + \psi, \qquad c = c_0 + \tilde{c}. \tag{4.1}$$

Substituting these into (1.3), we obtain

$$0 = \mathscr{H}\psi + N(\tilde{c}, \psi), \tag{4.2}$$

where $\mathscr{H}$ is the linearization of (1.3) at $\phi = \phi_\ell^{c_0}$ and N is the non-linear perturbation;

$$\begin{aligned} \mathscr{H}\psi &:= -\psi_{xx} + \mathrm{i}\ell\psi_x + \ell r_0^2\psi + \ell r_0^2 \mathrm{e}^{2\mathrm{i}\ell x}\bar{\psi}, \\ N(\tilde{c}, \psi) &:= \ell r_0 \mathrm{e}^{\mathrm{i}\ell x}(-\tilde{c} + |\psi|^2) + \mathrm{i}\tilde{c}\psi_x \\ &\quad - \mathrm{i}r_0 \mathrm{e}^{-\mathrm{i}\ell x}\psi\psi_x - \mathrm{i}r_0 \mathrm{e}^{\mathrm{i}\ell x}\bar{\psi}\psi_x - \mathrm{i}|\psi|^2\psi_x. \end{aligned}$$

In this section, the operators $\mathscr{H}$ and $N(\tilde{c}, \cdot)$ are treated as a mapping from H^2 to L^2. We now decompose (4.2) in terms of the kernel of $\mathscr{H}$ and its orthogonal complement. The kernel of $\mathscr{H}$ is three-dimensional (cf. PROPOSITION 1 (iv) and (vii)) spanned by $\mathrm{i}\mathrm{e}^{\mathrm{i}\ell x}$ and

$$\begin{aligned} \psi_1 &= (\ell + k)\mathrm{e}^{\mathrm{i}(\ell - k)x} - (\ell - k)\mathrm{e}^{\mathrm{i}(\ell + k)x}, \\ \psi_2 &= \mathrm{i}(\ell + k)\mathrm{e}^{\mathrm{i}(\ell - k)x} + \mathrm{i}(\ell - k)\mathrm{e}^{\mathrm{i}(\ell + k)x}. \end{aligned}$$

The 0-eigenfunction $\mathrm{i}\mathrm{e}^{\mathrm{i}\ell x}$ comes from the group action (the spatial translations) on the solution $\phi_\ell^{c_0}$. In order to eliminate the translational freedom, we introduce two function spaces X and Y, defined by

$$X := \left\{ \sum_{n\in\mathbf{Z}} a_n \mathrm{e}^{\mathrm{i}nx} \in H^2 \,\middle|\, a_n \in \mathbf{R} \right\}, \qquad Y := \left\{ \sum_{n\in\mathbf{Z}} a_n \mathrm{e}^{\mathrm{i}nx} \in L^2 \,\middle|\, a_n \in \mathbf{R} \right\}.$$

The space X is chosen because of the following reason.

$$X \subset \{\phi \in H^2(S^1, \mathbf{C}) \mid \phi(0) \in \mathbf{R}\}.$$

This inclusion means that the phase of each element $\phi \in X$ is fixed by requiring $\phi(0) \in \mathbf{R}$, while in general $\phi(0) \in \mathbf{R}$ may not be true for each element $\phi \in H^2$. From the definition of the operators $\mathscr{H}, N(\tilde{c}, \cdot)$ and that of the spaces X, Y, it follows that

$$\mathscr{H} : X \to Y, \qquad N(\tilde{c}, \cdot) : X \to Y.$$

From now on, we always treat $\mathscr{H}, N(\tilde{c}, \cdot)$ as mappings in this sense. The kernel of $\mathscr{H}$ is now 1-dimensional, spanned by ψ_1.

Now, in terms of the inner product on $L^2(S^1, \mathbf{C})$, we have

$$(\psi_1, \psi_1) = \mathrm{Re} \int_{S^1} \psi_1(x) \bar{\psi}_1(x) \mathrm{d}x = 4\pi(\ell^2 + k^2).$$

Let P be defined by

$$(I - P)\psi = \frac{1}{4\pi(\ell^2 + k^2)} (\psi_1, \psi)\psi_1 = A_1 \psi_1 \qquad \text{with } A_1 \in \mathbf{R},$$

and decompose $\psi \in X$ as $\psi = (I - P)\psi + P\psi$. By using $\varphi := (I - P)\psi$, $\Psi := P\psi$, one can rewrite (4.2) as follows.

$$(I - P)N(\tilde{c}, \varphi + \Psi) = 0, \qquad \mathscr{H}\Psi + PN(\tilde{c}, \varphi + \Psi) = 0. \tag{4.3}$$

Thanks to Proposition 1, the linear operator $\mathscr{H}$ is invertible on PX, and hence the implicit function theorem applied to the second equation of (4.3) implies the existence of a function $\Psi = \Psi(\tilde{c}, \varphi) = \Psi(\tilde{c}, A_1)$ defined for $(\tilde{c}, A_1)$ in a neighborhood $\mathcal{O}$ of $(0, 0) \in \mathbf{R}^2$ such that

$$0 \equiv \mathscr{H}\Psi(\tilde{c}, A_1) + N(\tilde{c}, A_1\psi_1 + \Psi(\tilde{c}, A_1)) \qquad \text{on } \mathcal{O}.$$

Substituting $\Psi = \Psi(\tilde{c}, A_1)$ into the first equation of (4.3), we now find that (4.2) is equivalent to

$$0 = (I - P)M(\tilde{c}, A_1), \tag{4.4}$$

where

$$M(\tilde{c}, A_1) := N(\tilde{c}, A_1\psi_1 + \Psi(\tilde{c}, A_1)). \tag{4.5}$$

The bifurcation function in (4.4) is explicitly given in terms of $(\tilde{c}, A_1)$ as follows.

PROPOSITION 2. *Let* $g(\tilde{c}, A_1) = (\psi_1, M(\tilde{c}, A_1))/(4\pi)$. *Then we have*

$$g(\tilde{c}, A_1) = [(\alpha\tilde{c} + \beta\tilde{c}^2) - \gamma A_1^2 + O(3)]A_1, \tag{4.6}$$

where

$$\alpha = 2k^2\ell, \qquad \beta = -\frac{8\ell^4 k^2}{(\ell^2 + k^2)^2}, \qquad \gamma = \frac{k^2}{\ell}(3\ell^4 + 6k^2\ell^2 - k^4),$$

and $O(3)$ *stands for the terms of order higher than or equal to* 3 *in* $(\tilde{c}, A_1)$.

The proof of PROPOSITION 2 is lengthy and explicit computations will be given in §4.2. We give here only some comments on the specific form of (4.6). It is easy to verify $\Psi(\tilde{c}, 0) = (\sqrt{\tilde{c} + c_0 - \ell} - r_0)e^{i\ell x}$ and $M(\tilde{c}, 0) \in \operatorname{span}[e^{i\ell x}]$. Therefore, $(I - P)M(\tilde{c}, 0) = 0$ and $g(\tilde{c}, A_1) = O(|A_1|)$. This explains why there is no $O(1)$-term in A_1 on the right hand side of (4.6). Moreover, $A_1 = 0$ corresponds to the semi-trivial solution $\phi_\ell^{\tilde{c}+c_0} = \phi_\ell^{c_0} + \Psi(\tilde{c}, 0)$.

By using (4.6), we prove THEOREM 4 as follows. We have found via (4.2), (4.3) and (4.4) that the solutions of (1.3) near the bifurcation points $(\phi, c) = (\phi_\ell^{c_0}, c_0)$ correspond, in one-to-one fashion, to the solutions of the equation

$$g(\tilde{c}, A_1) = 0 \qquad \text{for } (\tilde{c}, A_1) \in \mathcal{O}.$$

From (4.6), we find that the equation $g = 0$ possesses trivial solutions $(\tilde{c}, 0) \in \mathcal{O}$ which, as mentioned above, correspond to the semi-trivial solutions ϕ_ℓ^c for c near c_0. The formula (4.6) also reveals the existence of nontrivial solutions given by

$$c - c_0 = \tilde{c} = \frac{\gamma}{\alpha}A_1^2 + O(3) \qquad \text{for } A_1 \neq 0,$$

where $O(3)$ stands for the terms of order higher than or equal to 3 in A_1. The sign of the ratio γ/α determines the direction of the bifurcations, and one easily verifies the statements concerning the direction of bifurcation in THEOREM 4. The shape of bifurcated solution is also given by

$$\phi(x) = r(x)e^{i(\ell x + \theta(x))}, \tag{4.7}$$

where the amplitude $r(x) = r_0 + O(\sqrt{|\tilde{c}|})$ is given by

$$r(x) = \sqrt{(\sqrt{c_0 - \ell} + 2k\sqrt{|\alpha\tilde{c}/\gamma|}\cos kx)^2 + 4\ell^2|\alpha\tilde{c}/\gamma|\sin^2 kx + O(|\tilde{c}|)}$$

and the phase modulation $\theta(x) = O(\sqrt{|\tilde{c}|})$ is determined by

$$\tan\theta(x) = \frac{-2\ell\sqrt{|\alpha\tilde{c}/\gamma|}\,\sin kx}{\sqrt{c_0-\ell} + 2k\sqrt{|\alpha\tilde{c}/\gamma|}\,\cos kx} + O(|\tilde{c}|).$$

From these expressions, we find that $r(x)$ and $\theta(x)$ are $2\pi/k$-periodic and that the statements in THEOREM 4 follow in view of (4.7).

4.2. Computation of bifurcation function. The proof of (4.6) consists mostly of lengthy computations. Therefore, we will not exhibit the full detail, but only give a main flow of computations. In the sequel, we will compute the partial derivatives of $\Psi(\tilde{c}, A_1)$ and $M(\tilde{c}, A_1)$ with respect to $\tilde{c}$ and A_1. These differentiations are respectively denoted by ∂_0 and ∂_1.

We first give the formulae of the partial derivatives of $N(\tilde{c}, \psi)$. Note that $N(0,0) = 0$ and we will only need the derivatives evaluated at $(\tilde{c}, \psi) = (0,0)$. These are given by

$$\partial_0 N(0,0) = -r_0\ell e^{i\ell x}, \qquad \partial_0^m N(0,0) = 0 \quad \text{for } m \geq 2,$$

$$\partial_0\partial_\psi N(0,0)[u] = iu_x, \qquad \partial_0\partial_\psi^m N(0,0) = 0 \quad \text{for } m \geq 2,$$

$$\partial_\psi N(0,0)[u] = 0[u] = 0,$$

$$\partial_\psi^2 N(0,0)[u,v] = r_0\ell e^{i\ell x}(u\bar{v} + v\bar{u}) - ir_0 e^{-i\ell x}(uv_x + vu_x) - ir_0 e^{i\ell x}(\bar{u}v_x + \bar{v}u_x),$$

$$\partial_\psi^3 N(0,0)[u,v,w] = -i(u\bar{v} + v\bar{u})w_x - i(u\bar{w} + w\bar{u})v_x - i(v\bar{w} + w\bar{v})u_x,$$

$$\partial_\psi^m N(0,0) = 0 \quad \text{for } m \geq 4.$$

We now compute derivatives of $\Psi(\tilde{c}, A_1)$ at $(\tilde{c}, A_1) = (0,0)$ by the implicit differentiation of the defining relation of $\Psi(\tilde{c}, A_1)$;

$$\mathscr{H}\Psi + PN(\tilde{c},\, A_1\psi_1 + \Psi) = 0.$$

From $N(0,0) = 0$, we easily see that $\Psi(0,0) = 0$. By differentiating the defining equation with respect to $\tilde{c}$, we have

$$\mathscr{H}(\partial_0\Psi(0,0)) + P(-r_0\ell e^{i\ell x}) = 0.$$

Since $P(-r_0\ell e^{i\ell x}) = -r_0\ell e^{i\ell x}$ and $\mathscr{H}e^{i\ell x} = 2\ell(c_0-\ell)e^{i\ell x}$ (cf. PROPOSITION 1 (i)), we have $\partial_0\Psi(0,0) = e^{i\ell x}/(2r_0)$. Similar computations lead to

$$\partial_1\Psi(0,0) = 0, \qquad \partial_0^2\Psi(0,0) = -\frac{e^{i\ell x}}{4r_0^3}.$$

For the mixed derivative $\partial_0\partial_1\Psi$, we have

$$0 = \mathscr{H}(\partial_0\partial_1\Psi(0,0)) + \frac{2\ell^2 k}{\ell^2+k^2}[(\ell-k)e^{i(\ell-k)x} + (\ell+k)e^{i(\ell+k)x}],$$

and observing that (cf. PROPOSITION 1 (iv) and (vii))

$$(\ell - k)\mathrm{e}^{\mathrm{i}(\ell-k)x} + (\ell + k)\mathrm{e}^{\mathrm{i}(\ell+k)x}$$

is the eigenfunction of $\mathscr{H}$ corresponding to the eigenvalue $\ell^2 + k^2$, we find

$$\partial_0\partial_1\Psi(0,0) = -\frac{2\ell^2 k}{(\ell^2+k^2)^2}[(\ell - k)\mathrm{e}^{\mathrm{i}(\ell-k)x} + (\ell + k)\mathrm{e}^{\mathrm{i}(\ell+k)x}].$$

The equation of the derivative $\partial_1^2\Psi$ is given by

$$\begin{aligned} 0 &= \mathscr{H}(\partial_1^2\Psi(0,0)) + 4\ell r_0(\ell^2+k^2)\mathrm{e}^{\mathrm{i}\ell x} \\ &\quad - 2r_0(\ell^2-k^2)[(\ell - 2k)\mathrm{e}^{\mathrm{i}(\ell-2k)x} + (\ell + 2k)\mathrm{e}^{\mathrm{i}(\ell+2k)x}]. \end{aligned}$$

Since $\mathscr{H}$ maps the subspace $\mathrm{span}[\mathrm{e}^{\mathrm{i}\ell x}, \mathrm{e}^{\mathrm{i}(\ell-2k)x}, \mathrm{e}^{\mathrm{i}(\ell+2k)x}]$ into itself and is invertible on it, applying $\mathscr{H}^{-1}$, we obtain

$$\partial_1^2\Psi(0,0) = -\frac{2(\ell^2+k^2)}{r_0}\mathrm{e}^{\mathrm{i}\ell x} - \frac{r_0(\ell^2-k^2)}{k}[\mathrm{e}^{\mathrm{i}(\ell-2k)x} - \mathrm{e}^{\mathrm{i}(\ell+2k)x}].$$

We are now ready to compute derivatives of M up to third order by using the derivatives of Ψ, exhibited above, and the defining equation (4.5) of M. It is easy to see $M(0,0) = 0$ and

$$\partial_0 M(0,0) = -r_0\ell\mathrm{e}^{\mathrm{i}\ell x}, \qquad \partial_1 M(0,0) = 0.$$

The second derivatives of M are given by

$$\begin{aligned} \partial_0^2 M(0,0) &= \frac{\ell}{2r_0}\mathrm{e}^{\mathrm{i}\ell x}, \qquad \partial_0\partial_1 M(0,0) = 2k\ell[\mathrm{e}^{\mathrm{i}(\ell-k)x} + \mathrm{e}^{\mathrm{i}(\ell+k)x}], \\ \partial_1^2 M(0,0) &= 4r_0\ell(\ell^2+k^2)\mathrm{e}^{\mathrm{i}\ell x} \\ &\qquad - 2r_0(\ell^2-k^2)[(\ell - 2k)\mathrm{e}^{\mathrm{i}(\ell-2k)x} + (\ell + 2k)\mathrm{e}^{\mathrm{i}(\ell+2k)x}]. \end{aligned}$$

From these expressions, we find that the projection of the derivatives of M up to the second order vanish, except for the following one term;

$$(\psi_1,\, \partial_0\partial_1 M(0,0)) = 8\pi k^2\ell. \tag{4.8}$$

There are four third derivatives of M to be computed, and some of them have a very lengthy expression. The simplest one is $\partial_0^3 M(0,0) = -3\ell\mathrm{e}^{\mathrm{i}\ell x}/(2r_0^3)$, from which it follows that $(I-P)\partial_0^3 M(0,0) = 0$. A little more complicated is the following one:

$$\partial_0^2\partial_1 M(0,0) = -\frac{8\ell^4 k}{(\ell^2+k^2)^2}[\mathrm{e}^{\mathrm{i}(\ell-k)x} + \mathrm{e}^{\mathrm{i}(\ell+k)x}].$$

By using this expression, it is not so difficult to show

$$(\psi_1, \partial_0^2 \partial_1 M(0,0)) = 4\pi\beta. \tag{4.9}$$

The derivative $\partial_0 \partial_1^2 M(0,0)$ has a lengthy expression which is a linear combination of $e^{i\ell x}, e^{i(\ell-2k)x}$ with real coefficients. Therefore, it is projected to 0 by $I - P$.

There is one more third order derivative of M. The computation of this derivative involves lengthy terms, and we only exhibit the final result.

$$\begin{aligned}
&\partial_1^3 M(0,0) \\
&\quad = \left[6(\ell^2-k^2)^2 - 24k\ell(\ell^2+k^2) - \frac{3(\ell^2-k^2)^2}{k\ell}(\ell-k)(\ell+2k)\right] e^{i(\ell-k)x} \\
&\qquad - \left[6(\ell^2-k^2)^2 + 24k\ell(\ell^2+k^2) + \frac{3(\ell^2-k^2)^2}{k\ell}(\ell+k)(\ell-2k)\right] e^{i(\ell+k)x} \\
&\qquad + \{e^{i(\ell-3k)x}, e^{i(\ell+3k)x}\},
\end{aligned}$$

where $\{e^{i(\ell-3k)x}, e^{i(\ell+3k)x}\}$ stands for real linear combinations of $e^{i(\ell-3k)x}$ and $e^{i(\ell+3k)x}$. It is also not so difficult to find

$$(\psi_1, \partial_1^3 M(0,0)) = -\frac{24\pi}{\ell} k^2 (3\ell^4 + 6k^2\ell^2 - k^4). \tag{4.10}$$

The expressions (4.8), (4.9) and (4.10) prove (4.6) in Proposition 2.

5. Effects of subharmonic perturbation

In this paper, we have studied the stability of the solutions $\phi_\ell^c \in B_\ell$ and the bifurcations from B_ℓ under the class of perturbations belonging to

$$H_{2\pi}^1 := H^1(\mathbf{R}/(2\pi\mathbf{Z}), \mathbf{C}).$$

Subject to a larger class of perturbations, these solutions may become less stable and there may be more possibilities of bifurcations. To see this, let us take a positive integer j and consider

$$H_{2j\pi}^1 := H^1(\mathbf{R}/(2j\pi\mathbf{Z}), \mathbf{C})$$

as a class of perturbations. The perturbation class $H_{2j\pi}^1$ is called the *j-th subharmonic class* relative to $H_{2\pi}^1$. Obviously, each solution ϕ_ℓ^c belongs to $H_{2j\pi}^1$ for any $j \in \mathbf{N}$, since $H_{2\pi}^1 \subset H_{2j\pi}^1$. Moreover, we have that $H_{2j\pi}^1$ is spanned by

$$\{e^{i(n/j)x} \mid n \in \mathbf{Z}\}.$$

Therefore, we can consider that (1.1) is posed on $H^1_{2j\pi}$, and THEOREM 2 is valid on this phase space. Then, dealing with the eigenvalue problem for $\mathscr{H}_\ell^c$ on $H^1_{2j\pi}$, we obtain an analogue of PROPOSITION 1.

PROPOSITION 3. *For each $\ell \in \mathbf{Z}\backslash\{0\}$ and $2 \le j \in \mathbf{N}$, the eigenvalues of $\mathscr{H}_\ell^c$ on $H^1_{2j\pi}$ are given by*

$$\lambda = \lambda_{\ell,\pm}^{c,n/j} := \left(\frac{n}{j} - \ell\right)^2 + \ell(c-\ell) \pm \ell\sqrt{\left(\frac{n}{j} - \ell\right)^2 + (c-\ell)^2}$$

for $n \in \mathbf{Z}$, and items similar to those in PROPOSITION 1 *are true.*

(i) *For each $\ell \ge 1$,*

$$\lambda_{\ell,+}^{c,n/j} \ge \lambda_{\ell,+}^{c,\ell} = 2\ell(c-\ell) > 0 \qquad \forall n \in \mathbf{Z},$$

$$\lambda_{\ell,-}^{c,n/j} > 0 \quad \forall n \in \mathbf{Z}\backslash\{j\ell\}, \qquad \text{if } c > \ell + \frac{\ell^2 - (1/j)^2}{2\ell}.$$

(ii) *For each $\ell \ge 1$ and $k \in \{1, \dots, j\ell - 1\}$, there are two zero eigenvalues*

$$\lambda_{\ell,-}^{c,\ell\pm(k/j)} = 0 \qquad \text{at } c = \ell + \frac{\ell^2 - (k/j)^2}{2\ell},$$

except for the trivial one $\lambda_{\ell,-}^{c,\ell} = 0$.

(iii) *For each $\ell \le -1$, $\lambda_{\ell,-}^{c,n/j} > \lambda_{\ell,-}^{c,\ell} = 0$ $\forall n \in \mathbf{Z}\backslash\{j\ell\}$ and there are $2j|\ell| + 1$ negative eigenvalues $\lambda_{\ell,+}^{c,\ell\pm(k/j)} < 0$ corresponding to $k = 0, 1, \dots, -j\ell$.*

(iv) *For each $\ell \le -1$ and $\mathbf{N} \ni k > j|\ell|$, there are two zero eigenvalues*

$$\lambda_{\ell,+}^{c,\ell\pm(k/j)} = 0 \qquad \text{at } c = \ell + \frac{\ell^2 - (k/j)^2}{2\ell},$$

except for the trivial one $\lambda_{\ell,-}^{c,\ell} = 0$.

By using PROPOSITION 3, together with the arguments employed in §§2, 3 and 4 with $H^1_{2\pi}$ being replaced by $H^1_{2j\pi}$, we now obtain results similar to THEOREM 3 and THEOREM 4.

THEOREM 6 (Stability under subharmonic perturbation). *For each $\ell, j \in \mathbf{N}$, the travelling wave solution ϕ_ℓ^c appeared in* THEOREM 2 *is orbitally stable in $H^1_{2j\pi}$ if $c > \ell + \dfrac{\ell^2 - j^{-2}}{2\ell}$.*

When $j = 1$, THEOREM 6 is nothing but THEOREM 3. We recognize that as $j \in \mathbf{N}$ increases, the guaranteed stability region of the wave speed parameter c becomes smaller. In particular, ϕ_ℓ^c is stable on $H^1_{2j\pi}$ for all $j \in \mathbf{N}$, if $c > 3\ell/2$. In this sense, the orbital stability of ϕ_ℓ^c for $c > 3\ell/2$ is much stronger than the other cases. However, it is not clear how this strong type of stability is related to the stability under the perturbation class $H^1(\mathbf{R},\mathbf{C})$.

THEOREM 7 (Subharmonic bifurcation). *For each* $\mathbf{N} \ni j \geq 2$, $2j\pi$*-periodic travelling wave solutions of* (1.1) *bifurcate from* B_ℓ *for* $\ell \in \mathbf{Z}\backslash\{0\}$.

(i) *For* $\ell \geq 1$ *and for each* $k \in \{1,\dots,j\ell-1\}$, $2j\pi$*-periodic solutions of* $(j\ell,k)$*-type bifurcate from* B_ℓ *at* $c = \ell + (\ell^2 - (k/j)^2)/(2\ell)$. *The bifurcated branch exists locally for* $c > \ell + (\ell^2 - (k/j)^2)/(2\ell)$.

(ii) *For* $\ell \leq -1$ *and for each* $k \in \mathbf{N}$ $(k > j|\ell|)$, $2j\pi$*-periodic solutions of* $(j\ell,k)$*-type bifurcate from* B_ℓ *at* $c = \ell + (\ell^2 - (k/j)^2)/(2\ell)$. *The bifurcated branch exists locally for*

$$c > \ell + (\ell^2 - (k/j)^2)/(2\ell) \qquad \text{if } |\ell| < k/j < \sqrt{3+2\sqrt{3}}|\ell|,$$

$$c < \ell + (\ell^2 - (k/j)^2)/(2\ell) \qquad \text{if } k/j > \sqrt{3+2\sqrt{3}}|\ell|.$$

We observe that the larger the order of subharmonicity j is, the more bifurcation points appear. Indeed, we can easily prove that for $\ell \geq 1$ the bifurcation set

$$\bigcup_{j\geq 1}\left\{\ell + \frac{\ell^2 - (k/j)^2}{2\ell}\,\middle|\, k = 1,\dots,j\ell-1\right\}$$

of wave speed c is dense in the interval $(\ell, 3\ell/2)$ and there is no bifurcation point in $(3\ell/2,\infty)$. Similarly, for $\ell \leq -1$, the bifurcation set

$$\bigcup_{j\geq 1}\left\{\ell + \frac{\ell^2 - (k/j)^2}{2\ell}\,\middle|\, \mathbf{N} \ni k > j|\ell|\right\}$$

is dense in (ℓ,∞). In dissipative systems, bifurcations indicate inherent instabilities. Although our system is not dissipative, we may well suppose that for the wave speed parameter c in the range not covered by THEOREM 3 the solution ϕ_ℓ^c is orbitally unstable. However, to prove or disprove this statement remains a mathematically challenging problem.

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