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Spectral properties of a class of generalized Ruelle operators

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ABSTRACT. We consider a class of Ruelle type operators which play an important role in the study of singular perturbation of symbolic dynamics via thermodynamic formalism. We study the eigenvalues of those operators with maximal modulus and obtain necessary and sufficient conditions for them to be semisimple.

1. Introduction

Let $d \geq 2$ be an integer and $S = \{1, 2, \ldots, d\}$ a finite set endowed with the discrete topology. We write an element $\omega \in S^{\mathbf{Z}_+}$ as $\omega = \omega_0\omega_1\ldots$, where $\mathbf{Z}_+$ is the totality of nonnegative integers. We define a map $\sigma : S^{\mathbf{Z}_+} \to S^{\mathbf{Z}_+}$ called the shift transformation by $(\sigma\omega)_n = \omega_{n+1}$ for $\omega \in S^{\mathbf{Z}_+}$ and $n \in \mathbf{Z}_+$. Let $M = (M(ij))$ be a $d \times d$ matrix whose entries are either 0 or 1. We consider the set $\Sigma_M^+ = \{\omega \in S^{\mathbf{Z}_+} : M(\omega_n\omega_{n+1}) = 1 \text{ for any } n \in \mathbf{Z}_+\}$ and the shift $\sigma_M = \sigma|_{\Sigma_M^+}$. The topological dynamical system (Σ_M^+, σ_M) is called a subshift of finite type with transition matrix M. For integers $m \geq 0$, $n \geq 1$ and a word $w \in S^n$, put ${}_m[w] = \{\omega \in S^{\mathbf{Z}_+} : \omega_m\omega_{m+1}\ldots\omega_{m+n-1} = w\}$ and ${}_m[w]^M = {}_m[w] \cap \Sigma_M^+$. Such a set is called a cylinder set. A word $i_1i_2\ldots i_n \in S^n$ is called M-admissible if $M(i_1i_2)\cdot\ldots\cdot M(i_{n-1}i_n) = 1$. For $n \geq 1$, $W_n(M)$ denotes the totality of M-admissible words of S^n.

Let $A = (A(ij))$ and $B = (B(ij))$ be $d \times d$ transition matrices satisfying the following conditions.

(Σ.1) There exists an integer $n_0 \geq 1$ such that $A^{n_0} > 0$.
(Σ.2) $B(ij) = 1$ implies $A(ij) = 1$.
(Σ.3) Σ_B^+ is not empty.

Following the notion in [7], we can regard Σ_B^+ as the subshift obtained by the collapsing of Σ_A^+. We should note that the condition (Σ.3) in the present paper is much more general than the condition (Σ.3) in the previous paper [7].

For $\theta \in (0, 1)$, we define a metric d_θ on $S^{\mathbf{Z}_+}$ so that $d_\theta(\omega, \omega') = \theta^n$ if

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$\omega \neq \omega'$ and $n = \min\{n : \omega_n \neq \omega'_n\}$ and $d_\theta(\omega, \omega) = 0$. The metric topology induced by d_θ to $S^{\mathbf{Z}_+}$ coincides with the product topology on $S^{\mathbf{Z}_+}$ induced by the discrete topology of S. Clearly Σ_A^+ is a closed subset of $S^{\mathbf{Z}_+}$. Denote by $C(\Sigma_A^+)$ the totality of complex valued continuous functions on Σ_A^+ and by $C(\Sigma_A^+ \to \mathbf{R})$ the totality of real valued functions belonging to $C(\Sigma_A^+)$. Similarly, denote by $F_\theta(\Sigma_A^+)$ the totality of complex valued Lipschitz continuous functions with respect to d_θ and by $F_\theta(\Sigma_A^+ \to \mathbf{R})$ the totality of real valued functions belonging to $F_\theta(\Sigma_A^+)$. The spaces $C(\Sigma_A^+)$ endowed with the supremum norm $\|f\|_\infty = \sup_{\omega \in \Sigma_A^+} |f(\omega)|$, and $F_\theta(\Sigma_A^+)$ endowed with the norm $\|f\|_\theta = \|f\|_\infty + [f]_\theta$ are Banach spaces, where $[f]_\theta = \max_{i \in S} [f]_{\theta,i}$ with $[f]_{\theta,i} = \sup\{|f(\omega) - f(\omega')|/d_\theta(\omega, \omega') : \omega, \omega' \in {}_0[i]^A$ and $\omega \neq \omega'\}$ for $f \in F_\theta(\Sigma_A^+)$ and $i \in S$.

Let $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. We define an operator $\mathscr{L}_{B,\varphi}$ on $C(\Sigma_A^+)$ by

$$\mathscr{L}_{B,\varphi} f(\omega) = \sum_{i \in S : B(i\omega_0) = 1} e^{\varphi(i \cdot \omega)} f(i \cdot \omega),$$

where $i \cdot \omega$ denotes the concatenation of i and ω, i.e. $i \cdot \omega = i\omega_0\omega_1 \dots$ in Σ_A^+. The operator can be regarded as an operator on $F_\theta(\Sigma_A^+)$ naturally. We note that such an operator plays an important role in the study of singular perturbation of symbolic dynamics in the previous paper [7].

The two main purposes of this paper are the following:

(I) Giving a necessary and sufficient condition for semisimplicity of the eigenvalues of the operator $\mathscr{L}_{B,\varphi}$ with maximal modulus in terms of the orbit structure of the dynamics (Σ_B^+, σ_B) (Theorem 3.2).

(II) Showing a generalization of the Ruelle-Perron-Frobenius theorem for the operator $\mathscr{L}_{B,\varphi}$ under the condition in (I) (Theorem 3.3).

As auxiliary results, we obtain a decomposition of Σ_A^+ by using the pointwise exponential growth rate of $\mathscr{L}_{B,\varphi}^n 1$ (Theorem 5.1) and the information of detailed structure of the eigenspaces corresponding to the eigenvalues of the operator $\mathscr{L}_{B,\varphi}$ and the dual $\mathscr{L}_{B,\varphi}^*$ of $\mathscr{L}_{B,\varphi}$ with maximal modulus (Proposition 5.3, Proposition 5.4). More precisely, we give bases of those eigenspaces under the condition in (I).

In Section 2 we give some notions and facts which are necessary to state the main results. The statements of the main results are given in Section 3. In Section 4 we prove a generalization of the Ruelle-Perron-Frobenius theorem for $\mathscr{L}_{B,\varphi}$ under the transitivity condition $(\Sigma.3)_T$ on B. In Section 5 we investigate the exponential growth rate of $\mathscr{L}_{B,\varphi}^n 1$ and the detailed structure of eigenspaces corresponding to the eigenvalues of $\mathscr{L}_{B,\varphi}$ and $\mathscr{L}_{B,\varphi}^*$ with maximal modulus. Section 6 is devoted to the proofs of the main results. Finally we give some examples in Section 7.

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2. Preliminaries

In this section we give some notions and facts which are necessary to state our results. We assume the three conditions (Σ.1), (Σ.2) and (Σ.3) in the preceding section.

For non-empty subsets S' and S'' of S, let M be a $\#S' \times \#S''$ matrix with entries 0 or 1 indexed by $S' \times S''$ so that $M(ij) = 0$ if $B(ij) = 0$. Let $\varphi \in C(\Sigma_A^+ \to \mathbf{R})$. We define an operator $\mathscr{L}_{M,\varphi}$ on $C(\Sigma_A^+)$ by

$$\mathscr{L}_{M,\varphi} f(\omega) = \sum_{i \in S : M(i\omega_0) = 1} e^{\varphi(i \cdot \omega)} f(i \cdot \omega).$$

Note that if $ij \in S \times S \backslash S' \times S''$, we regard $M(ij)$ as 0. $\mathscr{L}_{M,\varphi}^* : M(\Sigma_A^+) \to M(\Sigma_A^+)$ denotes the dual operator of $\mathscr{L}_{M,\varphi}$ which is defined by $\mathscr{L}_{M,\varphi}^* m(f) = m(\mathscr{L}_{M,\varphi} f)$ for $m \in M(\Sigma_A^+)$ and $f \in C(\Sigma_A^+)$, where $M(\Sigma_A^+)$ denotes the totality of the complex Borel measures on Σ_A^+. If φ is an element of $F_\theta(\Sigma_A^+ \to \mathbf{R})$, then we can easily verify the inclusion $\mathscr{L}_{M,\varphi} F_\theta(\Sigma_A^+) \subset F_\theta(\Sigma_A^+)$.

By virtue of the theory of nonnegative matrices, the set S can be decomposed as $S = S(1) \cup S(2) \cup \cdots \cup S(m)$ for some $m \geq 1$ and ${}^t PBP =$

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ O & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{m-1m} \\ O & \cdots & O & B_{mm} \end{pmatrix} \tag{2.1}$$

so that for each $k \in \{1, 2, \ldots, m\}$ the submatrix $B_{kk} = (B(ij))_{i,j \in S(k)}$ of B is irreducible (i.e. for any $i, j \in S(k)$, $(B_{kk})^n(ij) > 0$ holds for some $n \geq 0$), where P is an appropriately chosen permutation matrix. Thus we may assume that B itself has the form as (2.1). Put $T = \{1, 2, \ldots, m\}$ for our convenience. Note that there exists $k \in T$ such that B_{kk} is not a 1×1 zero matrix (0) by virtue of the condition (Σ.3).

Put $\Sigma_k = \bigcup_{i \in S(k)} {}_0[i]^A$ for each $k \in T$. Let $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. For each $k \in T$, $\tilde{\lambda}_k$ denotes the spectral radius of the operator $\mathscr{L}_{B_{kk},\varphi}$ on $C(\Sigma_A^+)$ and put

$$\tilde{\lambda} = \max_{k \in T} \tilde{\lambda}_k.$$

It is shown in Section 5 that $\tilde{\lambda}$ is the spectral radius of the operator $\mathscr{L}_{B,\varphi}$ on $C(\Sigma_A^+)$. In addition it turns out to be an eigenvalue of $\mathscr{L}_{B,\varphi}$ on $F_\theta(\Sigma_A^+)$ (see Theorem 1.5 in [1]).

We introduce a partial order $\prec$ on T as follows: $k' \prec k$ if there exist $i \in S(k')$, $j \in S(k)$ and $n \geq 0$ such that $B^n(ij) > 0$ holds. Since the matrix B has the form (2.1), we see that $k' \prec k$ yields $k' \leq k$. We define disjoint subsets T_0, T_1 and T_2 of T by

$$T_0 = \{k \in T : \text{there exist } k', k'' \in T \text{ such that } k' \prec k, k \prec k'' \text{ and } \tilde{\lambda}_{k'} = \tilde{\lambda}_{k''} = \tilde{\lambda}\}$$

$$T_1 = \{k \in T \backslash T_0 : k' \prec k \text{ for some } k' \in T_0\}$$

$$T_2 = \{k \in T \backslash T_0 : k' \prec k \text{ does not hold for any } k' \in T_0\}.$$

Accordingly, S and Σ_A^+ are decomposed into the corresponding subsets

$$S_j = \bigcup_{k \in T_j} S(k) \quad \text{and} \quad \Sigma(j) = \bigcup_{k \in T_j} \Sigma_k \qquad \text{for each } j = 0, 1, 2, \text{ respectively.}$$

Note that the sets T_0, T_1 and T_2 depend on the function φ.

Let $C = C_\varphi$ be a $\#S_0 \times \#S_0$ matrix with entries 0 or 1 indexed by $S_0 \times S_0$ satisfying $C(ij) = B(ij)$ for each $i, j \in S_0$. Since B itself is assumed to have the form (2.1), there exist indexes $k(1) < k(2) < \cdots < k(m_0)$ in T such that C is expressed as

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m_0} \\ O & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_{m_0-1m_0} \\ O & \cdots & O & C_{m_0m_0} \end{pmatrix} \tag{2.2}$$

so that $m_0 = \#T_0$ and $C_{11} = B_{k(1)k(1)}, C_{22} = B_{k(2)k(2)}, \ldots, C_{m_0m_0} = B_{k(m_0)k(m_0)}$. It is easy to see that Σ_C^+ is σ-invariant, i.e. $\Sigma_C^+ = \sigma\Sigma_C^+$.

We consider the following condition on the matrix C:

$$(\Sigma\Phi) \qquad C \text{ has the form } C = \begin{pmatrix} C_{11} & O & \cdots & O \\ O & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & C_{m_0m_0} \end{pmatrix}.$$

We see that the condition $(\Sigma\Phi)$ holds if and only if the set $\{k \in T : \tilde{\lambda}_k = \tilde{\lambda}\}$ consists of incomparable elements with respect to the order $\prec$. In particular, $(\Sigma\Phi)$ implies that $\tilde{\lambda}_k = \tilde{\lambda}$ for any $k \in T_0$.

3. Main results

In this section, first we consider a spectral decomposition of $\mathscr{L}_{B,\varphi}$ for the maximal eigenvalues and next we state our main results. In what follows $\mathscr{L}(\mathscr{X})$ denotes the totality of bounded linear operators on a Banach space $\mathscr{X}$.

We recall that the essential spectral radius of an operator $\mathscr{L} \in \mathscr{L}(\mathscr{X})$ is the infimum of the set of numbers $r > 0$ such that if the set $\mathrm{Spec}(\mathscr{L}) \cap \{\lambda : |\lambda| > r\}$ is not empty, it consists of a finite number of eigenvalues with finite multiplicity, where $\mathrm{Spec}(\mathscr{L})$ denotes the spectrum of the operator $\mathscr{L}$. It is easy to verify the following fact.

Proposition 3.1. *Assume that the conditions* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied and let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then the essential spectral radius of the operator* $\mathscr{L}_{B,\varphi}$ *on* $F_\theta(\Sigma_A^+)$ *is not greater than* $\theta\tilde{\lambda}$.

Proof. The proposition follows immediately from Theorem 1.5 in [1]. In fact, it is shown that for any function $G \in F_\theta(\Sigma_A^+)$, the operator $\mathscr{M}_G$ on $C(\Sigma_A^+)$ defined by

$$\mathscr{M}_G f(\omega) = \sum_{i:A(i\omega_0)=1} G(i \cdot \omega) f(i \cdot \omega)$$

turns out to be a bounded operator on $F_\theta(\Sigma_A^+)$ and satisfies that the essential spectral radius of $\mathscr{M}_G$ on $F_\theta(\Sigma_A^+)$ is not greater than θr, where $r = \lim_{n\to\infty} \|\mathscr{M}_{|G|}^n 1\|_\infty^{1/n}$.

Putting $N = \bigcup_{ij:B(ij)=0} {}_0[ij]^A$ and $G = e^\varphi(1 - \chi_N)$, we have $\mathscr{M}_G = \mathscr{M}_{|G|} = \mathscr{L}_{B,\varphi}$ and thus the essential spectral radius of $\mathscr{L}_{B,\varphi}$ on $F_\theta(\Sigma_A^+)$ is not greater than θr. Furthermore, r is non-zero and becomes the spectral radius of the operator $\mathscr{L}_{B,\varphi}$ on $C(\Sigma_A^+)$ (see Proposition 5.2). Hence $r = \tilde{\lambda}$.

By virtue of the proposition above, the set $\{\lambda \in \mathrm{Spec}(\mathscr{L}_{B,\varphi}|_{F_\theta(\Sigma_A^+)}) : |\lambda| = \tilde{\lambda}\}$ can be written as $\{\tilde{\lambda} = \lambda_0, \lambda_1, \ldots, \lambda_{q-1}\}$, where $q \ge 1$ is an integer and λ_j's are distinct eigenvalues with finite multiplicity. By the general theory of linear operators (see [6]), we have the decomposition

$$\mathscr{L}_{B,\varphi} = \sum_{j=0}^{q-1} (\lambda_j \mathscr{P}_j + \mathscr{N}_j) + \mathscr{R} \tag{3.1}$$

of the operator $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ such that the following hold:

(1) For each j, $\mathscr{P}_j$ is the projection onto the generalized eigenspace corresponding to the eigenvalue λ_j.

(2) For each j, $\mathscr{N}_j$ is the nilpotent operator corresponding to the eigenvalue λ_j.

(3) For each j, $\mathscr{P}_j\mathscr{R} = \mathscr{R}\mathscr{P}_j = \mathscr{N}_j\mathscr{R} = \mathscr{R}\mathscr{N}_j = O$ and $\mathscr{P}_j\mathscr{P}_i = \mathscr{N}_j\mathscr{N}_i = O$ if $i \ne j$, where O is the zero element in $\mathscr{L}(F_\theta(\Sigma_A^+))$.

(4) The spectral radius of the operator $\mathscr{R}$ on $F_\theta(\Sigma_A^+)$ is less than $\tilde{\lambda}$.

Recall that an eigenvalue λ of an operator $\mathscr{L}$ is said to be semisimple if the dimension of the generalized eigenspace of λ is finite and coincides with that of the eigenspace of λ (see [6]). Now we are in a position to state our main results.

Theorem 3.2. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then the following are equivalent.*

(i) *The condition* $(\Sigma\Phi)$ *holds.*

(ii) $\sup_{n\geq 1} \tilde{\lambda}^{-n} \|\mathscr{L}_{B,\varphi}^n 1\|_\infty < +\infty$.

(iii) *All eigenvalues of the operator* $\mathscr{L}_{B,\varphi}$ *on* $F_\theta(\Sigma_A^+)$ *with maximal modulus are semisimple.*

(iv) *The eigenvalue* $\tilde{\lambda}$ *of* $\mathscr{L}_{B,\varphi}$ *on* $F_\theta(\Sigma_A^+)$ *is semisimple.*

Note that the substantial part of the theorem is the implication (iv) $\Rightarrow$ (i). By this theorem, if the condition $(\Sigma\Phi)$ holds then $\mathscr{L}_{B,\varphi}$ has the form

$$\mathscr{L}_{B,\varphi} = \sum_{j=0}^{q-1} \lambda_j \mathscr{P}_j + \mathscr{R} \tag{3.2}$$

and each $\mathscr{P}_j$ becomes the projection onto the eigenspace of λ_j. Moreover we have the following:

Theorem 3.3. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$, $(\Sigma.3)$ *and* $(\Sigma\Phi)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then as an element of* $\mathscr{L}(C(\Sigma_A^+))$, $\mathscr{L}_{B,\varphi}$ *has the decomposition* (3.2). *Moreover, for each* $j = 0, 1, \ldots, q-1$, *there exist a subset* $T_0(\lambda_j)$ *of* T_0 *and families* $\{h(\lambda_j, k) \in C(\Sigma_A^+) : k \in T_0(\lambda_j)\}$ *and* $\{\nu(\lambda_j, k) \in M(\Sigma_A^+) : k \in T_0(\lambda_j)\}$ *such that the following hold.*

(1) $\mathscr{P}_j$ *has the form*

$$\mathscr{P}_j(f) = \sum_{k \in T_0(\lambda_j)} \left(\int_{\Sigma_A^+} f \, d\nu(\lambda_j, k) \right) h(\lambda_j, k)$$

for $f \in C(\Sigma_A^+)$.

(2) *The eigenspace corresponding to the eigenvalue* λ_j *of the operator* $\mathscr{L}_{B,\varphi} \in \mathscr{L}(C(\Sigma_A^+))$ *is spanned by* $\{h(\lambda_j, k) : k \in T_0(\lambda_j)\}$.

(3) *For each* $k \in T_0(\lambda_j)$, $\nu(\lambda_j, k)$ *is an eigenvector corresponding to the eigenvalue* λ_j *of the dual* $\mathscr{L}_{B,\varphi}^*$ *with* $\int h(\lambda_j, k) d\nu(\lambda_j, k) = 1$.

(4) *For each* $k \in T_0(\lambda_j)$, $\operatorname{supp} h(\lambda_j, k) \cap \Sigma(0) = \Sigma_k$ *and* $\operatorname{supp} \nu(\lambda_j, k) \cap \Sigma(0) = \Sigma_{B_{kk}}^+$.

Note that the family of the functions $h(\lambda, k)$ and the family of the measures $\nu(\lambda, k)$ in Theorem 3.3 are exactly defined in Section 5. If the condition $(\Sigma\Phi)$ does not hold, each $\mathscr{P}_j$ may not be expressed by such a form above. But we have the following auxiliary result for the eigenvalue of $\mathscr{L}_{B,\varphi} : C(\Sigma_A^+) \to C(\Sigma_A^+)$ with maximal modulus. Recall that the period of a nonnegative irreducible matrix M with $M \neq (0)$ is defined as the greatest common divisor of $\{q > 0 : M^q(jj) > 0 \text{ for any } j\}$.

Proposition 3.4. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then for any eigenvalue* λ *of* $\mathscr{L}_{B,\varphi}$ *with maximal modulus there exists* $k \in T_0$ *with* $\tilde{\lambda} = \tilde{\lambda}_k$ *such that* $\lambda^p = \tilde{\lambda}^p$ *for the period* p *of the matrix* B_{kk}.

This proposition implies that there exist an integer $q_0 > 0$ and distinct elements $p_0, p_1, \ldots, p_{q-1}$ of $\{0, 1, \ldots, q_0 - 1\}$ such that $\lambda_j = \tilde{\lambda} e^{2\pi\sqrt{-1}p_j/q_0}$ for each $j = 0, 1, \ldots, q-1$.

Remark 3.5. Even if an eigenvalue λ of $\mathscr{L}_{B,\varphi}$ with $|\lambda| = \tilde{\lambda}$ other than $\tilde{\lambda}$ is semisimple, the condition $(\Sigma\Phi)$ is not necessarily fulfilled. We will give such an example in Section 7.

4. A generalization of Ruelle-Perron-Frobenius Theorem

We consider a generalization of the Ruelle-Perron-Frobenius theorem (Theorem 4.1) under the conditions $(\Sigma.1)$, $(\Sigma.2)$, and the transitivity condition $(\Sigma.3)_T$ below.

$(\Sigma.3)_T$ The set Σ_B^+ is not empty and if Σ^+ is the maximal σ-invariant subset of Σ_B^+ then the dynamics $(\Sigma^+, \sigma|_{\Sigma^+})$ is topologically transitive.

Note that if B has the form as in (2.1), then each matrix B_{kk} with $B_{kk} \neq (0)$ satisfies the condition $(\Sigma.3)_T$ and the maximal σ-invariant subset of $\Sigma_{B_{kk}}^+$ is $\Sigma_{B_{kk}}^+$ itself. We will use this fact in Section 5.

In what follows we assume the conditions $(\Sigma.1)$, $(\Sigma.2)$ and $(\Sigma.3)_T$. Let $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$ and C be the matrix with the form given in (2.2). Since Σ_C^+ is σ-invariant, Σ_C^+ is a subset of Σ^+. Furthermore, C is irreducible by the condition $(\Sigma.3)_T$ and thus $\Sigma_C^+ = \Sigma^+$ holds. Let $p > 0$ be the period of the matrix C. By the theory of nonnegative matrices, there exists a permutation matrix P such that

$$
{}^tPCP = \begin{pmatrix}
O & C_{12} & O & \cdots & O \\
O & O & C_{23} & \ddots & \vdots \\
\vdots & O & O & \ddots & O \\
O & & \ddots & \ddots & C_{p-1p} \\
C_{p1} & O & \cdots & O & O
\end{pmatrix} \quad \text{and}
$$

$$
{}^t P C^p P = \begin{pmatrix} \tilde{C}_1 & O & O & \cdots & O \\ O & \tilde{C}_2 & O & \ddots & \vdots \\ \vdots & O & \tilde{C}_3 & \ddots & O \\ O & & \ddots & \ddots & O \\ O & O & \cdots & O & \tilde{C}_p \end{pmatrix} \tag{4.1}
$$

so that each submatrix $\tilde{C}_j$ is aperiodic. Therefore we may assume that C itself has the form as the former matrix of (4.1). Denote by $S_{0,j}$ the index set of the matrix $\tilde{C}_{j+1}$ and put $X(j) = \bigcup_{i \in S_{0,j}} {}_0[i]^A$ for each $j = 0, 1, \dots, p-1$. Then we have $\Sigma(0) = \bigcup_{j=0}^{p-1} X(j)$, $\Sigma_C^+ = (X(0) \cup X(1) \cup \cdots \cup X(p-1)) \cap \Sigma_C^+$ and

$$
\sigma_C(X(j) \cap \Sigma_C^+) = X(j+1) \cap \Sigma_C^+ \pmod{p} \qquad \text{for each } j = 0, 1, \dots, p-1
$$

by the form (4.1).

Now we give the statement of the theorem.

THEOREM 4.1. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)_T$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then we obtain the spectral decomposition*

$$
\mathscr{L}_{B,\varphi} = \sum_{j=0}^{p-1} \lambda_{B,j} \mathscr{P}_{B,j} + \mathscr{R}_B \tag{4.2}
$$

of $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ *such that the following hold.*

(1) *There exists* $\lambda_B > 0$ *such that* $\lambda_{B,j} = \lambda_B e^{2\pi\sqrt{-1}j/p}$ *for each* j.

(2) *For each* j, $\mathscr{P}_{B,j}$ *is the projection onto the one-dimensional eigenspace corresponding to the eigenvalue* $\lambda_{B,j}$ *which is given by*

$$
\mathscr{P}_{B,j} f = \left(\int_{\Sigma_A^+} f \, d\nu_{B,j} \right) h_{B,j}.
$$

Here $\nu_{B,j} \in M(\Sigma_A^+)$ *is an eigenvector corresponding to the eigenvalue* $\lambda_{B,j}$ *of the operator* $\mathscr{L}_{B,\varphi}^*$ *and* $h_{B,j} \in F_\theta(\Sigma_A^+)$ *is an eigenfunction corresponding to the eigenvalue* $\lambda_{B,j}$ *of the operator* $\mathscr{L}_{B,\varphi}$ *with* $\int_{\Sigma_A^+} h_{B,j} \, d\nu_{B,j} = 1$. *In particular,* $\nu_B = \nu_{B,0}$ *is a Borel probability measure supported on* Σ_B^+ *and* $h_B = h_{B,0}$ *is a nonnegative function supported on* $\Sigma(0) \cup \Sigma(1)$. *Moreover,* $\nu_{B,j}|_{\Sigma(0)} = \sum_{k=0}^{p-1} \nu_B|_{X(k)} \kappa^{kj}$ *and* $h_{B,j}|_{\Sigma(0)} = \sum_{k=0}^{p-1} h_B|_{X(k)} \kappa^{-kj}$ *hold, where* $\kappa = e^{2\pi\sqrt{-1}/p}$.

(3) $\mathscr{P}_{B,j}\mathscr{R}_B = \mathscr{R}_B\mathscr{P}_{B,j} = O$ *for each* j *and* $\mathscr{P}_{B,i}\mathscr{P}_{B,j} = O$ *if* $i \neq j$.

(4) *The spectral radius of the operator* $\mathscr{R}_B$ *on* $F_\theta(\Sigma_A^+)$ *is less than* λ_B.

(5) *As an element of* $\mathscr{L}(C(\Sigma_A^+))$, $\mathscr{L}_{B,\varphi}$ *has the decomposition* (4.2). *In particular, for any* $f \in C(\Sigma_A^+)$

$$
\lim_{n\to\infty} \left\| \lambda_B^{-pn} \mathscr{L}_{B,\varphi}^{pn} f - \sum_{j=0}^{p-1} \nu_{B,j}(f) h_{B,j} \right\|_\infty = 0.
$$

We also obtain the following corollary to Theorem 4.1 for the triplet (λ_B, h_B, ν_B). We denote by $\mathscr{L}_{C,\varphi_C}$ the Ruelle operator on $C(\Sigma_C^+)$ of the potential $\varphi_C = \varphi|_{\Sigma_C^+}$. Put $\tilde{h}_C = h_B|_{\Sigma_C^+}$ and $\tilde{\nu}_C = \nu_B|_{\Sigma_C^+}$.

Corollary 4.2. *We assume the same conditions as in Theorem* 4.1. *Then* $\mathscr{L}_{C,\varphi_C}\tilde{h}_C = \lambda_B\tilde{h}_C$, $\mathscr{L}_{C,\varphi_C}^*\tilde{\nu}_C = \lambda_B\tilde{\nu}_C$ *and* $\tilde{\nu}_C(\tilde{h}_C) = 1$ *hold. Consequently,* $\log\lambda_B$ *and* $h_B\nu_B$ *become the topological pressure of* φ_C *and the Gibbs measure on* Σ_C^+ *of the potential* φ_C*, respectively.*

Our theorem might be a sort of folklore theorem but it is hard to find the literature with a complete proof for such a general case as we need. The special case of the theorem was proved in our previous paper [7], where we imposed the mixing condition $(\Sigma.3)_M$ below instead of $(\Sigma.3)_T$.

$(\Sigma.3)_M$ Σ_B^+ is not empty and if Σ^+ is the maximal σ-invariant subset of Σ_B^+ then the dynamics $(\Sigma^+, \sigma|_{\Sigma^+})$ is topologically mixing.

Stoyanov [10] also consider the special case when the transition matrices A and B are identical and irreducible.

In order to prove Theorem 4.1, we need some auxiliary results.

For the sake of convenience, for each $k, k' \in \mathbf{Z}$, we write $C_{kk'} = C_{jj'}$ if $k \equiv j \pmod p$ and $k' \equiv j' \pmod p$ for some $j, j' \in \{0, 1, \dots, p-1\}$. Similarly, for each $k \in \mathbf{Z}$, we write $X(k) = X(j)$ and $S_{0,k} = S_{0,j}$ if $k \equiv j \pmod p$ for some $j \in \{0, 1, \dots, p-1\}$.

As in [7], we can write S_0, S_1, S_2 and $\Sigma(0)$, $\Sigma(1)$, $\Sigma(2)$ in Section 2 as

$$S_0 = \{i \in S : {}_0[i]^C \neq \emptyset\},$$
$$S_1 = \{i \in S \backslash S_0 : \text{there exist } n \geq 1 \text{ and } j \in S_0 \text{ such that } B^n(ji) > 0\},$$
$$S_2 = \{i \in S \backslash S_0 : \text{for any } n \geq 1 \text{ and } j \in S_0, B^n(ji) = 0\},$$

and $\Sigma(j) = \bigcup_{i \in S_j} {}_0[i]^A$ for each $j = 0, 1, 2$. It is easy to see the following:

Lemma 4.3. *Let* $\omega = \omega_0\omega_1\dots\omega_{n-1}$ *be B-admissible. Then we have the following.*

(1) *If* $\omega_0, \omega_{n-1} \in S_0$*, then* $\omega_1, \omega_2 \dots, \omega_{n-2} \in S_0$.
(2) *If* $\omega_0 \in S_1$*, then* $\omega_1, \omega_2, \dots, \omega_{n-1} \in S_1$ *and* $n < d$.
(3) *If* $\omega_{n-1} \in S_2$*, then* $\omega_0, \omega_1, \dots, \omega_{n-2} \in S_2$ *and* $n < d$.
(4) *If* $n \geq d$*, then* $\omega_0 \in S_0 \cup S_2$ *and* $\omega_{n-1} \in S_0 \cup S_1$.
(5) $\Sigma_C^+ = \sigma^d \Sigma_B^+$.

Proof. For the proof see Lemma 2.2 and Proposition 2.4 in [7].

For $f \in C(\Sigma_A^+)$, we write $f_C = f|_{\Sigma_C^+}$. We also need the following:

Lemma 4.4. (1) *For any* $f \in C(\Sigma_A^+)$*, we have* $\mathscr{L}_{B,\varphi}^n f(\omega) = 0$ *for any* $n \geq d$ *and* $\omega \in \Sigma(2)$.

(2) *If* $f \in C(\Sigma_A^+)$ *satisfies* $\mathscr{L}_{B,\varphi} f = \lambda f$ *on* $\Sigma(0) \cup \Sigma(2)$ *for some* $\lambda \in \mathbf{C}$ *with* $\lambda \neq 0$, *then* $\mathscr{L}_{B,\varphi} f = \lambda f$ *on* $\Sigma(1)$ *if and only if* $f = \lambda^{-d} \mathscr{L}_{B,\varphi}^d (f \chi_{\Sigma(0)})$ *on* $\Sigma(1)$.

(3) *If* $f \in C(\Sigma_A^+)$ *satisfies* $\mathscr{L}_{B,\varphi} f = \lambda f$ *on* Σ_A^+ *for some* $\lambda \in C$, *then* $\mathscr{L}_{C,\varphi_C} f_C = \lambda f_C$.

PROOF. The assertions (1) and (3) follow from Lemma 3.3 (1) and (4) in [7], respectively. It remains to show (2). Assume that $\mathscr{L}_{B,\varphi} f = \lambda f$ on $\Sigma(0) \cup \Sigma(2)$ for some $f \in C(\Sigma_A^+)$ and $\lambda \in \mathbf{C} \backslash \{0\}$. Since $\mathscr{L}_{B,\varphi}^n f = \lambda^n f$ on $\Sigma(0) \cup \Sigma(2)$ for any $n \geq 1$, we have $f = 0$ on $\Sigma(2)$ by (1). Furthermore, for $n \geq d$ and $\omega \in \Sigma(0) \cup \Sigma(1)$, $\mathscr{L}_{B,\varphi}^n f(\omega) = \mathscr{L}_{B,\varphi}^n (f \chi_{\Sigma(0)})(\omega)$ by Lemma 4.3(4). Thus we have only to show that the fact $f = \lambda^{-d} \mathscr{L}_{B,\varphi}^d (f \chi_{\Sigma(0)})$ on $\Sigma(1)$ yields the equation $\mathscr{L}_{B,\varphi} f = \lambda f$. We have that for $\omega \in \Sigma(1)$, $\lambda f(\omega) = \lambda^{-d} \mathscr{L}_{B,\varphi}^d (\lambda f \chi_{\Sigma(0)})(\omega) = \lambda^{-d} \mathscr{L}_{B,\varphi}^d ((\mathscr{L}_{B,\varphi} f) \chi_{\Sigma(0)})(\omega) = \lambda^{-d} \mathscr{L}_{B,\varphi}^{d+1} f(\omega) = \mathscr{L}_{B,\varphi} f(\omega)$. Therefore the assertion (2) is valid.

For $c \geq 0$, we define a family Λ_c of functions by

$$\Lambda_c = \{f \in C(\Sigma_A^+) : 0 \leq f \leq 1 \text{ and if } \omega_0 = \omega_0' \text{ then } f(\omega) \leq f(\omega') e^{c d_\theta(\omega, \omega')}\}.$$

By the standard technique in thermodynamic formalism ([7], [8]), we see that for $c \geq c_1 = [\varphi]_\theta \theta / (1 - \theta)$ there exist $\lambda_B > 0$ and $g_B \in \Lambda_c$ such that $\mathscr{L}_{B,\varphi} g_B = \lambda_B g_B$ and $\|g_B\|_\infty = 1$. We state the outline of the proof. Choose any c with $c \geq [\varphi]_\theta \theta / (1 - \theta)$. For each $n \geq 1$, we can define a continuous operator $L_n : \Lambda_c \to \Lambda_c$ by $L_n f = \dfrac{\mathscr{L}_{B,\varphi}\left(f + \frac{1}{n}\right)}{\left\|\mathscr{L}_{B,\varphi}\left(f + \frac{1}{n}\right)\right\|_\infty}$. Note that Λ_c is a compact convex subset of $C(\Sigma_A^+)$. By the Schauder-Tychonoff theorem, L_n has a fixed point g_n in Λ_c. Namely, we have $\mathscr{L}_{B,\varphi}\left(g_n + \frac{1}{n}\right) = \lambda_n g_n$ and $\lambda_n = \left\|\mathscr{L}_{B,\varphi}\left(g_n + \frac{1}{n}\right)\right\|_\infty$. Note that $\|g_n\|_\infty = 1$. Choose any subsequence (n_k) and $g_B \in \Lambda_c$ such that $g_{n_k} \to g_B$ as $k \to \infty$ in $C(\Sigma_A^+)$. It is easy to check that $\lim_{k \to \infty} \lambda_{n_k} = \|\mathscr{L}_{B,\varphi} g_B\|_\infty = \lambda_B$, $\mathscr{L}_{B,\varphi} g_B = \lambda_B g_B$ and $\|g_B\|_\infty = 1$. The fact $\lambda_B > 0$ follows from the inequality $\lambda_n \geq e^{-\|\varphi\|_\infty}$ for any $n \geq 1$.

By virtue of Lemma 4.4 (1) and (2), we have $\operatorname{supp} g_B = \Sigma(0) \cup \Sigma(1)$. We write $g_0 = g_B + \chi_{\Sigma(2)}$, $\tilde{\varphi} = \varphi - \log g_0 \circ \sigma_A + \log g_0 - \log \lambda_B$ and $\tilde{\mathscr{L}} = \mathscr{L}_{B,\tilde{\varphi}}$. Put

$$\kappa_j = \exp(2\pi\sqrt{-1} j / p),$$

$$\xi_{j,0} = \chi_{X(0)} + \kappa_j^{-1} \chi_{X(1)} + \cdots + \kappa_j^{1-p} \chi_{X(p-1)} \qquad \text{and}$$

$$\xi_j = \xi_{j,0} + \kappa_j^{-d} (\tilde{\mathscr{L}}^d \xi_{j,0}) \chi_{\Sigma(1)}$$

for each $j \in \mathbf{Z}$. We have the following:

LEMMA 4.5. *For each* $j = 0, 1, \ldots, p-1$, κ_j *is an eigenvalue of* $\tilde{\mathscr{L}}$ *on* $C(\Sigma_A^+)$ *and* ξ_j *is an eigenfunction corresponding to* κ_j. *Furthermore, if* $f \in C(\Sigma_A^+)$ *with*

$f \neq 0$ and $\eta \in \mathbf{C}$ with $|\eta| = 1$ satisfy $\tilde{\mathscr{L}} f = \eta f$, then there exist $0 \leq j < p$ and a constant $c \in \mathbf{C}$ such that $\eta = \kappa_j$ and $f = c\xi_j$.

Proof. Note that if $i_1, i_2 \in S_0$ satisfy $B(i_1 i_2) = 1$ then $i_1 \in S_{0,j}$ and $i_2 \in S_{0,j+1}$ for some j. Therefore for each $j = 0, 1, \ldots, p-1$, $\tilde{\mathscr{L}} \chi_{X(j)}(\omega) = \chi_{X(j+1)}(\omega)$ holds for $\omega \in \Sigma(0)$. It is easy to check that $\mathscr{L}_{B,\varphi} \xi_j = \kappa_j \xi_j$ on $\Sigma(0)$. On the other hand, $\tilde{\mathscr{L}} \xi_j = \kappa_j \xi_j$ on $\Sigma(1)$ by Lemma 4.4(2). Thus ξ_j is an eigenfunction corresponding to the eigenvalue κ_j for each j.

Next we show that if $f \in C(\Sigma_A^+)$ with $f \neq 0$ satisfies $\tilde{\mathscr{L}} f = \kappa_j f$ for some j, then there exists a constant $c \in \mathbf{C}$ such that $f = c\xi_j$. Let $g = \mathrm{Re}\, f$. Since $\tilde{\mathscr{L}}$ is a positive operator, $\tilde{\mathscr{L}}^p g = g$ holds. Choose any $\omega^1, \omega^2 \in X(k)$ such that $\inf_{\omega \in X(k)} g(\omega) = g(\omega^1) = c_1(k)$ and $\sup_{\omega \in X(k)} g(\omega) = g(\omega^2) = c_2(k)$. Then for each $i = 1, 2$ and $n \geq 1$, we have

$$
\begin{aligned}
0 = g(\omega^i) - c_i(k) &= \sum_{u \in S^{np} : u \cdot \omega_0^i \in W_{np+1}(B)} e^{S_{np}\tilde{\varphi}(u \cdot \omega^i)} (g(u \cdot \omega^i) - c_i(k) \chi_{\Sigma(0)}(u \cdot \omega^i)) \\
&= \sum_{u \in S^{np} : u \cdot \omega_0^i \in W_{np+1}(C)} e^{S_{np}\tilde{\varphi}(u \cdot \omega^i)} (g(u \cdot \omega^i) - c_i(k)), \qquad (4.3)
\end{aligned}
$$

where $S_{np}\tilde{\varphi} = \sum_{k=0}^{np-1} \tilde{\varphi} \circ \sigma_A^k$. Since the function $g(u \cdot \omega^i) - c_i(k)$ is nonnegative if $i = 1$ and is nonpositive if $i = 2$, we have $g = c_i(k)$ on $\bigcup_{n=1}^{\infty} \sigma_B^{-np}\{\omega^i\}$ and so $g = c_i(k)$ on $\Sigma_C^+ \cap X(k)$. We notice $c_1(k) = c_2(k)$. Therefore $g = \mathrm{Re}\, f$ is constant on $X(k)$. By a similar argument, we have that $\mathrm{Im}\, f$ is constant on $X(k)$. Thus we see that f is constant $c(k) \in \mathbf{C}$ on $X(k)$ for each k. Now we show $f = c(0)\xi_j$. We note that $\mathscr{L}_{C,\tilde{\varphi}_C} f_C = \kappa_j f_C$ holds. By a basic property of the operator $\mathscr{L}_{C,\tilde{\varphi}_C}$, we have $c(k) = c(0)\kappa_j^{-k}$ for each $k = 1, 2, \ldots, p-1$ (see [10]). Therefore $f = c(0)\xi_j$ on $\Sigma(0)$. The equation $f = c(0)\xi_j$ on $\Sigma(1)$ follows from Lemma 4.4(2).

Finally, we show that if $\eta \in \mathbf{C}$ is an eigenvalue of $\tilde{\mathscr{L}}$ satisfying $|\eta| = 1$ then $\eta^p = 1$. Assume that $f \in C(\Sigma_A^+)$ with $f \neq 0$ and $\eta \in \mathbf{C}$ with $|\eta| = 1$ satisfy $\tilde{\mathscr{L}} f = \eta f$. By virtue of Lemma 4.4(1)(2), we have $f|_{\Sigma(2)} = 0$ and $f|_{\Sigma(0)} \neq 0$. Note that $|f| \leq \tilde{\mathscr{L}}^n |f|$ for any $n \geq 1$. Using a similar method to (4.3), we have that $|f| = \sup_{\omega \in \Sigma(0)} |f(\omega)|$ on Σ_C^+. Thus $f_C \neq 0$. By the (usual) Ruelle-Perron-Frobenius theorem (see [1], [2], [8], [10]) for the operator $\mathscr{L}_{C,\tilde{\varphi}_C}$, the equation $\mathscr{L}_{C,\tilde{\varphi}_C} f_C = \eta f_C$ yields $\eta^p = 1$.

Lemma 4.6. (1) *There exists a constant $c_2 > 0$ such that for all $f \in C(\Sigma_A^+)$ and $n \geq 1$, $\|\tilde{\mathscr{L}}^n f\|_\infty \leq c_2 \|f\|_\infty$.*

(2) *There exist constants $c_3, c_4 > 0$ such that for any $f \in F_\theta(\Sigma_A^+)$ and $n \geq d$,*

$$[\tilde{\mathscr{L}}^n f]_\theta \leq c_3 \theta^n [f]_\theta + c_4 \|f\|_\infty.$$

The proof is quite similar to that of Lemma 3.7 in [7]. So we omit it.

Now we are in a position to be able to apply the Ionescu Tulcea-Marinescu theorem in [5] to the operator $\tilde{\mathscr{L}}$ by virtue of Lemma 4.6. By Lemma 4.5, we have the decomposition

$$\tilde{\mathscr{L}} = \sum_{j=0}^{p-1} \kappa_j \tilde{\mathscr{P}}_j + \tilde{\mathscr{R}} \tag{4.4}$$

of the operator $\tilde{\mathscr{L}} \in \mathscr{L}(F_\theta(\Sigma_A^+))$, where (1) for each j, $\tilde{\mathscr{P}}_j$ is the projection onto the one-dimensional eigenspace corresponding to the eigenvalue κ_j which has the form $\tilde{\mathscr{P}}_j = \lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}(\bar{\kappa}_j)^k \tilde{\mathscr{L}}^k$, (2) $\tilde{\mathscr{P}}_j\tilde{\mathscr{R}} = \tilde{\mathscr{R}}\tilde{\mathscr{P}}_j = O$ for each j and $\tilde{\mathscr{P}}_i\tilde{\mathscr{P}}_j = O$ if $i \neq j$, and (3) the spectral radius of the operator $\tilde{\mathscr{R}}$ on $F_\theta(\Sigma_A^+)$ is less than 1. Since the Banach space $(F_\theta(\Sigma_A^+), \|\cdot\|_\theta)$ is densely embedded into the Banach space $(C(\Sigma_A^+), \|\cdot\|_\infty)$, the operator $\tilde{\mathscr{L}}$ as an element $\mathscr{L}(C(\Sigma_A^+))$ has the decomposition as (4.4). By the simplicity of κ_j, for any $f \in C(\Sigma_A^+)$ there exists a number $\mu_j(f) \in \mathbf{C}$ such that $\tilde{\mathscr{P}}_j f = \mu_j(f)\xi_j$ and $\mu_j(\xi_j) = 1$. We see that μ_j is a bounded linear functional on $C(\Sigma_A^+)$. Therefore, μ_j can be regarded as a complex Borel measure on Σ_A^+ by the Riesz Representation theorem. These measures have the following properties.

LEMMA 4.7. *For each* $j = 0, 1, \dots, p-1$, *we have the following.*

(1) $\tilde{\mathscr{L}}^*\mu_j = \kappa_j\mu_j$.

(2) *The measure* μ_0 *is positive and* $\operatorname{supp}\mu_0 = \Sigma_B^+$ *holds.*

(3) $\mu_j(f\chi_{X(k)}) = \kappa_j^k\mu_0(f\chi_{X(k)})$ *for each* $k = 0, 1, \dots, p-1$ *and* $f \in C(\Sigma_A^+)$.

PROOF. (1) By the decomposition (4.4), $\kappa_j\mu_j(f)\xi_j = \kappa_j\tilde{\mathscr{P}}_j f = \tilde{\mathscr{P}}_j(\kappa_j\tilde{\mathscr{P}}_j f) = \tilde{\mathscr{P}}_j(\tilde{\mathscr{L}}f) = \mu_j(\tilde{\mathscr{L}}f)\xi_j$ for any $f \in C(\Sigma_A^+)$. Thus we have $\tilde{\mathscr{L}}^*\mu_j = \kappa_j\mu_j$.

(2) Note that $(1/n)\sum_{k=0}^{n-1}(\bar{\kappa}_j)^k\tilde{\mathscr{L}}^k$ converges to $\tilde{\mathscr{P}}_j$ in $\mathscr{L}(F_\theta(\Sigma_A^+))$ as $n \to \infty$. Since $\tilde{\mathscr{L}}$ is a positive operator and $\kappa_0 = 1$ holds, $\tilde{\mathscr{P}}_0$ is positive and thus so is μ_0. We show $\operatorname{supp}\mu_0 = \Sigma_B^+$. For $\omega \in \Sigma_A^+\backslash\Sigma_B^+$ with $B(\omega_n\omega_{n+1}) = 0$ for some $n \geq 0$, we have $\mu_0({}_0[\omega_0\dots\omega_n]^A) = \mu_0(\mathscr{L}_{B,\varphi}^{n+1}\chi_{{}_0[\omega_0\dots\omega_n]^A}) = 0$. Therefore $\operatorname{supp}\mu_0 \subset \Sigma_B^+$. We show the converse inclusion. By $\mu_0(\xi_0) = \mu_0(\xi_0\chi_{\Sigma(0)}) = 1$, we see that $\mu_0({}_0[i]^A) > 0$ for some $i \in S_0$. Choose any $\omega \in \Sigma_B^+$ and $n \geq d$. Note that $\omega_{n-1} \in S_0$ by Lemma 4.3. By $C^{n_1+1}(\omega_{n-1}i) > 0$ for some $n_1 \geq 0$, $\mu_0({}_0[\omega_0\dots\omega_{n-1}]^A) = \mu_0(\tilde{\mathscr{L}}^{n+n_1}\chi_{{}_0[\omega_0\dots\omega_{n-1}]^A}) \geq e^{-(n+n_1)\|\tilde{\varphi}\|_\infty}\mu_0({}_0[i]^A) > 0$. Therefore $\omega \in \operatorname{supp}\mu_0$. Hence $\Sigma_B^+ \subset \operatorname{supp}\mu_0$.

(3) Note that $\mu_j(f\chi_{\Sigma(1)}) = 0$ for any $f \in C(\Sigma_A^+)$ by the definition of $\Sigma(1)$. Therefore, by putting $\tilde{\mu}_j = \mu_j|_{\Sigma_C^+}$, $\mathscr{L}_{C,\tilde{\varphi}_C}^*\tilde{\mu}_j = \kappa_j\tilde{\mu}_j$ holds for each $j = 1, 2, \dots, p-1$. Thus, the Ruelle-Perron-Frobenius theorem for the operator $\mathscr{L}_{C,\tilde{\varphi}_C}$ yields the assertion (3) (see [10]).

Now we can prove Theorem 4.1 and Corollary 4.2.

Proof of Theorem 4.1. Put

$$\mathscr{P}_{B,j}f = \left(\int g_0^{-1} f \, d\mu_j\right) g_0 \xi_j, \qquad \mathscr{R}_B f = \lambda_B g_0 \tilde{\mathscr{R}}(g_0^{-1} f),$$

$$\nu_{B,j} = \left(\int g_0^{-1} \, d\mu_0\right)^{-1} g_0^{-1} \mu_j, \qquad h_{B,j} = \left(\int g_0^{-1} \, d\mu_0\right) g_0 \xi_j,$$

$$\lambda_{B,j} = \lambda_B \kappa_j$$

for each $j = 0, 1, \dots, p-1$ and $f \in C(\Sigma_A^+)$. Then we can easily see the validity of the assertions (1)–(5) in the theorem.

Proof of Corollary 4.2. Note that $h_B \nu_B(f) = \tilde{h}_C \tilde{\nu}_C(f)$ for any $f \in C(\Sigma_A^+)$ and $\tilde{\nu}_C(\tilde{h}_C) = \nu_B(h_B) = 1$. The assertion follows immediately from Proposition 4.4(3) and the general theory of the thermodynamic formalism (see [2]).

We prove the following for our later convenience.

Proposition 4.8. *Under the assumptions as in Theorem* 4.1, *we have the following.*

(1) *For each* $\omega \in \Sigma(0) \cup \Sigma(1)$, $\lim_{n\to\infty} (\mathscr{L}_{B,\varphi}^n 1(\omega))^{1/n} = \lambda_B$.

(2) *If* $\nu \in M(\Sigma_A^+)$ *and* $\lambda \in \mathbf{C}$ *with* $|\lambda| = \lambda_B$ *satisfy* $\mathscr{L}_{B,\varphi}^* \nu = \lambda \nu$, *then there exist* $c \in \mathbf{C}$ *and* $0 \le j < p$ *such that* $\nu = c\nu_{B,j}$.

Proof. (1) Note that $\mathscr{L}_{B,\varphi}^n 1 = \mathscr{L}_{B,\varphi}^n \chi_{\Sigma(0)\cup\Sigma(1)}$ for any $n \ge d$. From the inequality

$$\frac{\lambda_B^n h_B}{\|h_B\|_\infty} = \frac{\mathscr{L}_{B,\varphi}^n h_B}{\|h_B\|_\infty} \le \mathscr{L}_{B,\varphi}^n \chi_{\Sigma(0)\cup\Sigma(1)} \le \frac{\mathscr{L}_{B,\varphi}^n h_B}{\inf_{\Sigma(0)\cup\Sigma(1)} h_B} = \frac{\lambda_B^n h_B}{\inf_{\Sigma(0)\cup\Sigma(1)} h_B},$$

the assertion is valid.

(2) Put $\nu_0 = g_0 \nu$ and $\kappa = \lambda/\lambda_B$. We have $\tilde{\mathscr{L}}^* \nu_0 = \kappa \nu_0$. Assume that $\kappa^p \ne 1$. Since $\kappa^{np} \nu_0(f) = \nu_0(\tilde{\mathscr{L}}^{np} f) = \sum_{i=0}^{p-1} \mu_i(f) \nu_0(\xi_i) + \nu_0(\tilde{\mathscr{R}}^{np} f)$ converges to $\sum_{i=0}^{p-1} \mu_i(f) \nu_0(\xi_i)$ as $n \to \infty$ for any $f \in C(\Sigma_A^+)$, ν_0 must be 0. On the other hand, we assume $\kappa = \kappa_j$ for some $0 \le j < p$. Note that $\nu_0(f\chi_{\Sigma(1)}) = 0$ for any $f \in C(\Sigma_A^+)$ and $\nu_0(X(k)) = \kappa_j^k \nu_0(X(0))$ for any k. Moreover, it is easy to see that $\nu_0(\xi_j) = p\nu_0(X(0))$ and $\nu_0(\xi_i) = 0$ if $i \ne j$. Thus we obtain that for any $f \in C(\Sigma_A^+)$,

$$\nu_0(f) = \nu_0(\kappa_j^{-n} \tilde{\mathscr{L}}^n f) = \sum_{i=0}^{p-1} (\kappa_i \kappa_j^{-1})^n \mu_i(f) \nu_0(\xi_i) + \kappa_j^{-n} \nu_0(\tilde{\mathscr{R}}^n f)$$

$$= p\nu_0(X(0))\mu_j(f) + \kappa_j^{-n} \nu_0(\tilde{\mathscr{R}}^n f) \to p\nu_0(X(0))\mu_j(f)$$

as $n \to \infty$. Hence $\nu = c\nu_{B,j}$ with $c = \nu_0(X(0)) p \mu_0(g_0^{-1})$.

5. Auxiliary results

In this section, using the pointwise exponential growth rate of $\mathscr{L}_{B,\varphi}^n 1$, we obtain a decomposition of the space Σ_A^+ and the detailed structure of eigenspaces of $\mathscr{L}_{B,\varphi}$ and $\mathscr{L}_{B,\varphi}^*$ with maximal modulus. Those results will be used to prove the main results.

First we state the following theorem.

THEOREM 5.1. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Then there exist a decomposition of* Σ_A^+ *as* $\Sigma_A^+ = Y(0) \cup Y(1) \cup \cdots \cup Y(r)$ *and numbers* $\eta_0 > \eta_1 > \cdots > \eta_r \geq -\infty$ *such that the following are valid.*

(1) *Each set* $Y(i)$ *is an open and closed subset of* Σ_A^+.

(2) *For any* $\omega \in Y(i)$, $\lim_{n\to\infty} \frac{1}{n} \log \mathscr{L}_{B,\varphi}^n 1(\omega) = \eta_i$, *where* $\log 0$ *is regarded as* $-\infty$.

Recall that for each $k \in T$, $\tilde{\lambda}_k$ denotes the spectral radius of the operator $\mathscr{L}_{B_{kk},\varphi}$ on $C(\Sigma_A^+)$ and $\Sigma_k = \bigcup_{i \in S(k)} {}_0[i]^A$, where $S(k)$ is the index set of the matrix B_{kk}. We put $\tilde{\lambda} = \max_{k\in T} \tilde{\lambda}_k$. As we mentioned in the preceding section, Theorem 4.1 is applicable to $\mathscr{L}_{B_{kk},\varphi}$ in the case when $B_{kk} \neq (0)$. We denote by $\lambda_{B_{kk}}$ the resulting eigenvalue. Notice that for each k, $\tilde{\lambda}_k = \lambda_{B_{kk}}$ if $B_{kk} \neq (0)$ and $\tilde{\lambda}_k = 0$ if $B_{kk} = (0)$. In order to prove Theorem 5.1, we need the following:

PROPOSITION 5.2. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Put* $T(k) = \{k' \in T : k' \prec k\}$ *for each* $k \in T$ *and define* $\lambda(\cdot) : T \to [0, \infty)$ *by* $\lambda(k) = \max_{k' \in T(k)} \tilde{\lambda}_{k'}$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathscr{L}_{B,\varphi}^n 1(\omega) = \log \lambda(k) \tag{5.1}$$

holds for each $k \in T$ *and* $\omega \in \Sigma_k$.

Once Proposition 5.2 is established, it is easy to prove Theorem 5.1. Indeed, let $\eta_0 > \eta_1 > \cdots > \eta_r$ be the distinct values of $\log \lambda(k)$'s. Put $Y(i) = \bigcup_{k\in T}\{\Sigma_k : \eta_i = \log \lambda(k)\}$ for each $i = 0, 1, \ldots, r$. Then the assertions (1) and (2) in Theorem 5.1 follow.

PROOF OF PROPOSITION 5.2. We write $\mathscr{L}_B = \mathscr{L}_{B,\varphi}$ for simplicity. First we consider the case when $\lambda(k) = 0$. In this case it is obvious that for any $k' \in T(k)$, $\tilde{\lambda}_{k'} = 0$ and so $B_{k'k'} = (0)$. Therefore $\mathscr{L}_B^n 1(\omega) = 0$ for any $n \geq d$ and $\omega \in \bigcup_{k' \in T(k)} \Sigma_{k'}$. Thus (5.1) holds for any $k \in T$ with $\lambda(k) = 0$.

Next we consider the case when $\lambda(k) > 0$. We prove the following inequalities which yield the validity of the assertion:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathscr{L}_B^n 1(\omega) \geq \log \lambda(k) \qquad \text{for each } \omega \in \Sigma_k, \text{ and} \tag{5.2}$$

$$\limsup_{n\to\infty} \frac{1}{n} \max_{\omega\in\Sigma_k} \log \mathscr{L}_B^n 1(\omega) \leq \log \lambda(k). \tag{5.3}$$

The inequality (5.2) is proved as follows. Note that $\lim_{n\to\infty}(1/n)\log \mathscr{L}_{B_{kk}}^n 1(\omega) = \log \tilde{\lambda}_k$ for each $\omega \in \Sigma_k$ by replacing B and $\Sigma(0)\cup\Sigma(1)$ with B_{kk} and Σ_k, respectively, in Proposition 4.8(1). Choose any $k \in T$ and $k' \in T(k)$ satisfying $\lambda(k) > 0$ and $\tilde{\lambda}_{k'} = \lambda(k)$. Then for any $\omega \in \Sigma_k$ and $\omega' \in \Sigma_{k'}$ there exists a word $w \in S^{d-1}$ such that $\omega_0' \cdot w \cdot \omega_0$ is B-admissible. Therefore we have

$$\mathscr{L}_B^{n+d} 1(\omega) = \mathscr{L}_B^d(\mathscr{L}_B^n 1)(\omega) \geq c_5 \mathscr{L}_B^n 1(\omega_0' \cdot w \cdot \omega) \geq c_5 \mathscr{L}_{B_{k'k'}}^n 1(\omega_0' \cdot w \cdot \omega),$$

where $c_5 = e^{-d\|\varphi\|_\infty}$. Furthermore,

$$\begin{aligned}\frac{1}{n+d} \log \mathscr{L}_B^{n+d} 1(\omega) &\geq \frac{1}{n+d} \log c_5 + \frac{1}{n+d} \log \mathscr{L}_{B_{k'k'}}^n 1(\omega_0' \cdot w \cdot \omega)\\ &\to \log \tilde{\lambda}_{k'} = \log \lambda(k)\end{aligned}$$

as $n \to \infty$. Thus the inequality (5.2) is valid.

The inequality (5.3) is proved for each $k \in T$ inductively, as follows. If $k = 1 \in T$, $k' \prec k$ yields $k' = k = 1$. This implies that $\lambda(1) = \tilde{\lambda}_1$ and $\mathscr{L}_B^n 1(\omega) = \mathscr{L}_{B_{11}}^n 1(\omega)$ for any $\omega \in \Sigma_1$ and $n \geq 1$. Therefore we see that $\limsup_{n\to\infty}(1/n)\log \max_{\omega\in\Sigma_1} \mathscr{L}_B^n 1(\omega) = \limsup_{n\to\infty}(1/n)\log\max_{\omega\in\Sigma_1} \mathscr{L}_{B_{11}}^n 1(\omega) = \log \lambda(1)$. The inequality (5.3) is valid when $k = 1$.

Next we prove that if (5.3) holds for each $1 \leq k' < k$, then so does for k. Let B_1 be a $d \times d$ matrix with entries 0 or 1 such that $B_1(ij) = B(ij)$ if $i \in S_k \cup \cdots \cup S_m$ and $B_1(ij) = 0$ otherwise. Put $B_2 = B - B_1$. Then $\mathscr{L}_B$ has the form $\mathscr{L}_B = \mathscr{L}_{B_1} + \mathscr{L}_{B_2}$. Since $B_1 B_2 = O$, we have $\mathscr{L}_{B_2}\mathscr{L}_{B_1} = O$. Furthermore, $\mathscr{L}_{B_1}^n 1(\omega) = \mathscr{L}_{B_{kk}}^n 1(\omega)$ for any $\omega \in \Sigma_k$ and $n \geq 0$. We have that for any $\omega \in \Sigma_k$,

$$\begin{aligned}\mathscr{L}_{B,\varphi}^n 1(\omega) = (\mathscr{L}_{B_1} + \mathscr{L}_{B_2})^n 1(\omega) &= \sum_{j=0}^n \mathscr{L}_{B_1}^{n-j} \mathscr{L}_{B_2}^j 1(\omega)\\ &\leq \sum_{j=0}^n \mathscr{L}_{B_{kk}}^{n-j} 1(\omega) \max_{\tilde{\omega}\in\Sigma_k} \mathscr{L}_{B_2}^j 1(\tilde{\omega}).\end{aligned}$$

Note that $\|\mathscr{L}_{B_{kk}}^n 1\|_\infty \leq c_6 \lambda(k)^n$ holds for any $n \geq 0$ and for some constant c_6. Indeed, if $\tilde{\lambda}_k > 0$, then it follows from the uniform boundedness of $\tilde{\lambda}_k^{-n}\|\mathscr{L}_{B_{kk}}^n 1\|_\infty$ for n. If $\tilde{\lambda}_k = 0$, then $\mathscr{L}_{B_{kk}}^n = O$ for any $n \geq d$ and thus it is valid by putting $c_6 = \max_{0\leq j\leq d}(\|\mathscr{L}_{B_{kk}}^j 1\|_\infty \lambda(k)^{-j})$. On the other hand, we claim that for any $\varepsilon > 0$ there exist numbers $n_0 \geq 1$ and $c_7 > 0$ such that

$\mathscr{L}_{B_2}^n 1(\omega) \le c_7(\lambda(k)e^{\varepsilon})^n$ for any $n \ge n_0$ and $\omega \in \Sigma_k$. Indeed, note that for $k' \in T(k)\backslash\{k\}$, the inequalities (5.3) and $\lambda(k') \le \lambda(k)$ are satisfied. We can choose $n_1 \ge 1$ so that $\mathscr{L}_B^n 1(\omega') \le (\lambda(k)e^{\varepsilon})^n$ for any $k' \in T(k)\backslash\{k\}$, $n \ge n_1$ and $\omega' \in \Sigma_{k'}$. Note that if $B_2(ij) = 1$ for some $j \in S(k)$ then $i \in S(k')$ for some $k' \in T(k)\backslash\{k\}$. Therefore we put $n_0 = n_1 + 1$ and $c_7 = \lambda(k)^{-1}\|\mathscr{L}_{B_2}1\|_\infty$. Consequently, we obtain

$$\begin{aligned}\frac{1}{n}\log\max_{\omega\in\Sigma_k}\mathscr{L}_B^n 1(\omega) &\le \frac{1}{n}\log\left(\sum_{j=0}^{n_0-1} c_6\|\mathscr{L}_{B_2}^j 1\|_\infty \lambda(k)^{n-j} + \sum_{j=n_0}^{n} c_6 c_7 \lambda(k)^n e^{\varepsilon j}\right)\\ &\le \log\lambda(k) + \frac{1}{n}\log c_8 + \frac{1}{n}\log\left(\sum_{j=0}^{n} e^{\varepsilon j}\right)\\ &\to \log\lambda(k) + \varepsilon\end{aligned}$$

as $n \to \infty$, where c_8 is a sufficiently large number depending on n_0. Therefore we have that the inequality (5.3) is valid for each $k \in T$. Now the proof of (5.1) is complete.

In the rest of this section, we give auxiliary results on eigenfunctions of $\mathscr{L}_{B,\varphi}$ and eigenvectors of $\mathscr{L}_{B,\varphi}^*$ with maximal modulus by using Proposition 5.2. Assume that two matrices A and B satisfy the conditions (Σ.1)–(Σ.3) and B has the form (2.1). Let $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. For each $k \in T$ with $B_{kk} \ne (0)$, p_k denotes the period of the matrix B_{kk}. Consider the triplet $(\lambda_{B_{kk},j}, h_{B_{kk},j}, \nu_{B_{kk},j}) \in \mathbf{R} \times C(\Sigma_A^+) \times M(\Sigma_A^+)$ for $j = 0, 1, \dots, p_k - 1$ obtained by putting $B = B_{kk}$ in Theorem 4.1. Then $\operatorname{supp} h_{B_{kk},j} = \Sigma_k$ and $\operatorname{supp} \nu_{B_{kk}} = \Sigma_{B_{kk}}^+$ hold. We need the following notation in the sequel. For each $k \in T_0$ and $\eta \in \mathbf{C}$ with $|\eta| = \tilde{\lambda}$, we put

$$(h_0(\eta,k), \nu_0(\eta,k)) = \begin{cases} (h_{B_{kk},j}, \nu_{B_{kk},j}) & \text{if } \eta = \lambda_{B_{kk},j} \text{ for some } 0 \le j \le p_k - 1\\ (0,0) & \text{otherwise.}\end{cases}$$

Note that for any $k \in T_1 \cup T_2$, the spectral radius $\tilde{\lambda}_k$ of the operator $\mathscr{L}_{B_{kk},\varphi}$ on $C(\Sigma_A^+)$ is strictly less than $\tilde{\lambda}$. We write T_1 as $\{k_1, \dots, k_s\}$ with $k_1 < \cdots < k_s$. For each $k \in T_0$ and $\eta \in \mathbf{C}$ with $|\eta| = \tilde{\lambda}$, let $h(\eta,k) \in C(\Sigma_A^+)$ be such that $h(\eta,k) = h_0(\eta,k) + h_1(\eta,k)$, where $h_1(\eta,k) = \sum_{i=1}^s h(\eta,k,k_i)$,

$$h(\eta,k,k_1) = (\eta I - \mathscr{L}_{B_{k_1k_1},\varphi})^{-1}\mathscr{L}_{B_{kk_1},\varphi}h_0(\eta,k) \qquad \text{and}$$

$$h(\eta,k,k_i) = (\eta I - \mathscr{L}_{B_{k_ik_i},\varphi})^{-1}\mathscr{L}_{B_{kk_i},\varphi}h_0(\eta,k) + \sum_{j=1}^{i-1}(\eta I - \mathscr{L}_{B_{k_ik_i},\varphi})^{-1}\mathscr{L}_{B_{k_jk_i},\varphi}h(\eta,k,k_j)$$

for each $i = 2, 3, \dots, s$, inductively. Similarly, we write T_2 as $\{l_1, \dots, l_t\}$ with $l_1 > \cdots > l_t$. Let $\nu(\eta, k) \in M(\Sigma_A^+)$ be such that $\nu(\eta, k) = \nu_0(\eta, k) + \nu_1(\eta, k)$, where $\nu_1(\eta, k) = \sum_{i=1}^{t} \nu(\eta, k, l_i)$,

$$\nu(\eta, k, l_1) = (\mathscr{L}_{B_{l_1 k}, \varphi}(\eta I - \mathscr{L}_{B_{l_1 l_1}, \varphi})^{-1})^* \nu_0(\eta, k) \qquad \text{and}$$

$$\nu(\eta, k, l_i) = (\mathscr{L}_{B_{l_i k}, \varphi}(\eta I - \mathscr{L}_{B_{l_i l_i}, \varphi})^{-1})^* \nu_0(\eta, k) + \sum_{j=1}^{i-1} (\mathscr{L}_{B_{l_i l_j}, \varphi}(\eta I - \mathscr{L}_{B_{l_i l_i}, \varphi})^{-1})^* \nu(\eta, k, l_j)$$

for each $i = 2, \dots, t$, inductively.

We have the following:

Proposition 5.3. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Assume that* $g \in C(\Sigma_A^+)$ *and* $\lambda \in \mathbf{C}$ *with* $|\lambda| = \tilde{\lambda}$ *satisfy* $\mathscr{L}_{B,\varphi} g = \lambda g$. *Then we have the following:*

(1) $g = 0$ *on* $\Sigma(2)$.

(2) *If* $g = 0$ *on* $\Sigma(0)$, *then* $g = 0$ *on* $\Sigma(1)$.

(3) *If the condition* $(\Sigma\Phi)$ *holds, then there exists a vector* (β_k) *in* $\mathbf{C}^{m_0}$ *such that* g *has the form* $g = \sum_{k \in T_0} \beta_k h(\lambda, k)$.

Proposition 5.4. *Assume that* $(\Sigma.1)$, $(\Sigma.2)$ *and* $(\Sigma.3)$ *are satisfied. Let* $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. *Assume that* $\nu \in M(\Sigma_A^+)$ *and* $\lambda \in \mathbf{C}$ *with* $|\lambda| = \tilde{\lambda}$ *satisfy* $\mathscr{L}_{B,\varphi}^* \nu = \lambda \nu$. *Then we have the following:*

(1) $\operatorname{supp} \nu \subset \Sigma_B^+$ *and* $\nu|_{\Sigma(1)} = 0$.

(2) *If* $\nu|_{\Sigma(0)} = 0$, *then* $\nu|_{\Sigma(2)} = 0$.

(3) *If the condition* $(\Sigma\Phi)$ *holds, then there exists a vector* (γ_k) *in* $\mathbf{C}^{m_0}$ *such that* $\nu = \sum_{k \in T_0} \gamma_k \nu(\lambda, k)$.

Proof of Proposition 5.3. As before, we write $\mathscr{L}_B = \mathscr{L}_{B,\varphi}$ for simplicity. Note that $(\mathscr{L}_B(f\chi_{\Sigma_k}))\chi_{\Sigma_{k'}} = \mathscr{L}_{B_{kk'}} f$ holds for each $k, k' \in T$ and $f \in C(\Sigma_A^+)$.

(1) If $k \in T_2$ and $k' \in T$ satisfy $k' \prec k$, then $k' \in T_2$. Therefore, $\mathscr{L}_B^n(g\chi_{\Sigma(2)})\chi_{\Sigma(2)} = (\mathscr{L}_B^n g)\chi_{\Sigma(2)} = \lambda^n g\chi_{\Sigma(2)}$ for any $n \geq 1$. Suppose $g \neq 0$ on $\Sigma(2)$. Then we have $\|g\chi_{\Sigma(2)}\|_\infty > 0$. It follows from Proposition 5.2 that

$$\tilde{\lambda} = |\lambda| = \left\| \mathscr{L}_B^n \left(\frac{g\chi_{\Sigma(2)}}{\|g\chi_{\Sigma(2)}\|_\infty} \right) \chi_{\Sigma(2)} \right\|_\infty^{1/n} \leq \|(\mathscr{L}_B^n 1)\chi_{\Sigma(2)}\|_\infty^{1/n} \to \lambda_0$$

as $n \to \infty$, where $\lambda_0 = \max_{k \in T_2} \tilde{\lambda}_k$. This contradicts the fact $\lambda_0 < \tilde{\lambda}$. Thus $g = 0$ on $\Sigma(2)$.

(2) We write T_1 as $\{k_1, k_2, \dots, k_s\}$ with $k_1 < k_2 < \cdots < k_s$. It is easy to see that $k \prec k_i$ yields $k \in T_0 \cup T_2 \cup \{k_1, k_2, \dots, k_i\}$ for each i. Therefore we have

$$\lambda g\chi_{\Sigma_{k_i}} = (\mathscr{L}_B g)\chi_{\Sigma_{k_i}} = \sum_{k\in T_0\cup T_2\cup\{k_1,k_2,\dots,k_i\}} \mathscr{L}_{B_{kk_i}} g = \sum_{k\in T_0} \mathscr{L}_{B_{kk_i}} g + \sum_{j=1}^{i} \mathscr{L}_{B_{k_jk_i}} g$$

by (1) for each i. Note that $\tilde{\lambda}_{k_1} < |\lambda| = \tilde{\lambda}$ by the definition of T_1. We have

$$g\chi_{\Sigma_{k_1}} = \sum_{k\in T_0} (\lambda I - \mathscr{L}_{B_{k_1k_1}})^{-1} \mathscr{L}_{B_{kk_1}} g \qquad \text{and} \tag{5.4}$$

$$g\chi_{\Sigma_{k_i}} = (\lambda I - \mathscr{L}_{B_{k_ik_i}})^{-1} \left(\sum_{k\in T_0} \mathscr{L}_{B_{kk_i}} g + \sum_{j=1}^{i-1} \mathscr{L}_{B_{k_jk_i}} g \right) \tag{5.5}$$

for each $i = 2, 3, \dots, s$. Assume that $g = 0$ on $\Sigma(0)$. Then $\mathscr{L}_{B_{kk'}} g = 0$ for any $k \in T_0$ and $k' \in T$. We have $g = 0$ on Σ_{k_i}, $i = 1, 2, \dots, s$, inductively and thus $g = 0$ on $\Sigma(1)$.

(3) Let $k \in T_0$. We see that there is no element $k' \in T_0$ such that $k \neq k'$ and $k' \prec k$. Therefore the relation $k' \prec k$ implies $k' \in T_2 \cup \{k\}$. We have $\mathscr{L}_{B_{kk},\varphi}(g\chi_{\Sigma_k}) = \lambda g\chi_{\Sigma_k}$ by (1). If $g \neq 0$ on Σ_k, then there exist an integer $0 \le j < p_k$ and a number $\beta_k \in \mathbf{C}$ such that $\lambda = \lambda_{B_{kk},j}$ and $g\chi_{\Sigma_k} = \beta_k h_0(\lambda, k)$ by Theorem 4.1 for the operator $\mathscr{L}_{B_{kk}}$. As a result, we see that g has the form $g = \sum_{k\in T_0} \beta_k h_0(\lambda, k)$ on $\Sigma(0)$ for some vector (β_k) in $\mathbf{C}^{m_0}$. We notice that the equations (5.4) and (5.5) yield the form $g = \sum_{k\in T_0} \beta_k h_1(\lambda, k)$ on $\Sigma(1)$. Thus we have $g = \sum_{k\in T_0} \beta_k h(\lambda, k)$.

PROOF OF PROPOSITION 5.4. (1) First, we show $\operatorname{supp} \nu \subset \Sigma_B^+$. For any $\omega \in \Sigma_A^+ \backslash \Sigma_B^+$, $B(\omega_n\omega_{n+1}) = 0$ for some $n \ge 0$. Then $\nu({}_0[\omega_0 \dots \omega_{n+1}]^A) = \lambda^{-n-1}\nu(\mathscr{L}_B^{n+1}\chi_{{}_0[\omega_0\dots\omega_{n+1}]^A}) = 0$. Thus, $\operatorname{supp} \nu \subset \Sigma_B^+$.

Next, we show $\nu|_{\Sigma(1)} = 0$. If $k' \in T_1$ satisfies $k \prec k'$ for some $k \in T_1$, then $k' \in T_1$. Therefore $\lambda^n \nu(f\chi_{\Sigma(1)}) = \nu(\mathscr{L}_B^n(f\chi_{\Sigma(1)})) = \nu(\chi_{\Sigma(1)}\mathscr{L}_B^n(f\chi_{\Sigma(1)}))$ for any $f \in C(\Sigma_A^+)$ and $n \ge 1$. Suppose $\nu(f\chi_{\Sigma(1)}) \neq 0$ for some $f \in C(\Sigma_A^+)$. Let B_1 be a $d \times d$ matrix with entries 0 or 1 such that $B_1(ij) = B(ij)$ if $i, j \in S_1$ and $B_1(ij) = 0$ otherwise. Then we notice that $\chi_{\Sigma(1)}\mathscr{L}_B^n(f\chi_{\Sigma(1)}) = \mathscr{L}_{B_1}^n f$ for any $n \ge 1$. If $\Sigma_{B_1}^+ = \varnothing$, then $\mathscr{L}_{B_1}^n = 0$ for any $n \ge d$. Therefore we may assume $\Sigma_{B_1}^+ \neq \varnothing$. Putting $B = B_1$ in Proposition 5.2, we have

$$\tilde{\lambda} = |\lambda| = \left| \frac{\nu(\mathscr{L}_{B_1}^n f)}{\nu(f\chi_{\Sigma(1)})} \right|^{1/n} \le \left(\frac{|\nu|(1)\|f\|_\infty}{|\nu(f\chi_{\Sigma(1)})|} \right)^{1/n} \|\mathscr{L}_{B_1}^n 1\|_\infty^{1/n} \to \lambda_1$$

as $n \to \infty$, where $|\nu|$ is the total variation of the measure ν and $\lambda_1 = \max_{k\in T_1} \tilde{\lambda}_k$. This contradicts the fact $\tilde{\lambda} > \lambda_1$. Thus, $\nu|_{\Sigma(1)} = 0$.

(2) We write T_2 as $\{l_1, l_2, \dots, l_t\}$ with $l_1 > l_2 > \dots > l_t$. Since $l_i \prec l$ means $l \in T_0 \cup T_1 \cup \{l_1, l_2 \dots, l_i\}$ for any i, we see

$$\lambda(g\chi_{\Sigma_{l_i}}) = \sum_{l\in T_0} \nu(\mathscr{L}_{B_{l_il}} g) + \sum_{j=1}^{i-1} \nu(\mathscr{L}_{B_{l_il_j}} g) + \nu(\mathscr{L}_{B_{l_il_i}} g)$$

for any $g \in C(\Sigma_A^+)$ and $i = 1, 2, \ldots, t$. For $f \in C(\Sigma_A^+)$, it follows from this equation that

$$\nu(f\chi_{\Sigma_{l_1}}) = \sum_{l \in T_0} \nu(\mathscr{L}_{B_{l_1 l}}(\lambda I - \mathscr{L}_{B_{l_1 l_1}})^{-1} f) \qquad \text{and} \tag{5.6}$$

$$\nu(f\chi_{\Sigma_{l_i}}) = \sum_{l \in T_0} \nu(\mathscr{L}_{B_{l_i l}}(\lambda I - \mathscr{L}_{B_{l_i l_i}})^{-1} f) + \sum_{j=1}^{i-1} \nu(\mathscr{L}_{B_{l_i l_j}}(\lambda I - \mathscr{L}_{B_{l_i l_i}})^{-1} f) \tag{5.7}$$

for each $i = 2, 3, \ldots, t$, by putting $g = (\lambda I - \mathscr{L}_{B_{l_i l_i}})^{-1} f$. Note that if $\nu|_{\Sigma_l} = 0$ then $\nu(\mathscr{L}_{B_{l_i l}} g) = 0$ for any i and $g \in C(\Sigma_A^+)$. Assume that $\nu|_{\Sigma(0)} = 0$. Then we have $\nu|_{\Sigma_{l_i}} = 0$ for $i = 1, 2, \ldots, t$, inductively. Thus $\nu|_{\Sigma(2)}$ must be 0.

(3) Assume that the condition $(\Sigma\Phi)$ is satisfied. We easily see that for any $l \in T_0$ and $f \in C(\Sigma_A^+)$, the equation $\lambda\nu(f\chi_{\Sigma_l}) = \nu(\mathscr{L}_{B_{ll}}(f\chi_{\Sigma_l}))$ holds. By virtue of Proposition 4.8(2), we have $\nu|_{\Sigma_l} = \gamma_l \nu_0(\lambda, l)$ for some $\gamma_l \in \mathbf{C}$. Thus $\nu|_{\Sigma(0)} = \sum_{l \in T_0} \gamma_l \nu_0(\lambda, l)$ is satisfied. We notice that (5.6) and (5.7) imply $\nu|_{\Sigma(2)} = \sum_{l \in T_0} \gamma_l \nu_1(\lambda, l)$. Hence we have $\nu = \sum_{l \in T_0} \gamma_l \nu(\lambda, l)$.

6. Proof of main results

This section is devoted to the proof of our main results.

Proof of Theorem 3.2. First we show that (i) implies (ii). Put

$$g = \sum_{k \in T_0} h(\tilde{\lambda}, k),$$

where each $h(\tilde{\lambda}, k)$ is defined in Section 5. The condition $(\Sigma\Phi)$ implies that g is an eigenfunction corresponding to the eigenvalue $\tilde{\lambda}$ of the operator $\mathscr{L}_{B,\varphi}$. In particular, we see that g is a nonnegative function whose support is $\Sigma(0) \cup \Sigma(1)$. We define a $d \times d$ matrix B_1 with entries 0 or 1 by $B_1(ij) = B(ij)$ if $i, j \in S_0 \cup S_1$ and $B_1(ij) = 0$ otherwise. Put $B_2 = B - B_1$. Then $\mathscr{L}_{B,\varphi}$ has the form $\mathscr{L}_{B,\varphi} = \mathscr{L}_{B_1,\varphi} + \mathscr{L}_{B_2,\varphi}$. We see that $B_1 B_2 = O$ from the definition of S_2. We have the inequality

$$\begin{aligned}\mathscr{L}_{B,\varphi}^n 1 = (\mathscr{L}_{B_1,\varphi} + \mathscr{L}_{B_2,\varphi})^n 1 &= \sum_{k=0}^{n} \mathscr{L}_{B_1,\varphi}^k \mathscr{L}_{B_2,\varphi}^{n-k} 1 \\ &\le \sum_{k=0}^{n} \Big(\|\mathscr{L}_{B_1,\varphi}^k 1\|_\infty \|\mathscr{L}_{B_2,\varphi}^{n-k} 1\|_\infty \Big) \end{aligned} \tag{6.1}$$

for any $n \ge 1$. It follows from Proposition 5.2 that

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathscr{L}_{B_2,\varphi}^n 1\|_\infty = \log \tilde{\lambda}_0,$$

where $\tilde{\lambda}_0 = \max_{k \in T_2} \tilde{\lambda}_k < \tilde{\lambda}$. Thus for any $\varepsilon > 0$ satisfying $\tilde{\lambda}_0 < e^{-\varepsilon}\tilde{\lambda}$ there exists a constant $c_9 \geq 1$ such that $\|\mathscr{L}^n_{B_2,\varphi} 1\|_\infty \leq c_9(e^{-\varepsilon}\tilde{\lambda})^n$ for any $n \geq 1$. On the other hand, for any $n \geq 1$ and $\omega \in \Sigma(0) \cup \Sigma(1)$, we have

$$\begin{aligned}\mathscr{L}^n_{B_1,\varphi} 1(\omega) &= \mathscr{L}^n_{B_1,\varphi} \chi_{\Sigma(0)\cup\Sigma(1)}(\omega) = \mathscr{L}^n_{B,\varphi} \chi_{\Sigma(0)\cup\Sigma(1)}(\omega) \\ &\leq \frac{1}{\inf_{\Sigma(0)\cup\Sigma(1)} g} \mathscr{L}^n_{B,\varphi} g(\omega) = \frac{1}{\inf_{\Sigma(0)\cup\Sigma(1)} g} \tilde{\lambda}^n g(\omega) \leq c_{10}\tilde{\lambda}^n,\end{aligned}$$

where $c_{10} = \|g\|_\infty / \inf_{\Sigma(0)\cup\Sigma(1)} g$. Consequently, the inequality (6.1) implies that

$$\|\mathscr{L}^n_{B,\varphi} 1\|_\infty \leq \sum_{k=0}^{n} c_9 c_{10} \tilde{\lambda}^n (e^{-\varepsilon})^{n-k} \leq \frac{c_9 c_{10}}{1 - e^{-\varepsilon}} \tilde{\lambda}^n$$

for any $n \geq 1$. Hence the condition (ii) is fulfilled.

Next, we show that (ii) implies (iii). Suppose that there exists an eigenvalue λ_{j_0} of the operator $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ that is not semisimple. Then the nilpotent $\mathscr{N}_{j_0}$ corresponding to λ_{j_0} is not a zero operator. Choose any $f \in F_\theta(\Sigma_A^+)$ such that $\mathscr{N}_{j_0} f \neq 0$. Put $0 < n_j \leq d$ such that $\mathscr{N}_j^{n_j} = O$ and $\mathscr{N}_j^{n_j - 1} \neq O$ for each j. The decomposition (3.1) of $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ yields the form

$$\mathscr{L}^n_{B,\varphi} = \sum_{j=0}^{q-1} \left(\lambda_j^n \mathscr{P}_j + \sum_{i=1}^{n_j - 1} \binom{n}{i} \lambda_j^{n-i} \mathscr{N}_j^i \right) + \mathscr{R}^n \tag{6.2}$$

for $n \geq d$. We obtain

$$\begin{aligned}\|\tilde{\lambda}^{-n} \mathscr{L}^n_{B,\varphi} f\|_\infty &\geq \sum_{j=0}^{q-1} \sum_{i=1}^{n_j - 1} \binom{n}{i} \tilde{\lambda}^{-i} \|\mathscr{N}_j^i f\|_\infty - \sum_{j=0}^{q-1} \|\mathscr{P}_j f\|_\infty - \tilde{\lambda}^{-n} \|\mathscr{R}^n f\|_\infty \\ &\geq n\tilde{\lambda}^{-1} \|\mathscr{N}_{j_0} f\|_\infty - \sum_{j=0}^{q-1} \|\mathscr{P}_j f\|_\infty - \tilde{\lambda}^{-n} \|\mathscr{R}^n f\|_\infty \to \infty\end{aligned}$$

as $n \to \infty$. Therefore the condition (ii) does not hold. Thus (ii) yields (iii).

The implication (iii) $\Rightarrow$ (iv) is trivial.

Finally, we show that (iv) implies (i). Assume that $(\Sigma\Phi)$ does not hold. Then there exist $k, k' \in T_0$ with $k \neq k'$ such that $\tilde{\lambda}_k = \tilde{\lambda}_{k'} = \tilde{\lambda}$ and $k \prec k'$. We show that the generalized eigenspace and the eigenspace do not coincide for the eigenvalue $\tilde{\lambda}$. For this purpose, we construct a function $h \in F_\theta(\Sigma_A^+)$ such that

$$(\tilde{\lambda} I - \mathscr{L}_{B,\varphi}) h \neq 0 \qquad \text{and} \qquad (\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^2 h = 0. \tag{6.3}$$

Let $e_0 \in T$ be a maximal element of the set

$$\{k \in T_0 : \text{there exists } k' \in T_0 \text{ such that } \tilde{\lambda}_k = \tilde{\lambda}_{k'} = \tilde{\lambda}, k \prec k' \text{ and } k \neq k'\}$$

in the sense that there is no element k of the set above such that $e_0 \prec k$ and $e_0 \neq k$. Put

$$T_{0,0} = \{k \in T_0 \backslash \{e_0\} : \tilde{\lambda}_k = \tilde{\lambda} \text{ and } e_0 \prec k\},$$

$$T_{0,1} = \{k \in T_0 \backslash (\{e_0\} \cup T_{0,0}) : e_0 \prec k\},$$

$$T_{0,2} = \{k \in T_0 : e_0 \prec k \text{ does not hold}\}.$$

Then T_0 is decomposed into the subsets $\{e_0\}$, $T_{0,0}$, $T_{0,1}$, $T_{0,2}$. Note that if $k \in T_{0,1}$ then $\tilde{\lambda}_k < \tilde{\lambda}$ and there exists $k' \in T_{0,0}$ such that $e_0 \prec k \prec k'$. We write $T_{0,1}$ as $\{e_1, e_2, \ldots, e_r\}$ with $e_1 < e_2 < \cdots < e_r$, and T_1 as $\{k_1, k_2, \ldots, k_s\}$ with $k_1 < k_2 < \cdots < k_s$.

We will construct such $h \in F_\theta(\Sigma_A^+)$ as the composition of the function h_k on Σ_k defined as follows:

(a) $h_k = 0$ if $k \in T_{0,2} \cup T_2$.

(b) $h_{e_0} = h_0(\tilde{\lambda}, e_0)$, which is a nonnegative eigenfunction corresponding to the eigenvalue $\tilde{\lambda}$ of the operator $\mathscr{L}_{B_{e_0 e_0}, \varphi}$ supported on Σ_{e_0}.

(c) $h_{e_i} = (\tilde{\lambda} I - \mathscr{L}_{B_{e_i e_i}, \varphi})^{-1} (\sum_{j=0}^{i-1} \mathscr{L}_{B_{e_j e_i}, \varphi} h_{e_j})$ for each $i = 1, 2, \ldots, r$, inductively.

(d) $h_{k_i} = \sum_{j=1}^{2} (\tilde{\lambda} I - \mathscr{L}_{B_{k_i k_i}, \varphi})^{-j} (\sum_{k \in \{e_0\} \cup T_{0,0} \cup T_{0,1} \cup \{k_1, \ldots, k_{i-1}\}} \mathscr{L}_{B_{k k_i}, \varphi} h_k)$ for each $i = 1, 2, \ldots, s$, inductively.

It remains to define h_k for $k \in T_{0,0}$. Let $k \in T_{0,0}$. By Theorem 4.1 for the operator $\mathscr{L}_{B_{kk}, \varphi}, \mathscr{L}_{B_{kk}, \varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ has the spectral decomposition

$$\mathscr{L}_{B_{kk}, \varphi} = \tilde{\lambda} \mathscr{P}_{B_{kk}} + \mathscr{E}_k$$

such that $\mathscr{P}_{B_{kk}}$ is the projection onto the eigenspace corresponding to $\tilde{\lambda}$ of the operator $\mathscr{L}_{B_{kk}, \varphi}$ and $\mathscr{E}_k$ satisfies $\mathscr{E}_k \mathscr{P}_{B_{kk}} = \mathscr{P}_{B_{kk}} \mathscr{E}_k = O$. We put $h_k = (\tilde{\lambda} I - \mathscr{E}_k)^{-1} g_k$ with $g_k = (I - \mathscr{P}_{B_{kk}})(\sum_{j=0}^{r} \mathscr{L}_{B_{e_j k}, \varphi} h_{e_j})$. The function h_k satisfies

$$(\tilde{\lambda} I - \mathscr{L}_{B_{kk}, \varphi}) h_k = (\tilde{\lambda} I - \mathscr{E}_k) h_k = g_k.$$

We set $h = \sum_{k \in T} h_k$.

It is easy to check that for each $k \in T_2 \cup \{e_0\} \cup T_{0,1} \cup T_{0,2}$, $(\tilde{\lambda} I - \mathscr{L}_{B,\varphi}) h = (\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^2 h = 0$ on Σ_k. For each $k \in T_{0,0}$ and $\omega \in \Sigma_k$, we have

$$\begin{aligned}(\tilde{\lambda} I - \mathscr{L}_{B,\varphi}) h(\omega) &= (\tilde{\lambda} I - \mathscr{L}_{B_{kk},\varphi}) h(\omega) - (\mathscr{L}_{B,\varphi} - \mathscr{L}_{B_{kk},\varphi}) h(\omega) \\ &= g_k(\omega) - \sum_{j=0}^{r} \mathscr{L}_{B_{e_j k}, \varphi} h_{e_j}(\omega) = -\mathscr{P}_{B_{kk}} \left(\sum_{j=0}^{r} \mathscr{L}_{B_{e_j k}, \varphi} h_{e_j} \right)(\omega).\end{aligned}$$

Note that the last term is non-zero since each function h_{e_j} is nonnegative and $e_0 \prec k$ holds. Therefore this equation implies that $(\tilde{\lambda} I - \mathscr{L}_{B,\varphi})h \neq 0$. On the other hand, we have $(\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^2 h(\omega) = 0$. It is not hard to see that for $k \in T_1$, $(\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^2 h = 0$ on Σ_k. Thus h satisfies (6.3). Consequently, $\tilde{\lambda}$ is not semisimple. We see that (iv) implies (i).

PROOF OF THEOREM 3.3. We use the notation $h(\lambda, k)$ and $v(\lambda, k)$ in Section 5. Choose any $\lambda \in \mathbf{C}$ which is an eigenvalue of the operator $\mathscr{L}_{B,\varphi} \in \mathscr{L}(C(\Sigma_A^+))$ satisfying $|\lambda| = \tilde{\lambda}$. Denote by $\mathscr{P} \in \mathscr{L}(C(\Sigma_A^+))$ the projection onto the eigenspace corresponding to the eigenvalue λ. By Proposition 5.3(3), the eigenspace is spanned by $\{h(\lambda, k) : k \in T_0(\lambda)\}$, where $T_0(\lambda) = \{k \in T_0 : h(\lambda, k) \neq 0\}$. Therefore for any $f \in C(\Sigma_A^+)$, there exist numbers $\mu_k(f) \in \mathbf{C}$ with $\mu_k(h(\lambda, k)) = 1$ such that $\mathscr{P} f = \sum_{k \in T_0(\lambda)} \mu_k(f) h(\lambda, k)$. We see that each μ_k is a linear functional and thus it is a complex Borel measure on Σ_A^+. By a similar argument, we have $\mathscr{L}_{B,\varphi}^* \mu_k = \lambda \mu_k$. By virtue of Proposition 5.4(3), $\mu_k = \gamma_k v(\lambda, k)$ holds for some $\gamma_k \in \mathbf{C}$ by $\operatorname{supp} \mu_k \cap \Sigma(0) \subset \Sigma_k$. Furthermore, $1 = \mu_k(h(\lambda, k)) = \gamma_k v(\lambda, k)(h(\lambda, k)) = \gamma_k$. Consequently we have the desired form $\mathscr{P}_j(f) = \sum_{k \in T_0(\lambda_j)} (\int f \, dv(\lambda_j, k)) h(\lambda_j, k)$ for each $j = 0, 1, \dots, q-1$ and $f \in C(\Sigma_A^+)$.

PROOF OF PROPOSITION 3.4. Assume that $\mathscr{L}_{B,\varphi} g = \lambda g$ for some $g \in C(\Sigma_A^+)$ with $g \neq 0$ and $\lambda \in \mathbf{C}$ with $|\lambda| = \tilde{\lambda}$. Then $g = 0$ on $\Sigma(2)$ by Proposition 5.3(1). We write T_0 as $\{m_1, m_2, \dots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$. Put $k = m_1$. We see that $k' \prec k$ yields $k' \in T_2 \cup \{k\}$. Therefore, we have

$$\mathscr{L}_{B_{kk},\varphi}(g\chi_{\Sigma_k}) = (\mathscr{L}_{B,\varphi} g)\chi_{\Sigma_k} = \lambda g \chi_{\Sigma_k}$$

on Σ_A^+. Since B_{kk} is irreducible and $\tilde{\lambda}_k = \tilde{\lambda}$ holds, if $g \neq 0$ on Σ_k then $\lambda^{p_k} = \tilde{\lambda}^{p_k}$ for the period p_k of the matrix B_{kk}. Thus either $\lambda^{p_k} = \tilde{\lambda}^{p_k}$ or $g = 0$ on Σ_k.

Let $k = m_i$ for some $i > 1$. Then $k' \prec k$ yields $k' \in T_2 \cup \{m_1, m_2, \dots, m_i\}$. If $g = 0$ on $\Sigma_{m_1} \cup \Sigma_{m_2} \cup \cdots \cup \Sigma_{m_{i-1}}$, then

$$\mathscr{L}_{B_{kk},\varphi}(g\chi_{\Sigma_k}) = \lambda g \chi_{\Sigma_k}$$

holds on Σ_A^+. If $\tilde{\lambda}_k < \tilde{\lambda}$, then $g\chi_{\Sigma_k}$ must be 0 by the above equation. If $\tilde{\lambda}_k = \tilde{\lambda}$, then either $\lambda^{p_k} = \tilde{\lambda}^{p_k}$ or $g = 0$ on Σ_k by the above argument. Inductively, we have that either $\lambda^{p_k} = \tilde{\lambda}^{p_k}$ for some $k \in T_0$ with $\tilde{\lambda}_k = \tilde{\lambda}$ or $g = 0$ on $\Sigma(0)$. By Proposition 5.3(2), if $g = 0$ on $\Sigma(0)$ then $g = 0$ on $\Sigma(1)$ and thus $g = 0$. This contradicts the fact $g \neq 0$. Hence $\lambda^{p_k} = \tilde{\lambda}^{p_k}$ holds for some $k \in T_0$ with $\tilde{\lambda}_k = \tilde{\lambda}$.

7. Examples

In this section, we give some typical examples which illustrate our results. Let A and B be 6×6 matrices with entries 0 or 1 as follows.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & * & * & * & * \\ 1 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ O & B_{22} \end{pmatrix}, \tag{7.1}$$

where

$$B_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B_{22} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the conditions (Σ.1), (Σ.2) and (Σ.3) are satisfied. We write $S(1) = \{1,2\}$ and $S(2) = \{3,4,5,6\}$. Put $\Sigma_k = \bigcup_{i \in S(k)} {}_0[i]$ for each $k = 1,2$. Choose any $\varphi \in F_\theta(\Sigma_A^+ \to \mathbf{R})$. Consider the triplet $(\tilde{\lambda}_k, \tilde{h}_k, \tilde{v}_k) = (\lambda_{B_{kk}}, h_{B_{kk}}, v_{B_{kk}}) \in \mathbf{R} \times C(\Sigma_A^+) \times M(\Sigma_A^+)$ for $k = 1,2$ obtained by putting $B = B_{kk}$ in Theorem 4.1. Since the period of B_{22} is 2, $-\tilde{\lambda}_2$ is an eigenvalue of the operator $\mathscr{L}_{B_{22},\varphi}$ and the corresponding eigenfunction has the form $\tilde{h}_{2,1} = \tilde{h}_2|_{X(0)} - \tilde{h}_2|_{X(1)}$ up to constant multiplier, where $X(0) = {}_0[3] \cup {}_0[4]$ and $X(1) = {}_0[5] \cup {}_0[6]$. Similarly, the corresponding eigenvector to the eigenvalue $-\tilde{\lambda}_2$ of the dual $\mathscr{L}^*_{B_{22},\varphi}$ has the form $\tilde{v}_{2,1} = \tilde{v}_2|_{X(0)} - \tilde{v}_2|_{X(1)}$ up to constant multiplier. Put $\tilde{\lambda} = \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$. We consider the following cases.

Example 7.1 (The case $\tilde{\lambda}_1 > \tilde{\lambda}_2$). We put $\varphi(\omega) = 1$ if $\omega \in \Sigma_1$ and $\varphi(\omega) = 0$ if $\omega \in \Sigma_2$. Then $\mathscr{L}_{B,\varphi} \in \mathscr{L}(C(\Sigma_A^+))$ has the decomposition

$$\mathscr{L}_{B,\varphi} f = \tilde{\lambda}\left(\int f \, dv(\tilde{\lambda}, 1)\right) h(\tilde{\lambda}, 1) + \mathscr{R} f,$$

where $h(\tilde{\lambda}, 1) = \tilde{h}_1 + (\tilde{\lambda} I - \mathscr{L}_{B_{22},\varphi})^{-1} \mathscr{L}_{B_{12},\varphi} \tilde{h}_1$ and $v(\tilde{\lambda}, 1) = \tilde{v}_1$. Indeed, we easily see that $T_0 = \{1\}$ and $C = B_{11}$. Therefore the condition ($\Sigma\Phi$) is satisfied. Since B_{11} is aperiodic, the set of eigenvalues of $\mathscr{L}_{B,\varphi}$ with maximal modulus is $\{\tilde{\lambda}\}$. Note that $T_1 = \{2\}$ and $T_2 = \varnothing$ if $B_{12} \neq O$ and that $T_1 = \varnothing$ and $T_2 = \{2\}$ if $B_{12} = O$. Thus Theorem 3.3 yields the assertion.

Example 7.2 (The case $\tilde{\lambda}_1 < \tilde{\lambda}_2$). We put $\varphi(\omega) = 0$ if $\omega \in \Sigma_1$ and $\varphi(\omega) = 1$ if $\omega \in \Sigma_2$. Then $\mathscr{L}_{B,\varphi} \in \mathscr{L}(C(\Sigma_A^+))$ has the decomposition

$$\mathscr{L}_{B,\varphi} f = \sum_{j=0}^{1} (-1)^j \tilde{\lambda} \left(\int f \, d\nu((-1)^j \tilde{\lambda}, 2) \right) h((-1)^j \tilde{\lambda}, 2) + \mathscr{R} f,$$

where $h(\tilde{\lambda}, 2)$, $\nu(\tilde{\lambda}, 2)$, $h(-\tilde{\lambda}, 2)$, and $\nu(-\tilde{\lambda}, 2)$ have the form

$$h(\tilde{\lambda}, 2) = \tilde{h}_2, \qquad \nu(\tilde{\lambda}, 2) = \tilde{\nu}_2 + (\mathscr{L}_{B_{12},\varphi}(\tilde{\lambda} I - \mathscr{L}_{B_{11},\varphi})^{-1})^* \tilde{\nu}_2,$$

$$h(-\tilde{\lambda}, 2) = \tilde{h}_{2,1}, \qquad \nu(-\tilde{\lambda}, 2) = \tilde{\nu}_{2,1} + (\mathscr{L}_{B_{12},\varphi}(-\tilde{\lambda} I - \mathscr{L}_{B_{11},\varphi})^{-1})^* \tilde{\nu}_{2,1},$$

respectively. Indeed, we have $T_0 = \{2\}$, $T_1 = \emptyset$ and $T_2 = \{1\}$. Note that B_{22} is irreducible whose period is 2. The assertion is valid from Theorem 3.3.

Example 7.3 (The case $\tilde{\lambda}_1 = \tilde{\lambda}_2$). We put $\varphi = 1$. We have the following:
(a) Assume that $B_{12} = O$. Then $\mathscr{L}_{B,\varphi} \in \mathscr{L}(C(\Sigma_A^+))$ has the form

$$\mathscr{L}_{B,\varphi} f = \tilde{\lambda} \left(\int f \, d\nu(\tilde{\lambda}, 1) \right) h(\tilde{\lambda}, 1) + \sum_{j=0}^{1} (-1)^j \tilde{\lambda} \left(\int f \, d\nu((-1)^j \tilde{\lambda}, 2) \right) h((-1)^j \tilde{\lambda}, 2) + \mathscr{R} f,$$

where $h(\tilde{\lambda}, k) = \tilde{h}_k$ and $\nu(\tilde{\lambda}, k) = \tilde{\nu}_k$ for $k = 1, 2$, and $h(-\tilde{\lambda}, 2) = \tilde{h}_{2,1}$ and $\nu(-\tilde{\lambda}, 2) = \tilde{\nu}_{2,1}$.

(b) Assume that $B_{12} \neq O$. Then the set of eigenvalues of $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$ with maximal modulus is equal to $\{\tilde{\lambda}, -\tilde{\lambda}\}$. In particular, $\tilde{\lambda}$ is not semisimple and $-\tilde{\lambda}$ is simple (consequently semisimple).

Indeed, we note that $T_0 = \{1, 2\}$ and $T_1 = T_2 = \emptyset$. First we assume $B_{12} = O$. Since the condition $(\Sigma\Phi)$ is satisfied, the assertion (a) follows from Theorem 3.3. Next we assume $B_{12} \neq O$. We see that $(\Sigma\Phi)$ does not hold. By Theorem 3.2, $\tilde{\lambda}$ is not a semisimple eigenvalue of the operator $\mathscr{L}_{B,\varphi} \in \mathscr{L}(F_\theta(\Sigma_A^+))$. On the other hand, the set of eigenvalues of $\mathscr{L}_{B,\varphi}$ with maximal modulus is equal to either $\{\tilde{\lambda}\}$ or $\{\tilde{\lambda}, -\tilde{\lambda}\}$ by Proposition 3.4. Since $\mathscr{L}_{B,\varphi} \tilde{h}_{2,1} = \mathscr{L}_{B_{22},\varphi} \tilde{h}_{2,1} = -\tilde{\lambda} \tilde{h}_{2,1}$ holds, we see that $-\tilde{\lambda}$ is an eigenvalue of $\mathscr{L}_{B,\varphi}$. It remains to show that $-\tilde{\lambda}$ is a simple eigenvalue of the operator $\mathscr{L}_{B,\varphi}$. Assume that $f \in F_\theta(\Sigma_A^+)$ with $f \neq 0$ satisfies $(-\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^n f = 0$ for some $n \geq 1$. Note that $\mathscr{L}_{B,\varphi}^k f = \mathscr{L}_{B_{11},\varphi}^k f$ on Σ_1 for any $k \geq 1$. Since $-\tilde{\lambda}$ is not an eigenvalue of the operator $\mathscr{L}_{B_{11},\varphi}$, we have $f = 0$ on Σ_1. Thus $(-\tilde{\lambda} I - \mathscr{L}_{B,\varphi})^n f = (-\tilde{\lambda} I - \mathscr{L}_{B_{22},\varphi})^n f = 0$. The simplicity of $-\tilde{\lambda}$ of the operator $\mathscr{L}_{B_{22},\varphi}$ yields $n = 1$. Hence $-\tilde{\lambda}$ is simple.

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