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The exactness of the log homotopy sequence

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ABSTRACT. We develop the theory of log homotopy exact sequences associated to proper log smooth morphisms and morphisms whose characteristic sheaves are locally constant with stalks isomorphic to the monoid of natural numbers. In the process of developing this theory, we also show the existence of a logarithmic version of the Stein factorization and develop the theory of algebraization of log formal schemes.

1. Introduction

In the study of the geometry of log schemes, the following objects often appear:

(i) a proper log smooth fibration over a log regular base log scheme,
(ii) a morphism (of log schemes) whose characteristic sheaf is locally constant with stalk isomorphic to $\mathbf{N}$.

In this paper, the behavior of the log fundamental group for such an object is studied; in particular, it is shown that the homotopy sequence associated to such a morphism is *exact*.

This paper is organized as follows.

In Section 2, we prove the existence of *a logarithmic version of the Stein factorization* under some hypotheses (cf. Definition 3, Theorem 1, also Remark 3). In [5], Exposé X, Corollaire 1.4, the exactness of the homotopy sequence associated to a proper separable morphism is proven. In this proof, the existence of the Stein factorization plays an essential role. Therefore, to prove a logarithmic analogue of the exactness of the homotopy sequence, we consider the existence of a logarithmic analogue of the Stein factorization.

In Section 3, we prove a logarithmic analogue of [5], Exposé X, Corollaire 1.4, i.e., the exactness of the *log homotopy sequence* by means of the existence of the log Stein factorization (cf. Theorem 2). Moreover, a logarithmic analogue of the fact that the fundamental group of the product of schemes is naturally isomorphic to the product of the fundamental groups of these schemes (cf. [5], Exposé X, Corollaire 1.7) is proven (cf. Proposition 3).

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In Section 4, we define the notion of a log structure on a formal scheme and establish a theory of *algebraizations* of log formal schemes. One can develop a theory of algebraizations of log formal schemes (cf. Theorem 3) in a similar fashion to the classical theory of algebraizations of formal schemes (for example, the theory considered in [2], § 5). This algebraization theory of formal log schemes implies a logarithmic analogue of the fact that the fundamental group of a proper smooth scheme over a "complete base" is naturally isomorphic to the fundamental group of the closed fiber (cf. [5], Exposé X, Théorème 2.1, also [16], Théorème 2.2, (a)) (cf. Corollary 1). This result is used in the next section.

In Section 5, we define the notion of a *morphism of type* $\mathbf{N}^{\oplus n}$ and consider fundamental properties of such a morphism. Roughly speaking, a morphism of log schemes is of type $\mathbf{N}^{\oplus n}$ if the relative characteristic sheaf is locally constant with stalk isomorphic to $\mathbf{N}^{\oplus n}$. The main result of this section is the fact that at the level of anabelioids (i.e., Galois categories) (determined by ket coverings), certain morphisms of type $\mathbf{N}^{\oplus n}$ can be regarded as "$\mathbf{G}_m^{\times n}$-fibrations" (cf. Theorem 5). Moreover, as in [11], Lemma 4.4, we give a sufficient condition for the homomorphism from the log fundamental group of the fiber of the "$\mathbf{G}_m^{\times n}$-fibration" determined by such a morphism of type $\mathbf{N}^{\oplus n}$ to the log fundamental group of the total space of the "$\mathbf{G}_m^{\times n}$-fibration" to be injective (cf. Proposition 4).

In Appendix A, we prove analogues for the étale site of the results given in [9] for the Zariski site, since such analogues will be necessary in the present paper.

Finally, in Appendix B, we prove the well-known fact that the category of ket coverings of a connected fs log scheme is a *Galois category*; this implies, in particular, the existence of log fundamental groups (cf. Theorem B.1, also Theorem B.2). The log fundamental group has already been constructed by several people (e.g., [1]; [6], 4.6; [15], 3.3; [16], 1.2). Since, however, at the time of writing, a proof of this fact was not available in published form, and, moreover, various facts used in the proof of this fact are necessary elsewhere in this paper, we decided to give a proof of this fact. Moreover, although other authors approach the problem of showing this fact by considering the category of locally constant sheaves on the Kummer log étale site, we take a more direct approach to this problem which allows us to avoid the use of locally constant sheaves on the Kummer log étale site.

Notations and Terminologies

Sets:

We shall assume that the underlying topological space of a *connected* scheme is not empty. In particular, if a morphism is geometrically connected, then it is surjective.

Numbers:

We shall denote by $\mathbf{N}$ the monoid of rational integers $n \geq 0$, by $\mathbf{Z}$ the ring of rational integers, by $\mathbf{Q}$ the field of rational numbers, by $\hat{\mathbf{Z}}$ (respectively, $\mathbf{Z}_l$) the profinite completion of $\mathbf{Z}$ (respectively, pro-l completion of $\mathbf{Z}$ for a prime number l), and by $\mathbf{Q}_l$ the field of fractions of $\mathbf{Z}_l$.

Let Σ be a set of prime numbers, and n an integer. Then we shall say that n is a *Σ-integer* if the prime divisors of n are in Σ.

Groups:

Let G be a profinite group, and Σ a non-empty set of prime numbers. We shall refer to the quotient

$$\varprojlim G/H$$

of G (where the projective limit is over all open normal subgroups $H \subseteq G$ such that the index $[G : H]$ of H is a Σ-integer) as the *maximal pro-Σ quotient of* G. We shall denote by $G^{(\Sigma)}$ the maximal pro-Σ quotient of G.

Log schemes:

For a log scheme $X^{\log}$, we shall denote by $\mathscr{M}_X$ (respectively, X) the sheaf of monoids that defines the log structure (respectively, the underlying scheme) of $X^{\log}$. For a morphism $f^{\log}$ of log schemes, we shall denote by f the underlying morphism of schemes.

Let $\mathscr{P}$ be a property of schemes [for example, "quasi-compact", "connected", "normal", "regular"] (respectively, morphisms of schemes [for example, "proper", "finite", "étale", "smooth"]). Then we shall say that a log scheme (respectively, a morphism of log schemes) satisfies $\mathscr{P}$ if the underlying scheme (respectively, the underlying morphism of schemes) satisfies $\mathscr{P}$.

For fs log schemes $X^{\log}$, $Y^{\log}$, and $Z^{\log}$, we shall denote by $X^{\log} \times_{Y^{\log}} Z^{\log}$ the fiber product of $X^{\log}$ and $Z^{\log}$ over $Y^{\log}$ in the category of *fs log schemes*. In general, the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is *not naturally isomorphic* to $X \times_Y Z$. However, since strictness (a morphism $f^{\log} : X^{\log} \to Y^{\log}$ is called *strict* if the induced morphism $f^*\mathscr{M}_Y \to \mathscr{M}_X$ on X is an isomorphism) is stable under base-change in the category of *arbitrary* log schemes, if $X^{\log} \to Y^{\log}$ is strict, then the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is naturally isomorphic to $X \times_Y Z$. Note that since the natural morphism from the saturation of a fine log scheme to the original fine log scheme is finite, properness and finiteness are stable under fs base-change.

If there exist both schemes and log schemes in a commutative diagram, then we regard each scheme in the diagram as the log scheme obtained by equipping the scheme with the trivial log structure.

We shall refer to the largest open subset (possibly empty) of the underlying scheme of a log scheme on which the log structure is trivial as the *interior* of the log scheme.

We shall refer to a Kummer log étale (respectively, finite Kummer log étale) morphism of fs log schemes as a *ket* morphism (respectively, a *ket covering*).

Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log}: X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ induced by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

2. The log Stein factorization

In this section, we show the existence of a logarithmic version of the Stein factorization.

Definition 1. Let $X^{\log}$ be an fs log scheme, and $\bar{x} \to X$ a geometric point.

(i) We shall refer to the strict morphism $\bar{x}^{\log} \to X^{\log}$ whose underlying morphism of schemes is $\bar{x} \to X$ as the *strict geometric point* over $\bar{x} \to X$.

(ii) We shall refer to $\bar{x}_1^{\log} \to X^{\log}$ as a *reduced covering point* over the strict geometric point $\bar{x}^{\log} \to X^{\log}$ or, alternatively, over the geometric point $\bar{x} \to X$, if it is obtained as a composite

$$\bar{x}_1^{\log} \to \overline{x'}_1^{\log} \to \bar{x}^{\log} \to X^{\log},$$

where $\bar{x}^{\log} \to X^{\log}$ is the strict geometric point over $\bar{x} \to X$, $\overline{x'}_1^{\log} \to \bar{x}^{\log}$ is a connected ket covering, and $\bar{x}_1^{\log} \to \overline{x'}_1^{\log}$ is a strict morphism of fs log schemes for which the underlying morphism of schemes determines an isomorphism $\bar{x}_1 \simeq \overline{x'}_{1,\mathrm{red}}$. Note that, in general, $\bar{x}_1^{\log} \to \bar{x}^{\log}$ is *not* a ket covering. (See Remark 1 below.)

Remark 1. The underlying scheme of the domain of a strict geometric point $\bar{x}^{\log} \to X^{\log}$ is the spectrum of a separably closed field. However, in general, the underlying scheme of the domain of a connected ket covering $\overline{x'}_1^{\log} \to \bar{x}^{\log}$ is not the spectrum of a separably closed field. On the other hand, if we denote by $\bar{x}_1^{\log}$ the log scheme obtained by equipping $\overline{x'}_{1,\mathrm{red}}$ with the log structure induced by the log structure of $\overline{x'}_1^{\log}$ (i.e., the natural morphism $\bar{x}_1^{\log} \to X^{\log}$ is a reduced covering point over $\bar{x}^{\log} \to X^{\log}$), then the following hold.

(i) The underlying scheme of $\bar{x}_1^{\log}$ is the spectrum of a separably closed field (cf. Proposition B.2).

(ii) There is a natural equivalence between the category of ket coverings of $\bar{x}_1^{\log}$ and the category of ket coverings of $\overline{x'}_1^{\log}$ (cf. Proposition B.6). In particular, $\pi_1(\overline{x'}_1^{\log}) \simeq \pi_1(\bar{x}_1^{\log})$. (Concerning the log fundamental group, see Theorem B.1.)

(iii) The natural morphism $\bar{x}_1^{\log} \to \overline{x'}_1^{\log}$ is a homeomorphism on the underlying topological spaces and remains so after any base-change in the category of *fs log schemes* over $\overline{x'}_1^{\log}$. Indeed, this follows from the fact that this morphism is strict, together with the fact that the underlying morphism of schemes is a universal homeomorphism.

The following technical lemma follows immediately from Proposition B.6.

Lemma 1. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring. Then for a strict geometric point $\bar{x}^{\log} \to X^{\log}$ for which the image of the underlying morphism of schemes is the closed point of X, and any reduced covering point $\bar{x}_1^{\log} \to X^{\log}$ over $\bar{x}^{\log} \to X^{\log}$, there exists a ket covering $Y^{\log} \to X^{\log}$ and a strict geometric point $\bar{y}^{\log} \to Y^{\log}$ such that $\bar{y}^{\log} \to Y^{\log} \to X^{\log}$ factors as a composite $\bar{y}^{\log} \to \bar{x}_1^{\log} \to X^{\log}$, where the morphism $\bar{y}^{\log} \to \bar{x}_1^{\log}$ is a reduced covering point over the strict geometric point $\bar{x}_1^{\log} \to \bar{x}_1^{\log}$ given by the identity morphism of $\bar{x}_1^{\log}$.*

Lemma 2. *Let $X^{\log}$ be an fs log scheme equipped with the trivial log structure, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. Then the morphism $X' \to X$ that appears in the Stein factorization $Y \to X' \to X$ of f is finite étale.*

Proof. By [5], Exposé X, Proposition 1.2, it is enough to show that f is proper and separable. The properness of f is assumed in the statement of Lemma 2. Since the log structure of $X^{\log}$ is trivial, $f^{\log}$ is integral (cf. [8], Proposition 4.1). Since an integral log smooth morphism is flat (cf. [8], Theorem 4.5), f is flat. For the rest of the proof of the separability of f, by base-changing, we may assume that $X = \operatorname{Spec} k$, where k is a field whose characteristic we denote by p. Then étale locally on Y, there exist an fs monoid P whose associated group P^{gp} is p-torsion-free if p is not zero, and an étale morphism $Y \to \operatorname{Spec} k[P]$ over k (cf. [8], Theorem 3.5). On the other hand, $k[P] \otimes_k K \subseteq k[P^{\mathrm{gp}}] \otimes_k K$, and $k[P^{\mathrm{gp}}] \otimes_k K = K[P^{\mathrm{gp}}]$ is reduced for any extension field K of k by the assumption on P^{gp}; thus, $k[P] \otimes_k K$, hence also $Y \otimes_k K$, is reduced. Therefore, f is separable.

Lemma 3. *Let $X^{\log}$ be a log regular log scheme (cf. Definition* A.1*), $U_X \subseteq X$ the interior of $X^{\log}$, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. If we denote by $Y \times_X U_X \to V \to U_X$ the Stein factorization of $f|_{Y \times_X U_X}$, then the following hold.*

(i) *$V \to U_X$ is finite étale.*

(ii) *The normalization of X in V is tamely ramified over the generic points of $X \setminus U_X$.*

Proof. Since log smoothness and properness are stable under base-change, assertion (i) follows from Lemma 2. For assertion (ii), since normalization and the operation of taking Stein factorization commute with étale localization, we may assume that X is the spectrum of a strictly henselian discrete valuation ring R, and the log structure of $X^{\log}$ is defined by the closed point of X (cf. Proposition A.6). Moreover, for assertion (ii), we may assume that V is connected. Let us write k for the residue field of R, $g: X' \to X$ for the normalization of X in V, and $R' \stackrel{\text{def}}{=} \Gamma(X', \mathcal{O}_{X'})$. Note that since $V \to U_X$ is finite étale, $X' \to X$ is finite; in particular, R' is a strictly henselian discrete valuation ring. Let $X'^{\log}$ be the log scheme obtained by equipping X' with the log structure defined by the closed point of X', and k' the residue field of R'.

First, I *claim* that $f^{\log}: Y^{\log} \to X^{\log}$ factors through the morphism $g^{\log}: X'^{\log} \to X^{\log}$. Indeed, since $Y^{\log}$ is log regular (cf. Proposition A.5), Y is normal (cf. Proposition A.3). Thus, if we denote by $Y \to Z \to X$ the Stein factorization of f, then Z is normal; in particular, the morphism $Z \to X$ factors through $X' \to X$. Therefore, it follows that the morphism f factors as the composite $Y \stackrel{f'}{\to} X' \to X$. Moreover, since $Y^{\log}$ is log regular, and the interior of $Y^{\log}$ is included in $f'^{-1}(U_{X'})$ $(= f^{-1}(U_X))$, where $U_{X'}$ is the interior of $X'^{\log}$, it follows from Proposition A.6 that the morphism f' extends to a morphism of log schemes, i.e., the morphism $f^{\log}$ factors as the composite $Y^{\log} \xrightarrow{f'^{\log}} X'^{\log} \longrightarrow X^{\log}$.

Next, I *claim* that the field extension $k \subseteq k'$ induced by g is separable, i.e., the morphism $k \to k'$ is an isomorphism. Indeed, this follows from Lemma 4 below, together with the first *claim*.

Let e be the ramification index of the finite flat extension $R \to R'$. Finally, I *claim* that e is prime to the characteristic of k. Indeed, assume that $e = 0$ in k. Then the k-algebra $R' \otimes_R k$ is isomorphic to $k[t]/(t^e)$, where t is an indeterminate, and by the definitions, the $R' \otimes_R k$-module $\Omega^1_{X'^{\log}/X} \otimes_R k$ is isomorphic to the quotient of $k[t]/(t^e)dt \oplus k[t]/(t^e)$ by the $k[t]/(t^e)$-submodule generated by $(nt^{n-1}\,dt, -nt^n)$, where $n \in \mathbf{N}$, i.e., the $R' \otimes_R k$-module $\Omega^1_{X'^{\log}/X} \otimes_R k$ is a free $R' \otimes_R k$-module of rank 1 with basis consisting of an element "$dt/t = (0,1)$". In a similar vein, the k-module $\Omega^1_{X^{\log}/X} \otimes_R k$ is isomorphic to k. Moreover, the morphism of $R' \otimes_R k$-modules

$$\phi : k[t]/(t^e) \simeq (\Omega^1_{X^{\log}/X} \otimes_R k) \otimes_k (R' \otimes_R k) \to \Omega^1_{X'^{\log}/X} \otimes_R k \simeq k[t]/(t^e)dt/t$$

induced by $g^{\log}$ maps $1 \in k[t]/(t^e)$ to $e \cdot dt/t + u^{-1}\,du = u^{-1}\,du$, where $u \in (k[t]/(t^e))^*$ is a unit of $k[t]/(t^e)$; thus, the image of the morphism $\phi \otimes_{R' \otimes_R k} k$

vanishes. On the other hand, since the morphism $f^{\log}$ is log smooth, it follows from [8], Proposition 3.12, that the natural morphism $f^*\Omega^1_{X^{\log}/X} \to \Omega^1_{Y^{\log}/X}$ is injective, and its image is locally a direct summand of $\Omega^1_{Y^{\log}/X}$. Therefore, since $f^{\log}$ factors through $g^{\log}$ by the first *claim*, it follows that the morphism $\phi \otimes_{R' \otimes_R k} k$ is also *injective*. Thus, we obtain a contradiction. This completes the proof of Lemma 3.

Lemma 4. *Let $X^{\log}$ be an fs log scheme whose underlying scheme is the spectrum of a field k, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a log smooth morphism. Then the subset $S_{Y/X} \subseteq Y$ of Y consisting of closed points y such that the field extension $k \subseteq k(y)$ is separable is dense in Y.*

Proof. To prove Lemma 4, by base-changing, we may assume that k is separably closed. First, observe that it is enough to show that for any geometric point $\bar{y} \to Y$ of Y, there exists an étale neighborhood $U \to Y$ of $\bar{y} \to Y$ such that $S_{U/X} \subseteq U$ is dense in U.

Let $\bar{y} \to Y$ be a geometric point of Y, and $P \to k$ a clean chart of $X^{\log}$ (cf. Definition B.1, (ii)). Then it follows from [8], Theorem 3.5, that there exists a chart

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ k & \longrightarrow & A \end{array}$$

of $U^{\log} \to X^{\log}$, where $U \stackrel{\text{def}}{=} \operatorname{Spec} A \to Y$ is an étale neighborhood of the geometric point $\bar{y} \to Y$, $U^{\log}$ is the log scheme obtained by equipping U with the log structure induced by the log structure of $Y^{\log}$, and Q an fs monoid, such that the morphism $k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \to A$ induced by the above diagram is *étale*; in particular, $U \to \operatorname{Spec} k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ is *open*. Thus, to prove the assertion, we may assume that $Y^{\log}$ is the log scheme obtained by equipping $\operatorname{Spec} k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ with the log structure determined by the chart $Q \to k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$.

Let k_0 be the prime field which is included in k. Then since the chart $P \to k$ is *clean*, this chart factors as $P \xrightarrow{\alpha} k_0 \subseteq k$. Let $X_0^{\log}$ (respectively, $Y_0^{\log}$) be the log scheme obtained by equipping $X_0 \stackrel{\text{def}}{=} \operatorname{Spec} k_0$ (respectively, $Y_0 \stackrel{\text{def}}{=} \operatorname{Spec} k_0 \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$) with the log structure determined by the chart $P \xrightarrow{\alpha} k_0$ (respectively, $Q \to k_0 \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$). Then we have a *cartesian* diagram

$$\begin{array}{ccc} Y^{\log} & \longrightarrow & Y_0^{\log} \\ \downarrow & & \downarrow \\ X^{\log} & \longrightarrow & X_0^{\log}. \end{array}$$

Let $\operatorname{Spec} k(y_0) \to Y_0$ be the natural morphism determined by a closed point $y_0 \in Y_0$ of Y_0. Then since the field k_0 is perfect, the morphism $\operatorname{Spec} k(y_0) \times_{Y_0} Y \simeq \operatorname{Spec} k(y_0) \times_{X_0} X \xrightarrow{\mathrm{pr}_2} X$ is étale. (Note that since $y_0 \in Y_0$ is a closed point, the composite $\operatorname{Spec} k(y_0) \to Y_0 \to X_0$ is finite.) Thus, the image of $\operatorname{Spec} k(y_0) \times_{Y_0} Y \to Y$ is included in $S_{Y/X} \subseteq Y$. Therefore, the assertion follows from the fact that the subset of Y_0 consisting of the closed points is *dense* in Y_0, together with the *openness* of the morphism $Y \to Y_0$ (cf. [3], Corollaire 2.4.10).

Definition 2. Let $X^{\log}$ and $Y^{\log}$ be fs log schemes. Then we shall say that a morphism $f^{\log} : Y^{\log} \to X^{\log}$ is *log geometrically connected* if for any reduced covering point $\bar{x}_1^{\log} \to \bar{x}^{\log}$ over any strict geometric point $\bar{x}^{\log} \to X^{\log}$, the fiber product $Y^{\log} \times_{X^{\log}} \bar{x}_1^{\log}$ is connected.

Note that it follows from Remark 1, (iii), that this condition is equivalent to the condition that for any connected ket covering $\overline{x'}^{\log} \to \bar{x}^{\log}$ of a strict geometric point $\bar{x}^{\log} \to X^{\log}$, $Y^{\log} \times_{X^{\log}} \overline{x'}^{\log}$ is connected.

Remark 2. In log geometry, there exists the notion of a *log geometric point*. In fact, one can regard a log geometric point as a *limit* of ket coverings over a strict geometric point. Thus, one natural way to define log geometric connectedness is by the condition that every base-change via a log geometric point is connected. However, in general, a log geometric point is not a fine log scheme. Hence we can not perform such a base-change in the category of fs log schemes.

Theorem 1. *Let $X^{\log}$ be a log regular log scheme, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. If we denote by $Y \xrightarrow{f'} X' \xrightarrow{g} X$ the Stein factorization of f, then X' admits a log structure that satisfies the following conditions.*

(i) *There exists a ket covering $X'^{\log} \to X^{\log}$ whose underlying morphism of schemes is g.*

(ii) *$Y^{\log} \to X'^{\log}$ is log geometrically connected.*

Proof. Let $U_X \subseteq X$ be the interior of $X^{\log}$. If we denote by $Y \times_X U_X \to V \to U_X$ the Stein factorization of $Y \times_X U_X \to U_X$, then by Lemma 3, $V \to U_X$ is finite étale, and the normalization Z of X in V is tamely ramified over the generic points of $X \backslash U_X$. Hence Z admits a log structure that determines a ket covering $Z^{\log} \to X^{\log}$ by the log purity theorem in [10]. (Concerning the *log purity theorem*, see Remark B.2.) Now $Y^{\log}$ is log regular, hence normal (cf. Proposition A.3); thus, X' is normal. Therefore, $X' \to X$ factors through Z. Since both $X' \times_X U_X$ and $Z \times_X U_X$ are naturally isomorphic to V, we have $X' \simeq Z$. This completes the proof of assertion (i).

For assertion (ii), since the operation of taking Stein factorization commutes with étale base-change, by base-changing, we may assume that both X and X' are the spectra of strictly henselian local rings. Moreover, by Lemma 1, it is enough to show that for any connected ket covering $X_1^{\log} \to X^{\log}$ and any strict geometric point $\bar{x}^{\log} \to X'^{\log} \times_{X^{\log}} X_1^{\log}$ for which the image of the underlying morphism of schemes is a closed point, $Y^{\log} \times_{X'^{\log}} \bar{x}^{\log}$ is connected.

Let us denote by $Y_1^{\log}$ the fiber product $Y^{\log} \times_{X^{\log}} X_1^{\log}$. Since log smoothness and properness are stable under base-change, $Y_1^{\log} \to X_1^{\log}$ is log smooth and proper. By assertion (i), if we denote by $Y_1 \to X_1' \to X_1$ the Stein factorization of $Y_1 \to X_1$, then X_1' admits a log structure such that the resulting morphism $X_1'^{\log} \to X_1^{\log}$ is a ket covering. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} Y_1^{\log} & \longrightarrow & X_1'^{\log} & \longrightarrow & X_1^{\log} \\ \downarrow & & \downarrow & & \downarrow \\ Y^{\log} & \longrightarrow & X'^{\log} & \longrightarrow & X^{\log}. \end{array}$$

Now I claim that the right-hand square in the above commutative diagram is cartesian. Note that it follows formally from this claim that the left-hand square is also cartesian. In particular, it follows from this claim, together with the connectedness property of the Stein factorization, that $Y^{\log} \times_{X'^{\log}} \bar{x}^{\log} = Y_1^{\log} \times_{X_1'^{\log}} \bar{x}^{\log}$ is connected for any strict geometric point $\bar{x}^{\log} \to X_1'^{\log}$ whose image of the underlying morphism of schemes lies on a closed point of $X_1'^{\log}$.

The claim of the preceding paragraph may be verified as follows. If we base-change by $U_X \to X^{\log}$, then we obtain a commutative diagram

$$\begin{array}{ccccc} Y_1^{\log} \times_{X^{\log}} U_X & \longrightarrow & X_1'^{\log} \times_{X^{\log}} U_X & \longrightarrow & X_1^{\log} \times_{X^{\log}} U_X \\ \downarrow & & \downarrow & & \downarrow \\ Y^{\log} \times_{X^{\log}} U_X & \longrightarrow & X'^{\log} \times_{X^{\log}} U_X & \longrightarrow & U_X. \end{array}$$

Since $U_X \to X^{\log}$ is a strict morphism, and the log structures of U_X and $X_1^{\log} \times_{X^{\log}} U_X$ are trivial, the underlying scheme of $Y_1^{\log} \times_{X^{\log}} U_X$ is $Y_1 \times_X U_X$. Moreover, $X_1^{\log} \times_{X^{\log}} U_X \to U_X$ is finite étale, hence flat. Thus, the underlying morphism of schemes of $Y_1^{\log} \times_{X^{\log}} U_X \to (X'^{\log} \times_{X^{\log}} X_1^{\log}) \times_{X^{\log}} U_X \to X_1^{\log} \times_{X^{\log}} U_X$ is the Stein factorization of the underlying morphism of schemes of $Y_1^{\log} \times_{X^{\log}} U_X \to X_1^{\log} \times_{X^{\log}} U_X$; in particular, $X_1'^{\log} \times_{X^{\log}} U_X \simeq (X'^{\log} \times_{X^{\log}} X_1^{\log}) \times_{X^{\log}} U_X$ over U_X. Therefore, $X_1'^{\log} \simeq X'^{\log} \times_{X^{\log}} X_1^{\log}$ by Proposition B.7.

Definition 3. In the notation of Theorem 1, we shall refer to $Y^{\log} \to X'^{\log} \to X^{\log}$ as the *log Stein factorization* of $f^{\log}$. This name is motivated by condition (ii) in the statement of Theorem 1.

Proposition 1. *The operation of taking log Stein factorization commutes with base-change by a morphism which satisfies the following condition* $(*)$.

$(*)$ *The domain is a log regular log scheme, and the restriction to the interior is flat.*

(For example, log smooth morphisms satisfy $(*)$.*)*

Proof. Let $X^{\log}$ be a log regular log scheme, $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism, and $g^{\log} : X_1^{\log} \to X^{\log}$ a morphism which satisfies the condition $(*)$ in the statement of Proposition 1. Let us denote by $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ the base-change of $f^{\log}$ by $g^{\log}$, and by $Y^{\log} \to X'^{\log} \to X^{\log}$ (respectively, $Y_1^{\log} \to X_1'^{\log} \to X_1^{\log}$) the log Stein factorization of $f^{\log}$ (respectively, $f_1^{\log}$). Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccc} Y_1^{\log} & \longrightarrow & X_1'^{\log} & \longrightarrow & X_1^{\log} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g^{\log}} \\ Y^{\log} & \longrightarrow & X'^{\log} & \longrightarrow & X^{\log}. \end{array}$$

If we denote by $X_2^{\log}$ the fiber product $X_1^{\log} \times_{X^{\log}} X'^{\log}$, then the above commutative diagram determines a morphism $X_1'^{\log} \to X_2^{\log}$. Our claim is that this morphism is an isomorphism.

Let $U_1 \subseteq X_1$ be the interior of $X_1^{\log}$. Since the log structure of U_1 is trivial, $U_1 \to X^{\log}$ is strict. Therefore, the underlying scheme of $Y_1^{\log} \times_{X_1^{\log}} U_1$ is $Y \times_X U_1$, and the factorization induced on the underlying schemes by the factorization $Y_1^{\log} \times_{X_1^{\log}} U_1 \to X_1'^{\log} \times_{X_1^{\log}} U_1 \to U_1$ is the Stein factorization of the underlying morphism of $Y_1^{\log} \times_{X_1^{\log}} U_1 \to U_1$. On the other hand, it follows from the flatness of $U_1 \to X$ that the factorization induced on the underlying schemes by the factorization $Y_1^{\log} \times_{X_1^{\log}} U_1 \to X_2^{\log} \times_{X_1^{\log}} U_1 \to U_1$ is also the Stein factorization of the underlying morphism $Y_1^{\log} \times_{X_1^{\log}} U_1 \to U_1$. Thus, we obtain $X_1'^{\log} \times_{X_1^{\log}} U_1 \simeq X_2^{\log} \times_{X_1^{\log}} U_1$. Now $X_1'^{\log} \to X_1^{\log}$ and $X_2^{\log} \to X_1^{\log}$ are ket coverings; thus, by Proposition B.7, $X_1'^{\log} \simeq X_2^{\log}$.

Remark 3. In this section, we only consider the log Stein factorization in the case where the base log scheme is log regular. However, if a morphism $f^{\log} : Y^{\log} \to X^{\log}$ of fs log schemes admits a *cartesian diagram*

$$\begin{array}{ccc} Y^{\log} & \xrightarrow{f^{\log}} & X^{\log} \\ \downarrow & & \downarrow \\ Y_1^{\log} & \xrightarrow[f_1^{\log}]{} & X_1^{\log}, \end{array}$$

where

- $X_1^{\log}$ is a log regular log scheme,
- $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ is a proper log smooth morphism, and
- the right-hand vertical arrow $X^{\log} \to X_1^{\log}$ is strict,

then the factorization $Y^{\log} \to X_1'^{\log} \times_{X_1^{\log}} X^{\log} \to X^{\log}$ obtained by base-changing the log Stein factorization $Y_1^{\log} \to X_1'^{\log} \to X_1^{\log}$ of $f_1^{\log}$ via $X^{\log} \to X_1^{\log}$ satisfies the following.

- $Y^{\log} \to X_1'^{\log} \times_{X_1^{\log}} X^{\log}$ is log geometrically connected.
- $X_1'^{\log} \times_{X_1^{\log}} X^{\log} \to X^{\log}$ is a ket covering.

3. The log homotopy exact sequence

In this section, we prove a logarithmic analogue of [5], Exposé X, Corollaire 1.4, i.e., the exactness of the *log homotopy sequence*.

Proposition 2. *Let $X^{\log}$ be a connected log regular log scheme, $Y^{\log}$ an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. Then the following conditions are equivalent.*

(i) $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.

(ii) *If we denote the Stein factorization of f by $Y \to X' \to X$, then the morphism $X' \to X$ is an isomorphism (i.e., f is geometrically connected).*

(iii) *If we denote the log Stein factorization of $f^{\log}$ by $Y^{\log} \to X'^{\log} \to X^{\log}$, then the morphism $X'^{\log} \to X^{\log}$ is an isomorphism (i.e., $f^{\log}$ is log geometrically connected).*

(iv) *Y is connected, and $f^{\log}$ induces a surjection $\pi_1(Y^{\log}) \to \pi_1(X^{\log})$. (Concerning the log fundamental group, see Theorem* B.1.*)*

Moreover, the above four conditions imply the following condition.

(v) *Y is connected, and f induces a surjection $\pi_1(Y) \to \pi_1(X)$.*

Proof. The equivalence of the first three conditions is immediate from the constructions of the Stein and log Stein factorizations.

Assume the first three conditions. Then since f is surjective (by condition (i)), geometrically connected (by condition (ii)), and proper, it follows that Y is connected. Now let $X_1^{\log} \to X^{\log}$ be a connected ket covering, and $f_1^{\log} : Y_1^{\log} \to X_1^{\log}$ the base-change $Y^{\log} \times_{X^{\log}} X_1^{\log} \to X_1^{\log}$. Then f_1 is also surjective and proper. Moreover, it follows from Proposition 1 that f_1 is geometrically connected. Thus, Y_1 is connected. This completes the proof that the first three conditions imply condition (iv).

Next, we show that condition (iv) implies condition (iii). Assume that $f^{\log}$ induces a surjection $\pi_1(Y^{\log}) \to \pi_1(X^{\log})$. If we denote by $Y^{\log} \to X'^{\log} \to X^{\log}$ the log Stein factorization of $f^{\log}$, then since Y is connected, and $Y \to X'$ is surjective, X' is connected. Moreover, it follows from Theorem 1, (i), that $X'^{\log} \to X^{\log}$ is a ket covering. By condition (iv), $Y^{\log} \times_{X^{\log}} X'^{\log} \to Y^{\log}$ is also a *connected* ket covering. However, this covering has a section, hence $Y^{\log} \times_{X^{\log}} X'^{\log} \simeq Y^{\log}$. Thus, by applying the general theory of Galois categories to $\text{Két}(X'^{\log})$ and $\text{Két}(Y^{\log})$, we obtain $X'^{\log} \simeq X^{\log}$. (Concerning $\text{Két}(X^{\log})$, see Definition B.4, (i), also Theorem B.1.)

Finally, we show that condition (iv) implies condition (v). It is immediate that the morphism $X^{\log} \to X$ determined by the morphism of sheaves of monoids $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ induces a *surjection* $\pi_1(X^{\log}) \to \pi_1(X)$. Thus, condition (v) follows from condition (iv), the fact that $\pi_1(X^{\log}) \to \pi_1(X)$ is surjective, and the existence of the commutative diagram

$$\begin{array}{ccc} \pi_1(Y^{\log}) & \longrightarrow & \pi_1(X^{\log}) \\ \Big\downarrow & & \Big\downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X). \end{array}$$

Remark 4. In the statement of Proposition 2, condition (v) does not imply condition (iv). Indeed, let R be a strictly henselian discrete valuation ring, K the field of fractions of R, L a tamely ramified extension of K, and R_L the integral closure of R in L. If we denote by $(\text{Spec}\, R)^{\log}$ (respectively, $(\text{Spec}\, R_L)^{\log}$) the log scheme obtained by equipping $\text{Spec}\, R$ (respectively, $\text{Spec}\, R_L$) with the log structure defined by the closed point, then the natural morphism $(\text{Spec}\, R_L)^{\log} \to (\text{Spec}\, R)^{\log}$ satisfies condition (v) (since $\pi_1(\text{Spec}\, R) = 1$), but $\pi_1((\text{Spec}\, R_L)^{\log}) \to \pi_1((\text{Spec}\, R)^{\log})$ is not surjective unless $K = L$ (since $(\text{Spec}\, R_L)^{\log} \to (\text{Spec}\, R)^{\log}$ is a connected ket covering).

Next, we show the exactness of the *log homotopy sequence*.

Theorem 2. *Let $X^{\log}$ be a connected log regular log scheme, $Y^{\log}$ a connected fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a proper log smooth morphism. Moreover, we assume one of conditions* (i), (ii), (iii), *and* (iv) *in Proposition* 2. *Then, for any strict geometric point $\bar{x}^{\log} \to X^{\log}$, the following sequence is exact:*

$$\varprojlim \pi_1(Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \xrightarrow{\ s\ } \pi_1(Y^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1.$$

Here, the projective limit is over all reduced covering points $\bar{x}_\lambda^{\log} \to \bar{x}^{\log}$, and s is induced by the natural projections $Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to Y^{\log}$.

Proof. Note that by condition (iii) in Proposition 2 and the connectedness property of the log Stein factorization, $Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ is connected for any reduced covering point $\bar{x}_\lambda^{\log} \to \bar{x}^{\log}$ over $\bar{x}^{\log}$.

Next, observe that the surjectivity of $\pi_1(f^{\log})$ follows from condition (iv) in Proposition 2. Moreover, the assertion that $\pi_1(f^{\log}) \circ s = 1$ is verified as follows. To prove this fact, it is enough to show that for any ket covering $X'^{\log} \to X^{\log}$, there exists a reduced covering point $\bar{x}_{\lambda}^{\log} \to X^{\log}$ over the strict geometric point $\bar{x}^{\log} \to X^{\log}$ such that the ket covering

$$(Y^{\log} \times_{X^{\log}} \bar{x}_{\lambda}^{\log}) \times_{X^{\log}} X'^{\log} \to Y^{\log} \times_{X^{\log}} \bar{x}_{\lambda}^{\log}$$

is trivial. On the other hand, it follows immediately from Proposition B.2 that there exists a reduced covering point $\bar{x}_{\lambda}^{\log} \to X^{\log}$ over the strict geometric point $\bar{x}^{\log} \to X^{\log}$ such that

$$X'^{\log} \times_{X^{\log}} \bar{x}_{\lambda}^{\log} \simeq \bar{x}_{\lambda}^{\log} \sqcup \cdots \sqcup \bar{x}_{\lambda}^{\log}.$$

This completes the proof of the fact that $\pi_1(f^{\log}) \circ s = 1$.

Hence it is enough to show that the kernel of $\pi_1(f^{\log})$ is generated by the image of s. By the general theory of profinite groups, it is enough to show that for an open subgroup G of $\pi_1(Y^{\log})$, if G contains the image of s, then G contains the kernel of $\pi_1(f^{\log})$. Let $Y_1^{\log} \to Y^{\log}$ be the connected ket covering corresponding to G. Then since G contains the image of s, there exists a reduced covering point $\bar{x}_{\lambda}^{\log} \to \bar{x}^{\log}$ such that $Y_1^{\log} \times_{X^{\log}} \bar{x}_{\lambda}^{\log} \to Y^{\log} \times_{X^{\log}} \bar{x}_{\lambda}^{\log}$ has a (ket) section. Since $Y_1^{\log} \to Y^{\log}$ is finite and log étale, it follows that $Y_1^{\log} \to X^{\log}$ is proper and log smooth. Let $Y_1^{\log} \to X_1^{\log} \to X^{\log}$ be the log Stein factorization of this morphism, and $Y_2^{\log}$ the fiber product $Y^{\log} \times_{X^{\log}} X_1^{\log}$. Thus, we have a commutative diagram

$$\begin{array}{ccccc}
Y_1^{\log} & \longrightarrow & Y_2^{\log} & \longrightarrow & X_1^{\log} \\
\| & & \downarrow & & \downarrow \\
Y_1^{\log} & \longrightarrow & Y^{\log} & \xrightarrow[f^{\log}]{} & X^{\log},
\end{array}$$

where the right-hand square is cartesian. Now it is enough to prove that $Y_1^{\log} \to Y_2^{\log}$ is an isomorphism. To prove this, it is enough to show the following.

(i) $Y_2^{\log}$ is connected.

(ii) $Y_1^{\log} \to Y_2^{\log}$ is a ket covering.

(iii) $Y_1^{\log} \to Y_2^{\log}$ has rank one at some point. (We shall say that a ket covering $Y^{\log} \to X^{\log}$ of an fs log scheme has *rank one at some point* if there exists a log geometric point of $X^{\log}$ such that, for the fiber functor F of Két$(X^{\log})$ defined by the log geometric point [cf. Definition B.4, (ii)], the cardinality of $F(Y^{\log})$ is one.)

The first assertion follows from condition (iv) in Proposition 2, and the second assertion follows from the fact that $Y_1^{\log} \to Y^{\log}$ and $Y_2^{\log} \to Y^{\log}$ are ket coverings and Proposition B.4. Hence, in the rest of the proof, we show the third assertion.

By replacing the reduced covering point $\bar{x}_\lambda^{\log} \to \bar{x}^{\log}$ by the composite $\bar{x}_{\lambda'}^{\log} \to \bar{x}_\lambda^{\log} \to \bar{x}^{\log}$, where $\bar{x}_{\lambda'}^{\log} \to \bar{x}_\lambda^{\log}$ is a reduced covering point, if necessary, we may assume that $X_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ splits as a disjoint union of copies of $\bar{x}_\lambda^{\log}$. If we base-change the above commutative diagram by $\bar{x}_\lambda^{\log} \to X^{\log}$, then we obtain the following commutative diagram

$$\begin{array}{ccccc} Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} & \longrightarrow & \overbrace{(Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \sqcup \cdots \sqcup (Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log})}^{n} & \longrightarrow & \overbrace{\bar{x}_\lambda^{\log} \sqcup \cdots \sqcup \bar{x}_\lambda^{\log}}^{n} \\ \| & & \downarrow & & \downarrow \\ Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} & \longrightarrow & Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} & \longrightarrow & \bar{x}_\lambda^{\log}, \end{array}$$

where the right-hand square is cartesian. By the general theory of Galois categories, it is enough to show that

$$Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to Y_2^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} (= \overbrace{(Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \sqcup \cdots \sqcup (Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log})}^{n})$$

has rank one at some point.

Now $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ has a (ket) section; thus, one of the connected components of $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ is isomorphic to $Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$. Since $Y_1^{\log} \to Y_2^{\log}$ is a surjective ket covering,

$$Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to \overbrace{(Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \sqcup \cdots \sqcup (Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log})}^{n}$$

is surjective (cf. [12], Proposition 2.2.2). On the other hand, the number of connected components of $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ is n by the connectedness property of the log Stein factorization $Y_1^{\log} \to X_1^{\log} \to X^{\log}$. Thus, $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to Y_2^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ induces a bijection between the set of connected components of $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ and that of $Y_2^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$. Since one of the connected components of $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ is isomorphic to $Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$, $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log} \to Y_2^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$ is an isomorphism on at least one connected component of $Y_1^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$, which is isomorphic to $Y^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}$. This completes the proof of the third assertion.

Proposition 3. *Let k be a field, $X^{\log}$ a log smooth proper log geometrically connected fs log scheme over k, and $Y^{\log}$ a connected log regular log scheme over k with the interior U_Y. Let $p_1^{\log} : X^{\log} \times_k Y^{\log} \to X^{\log}$ (respectively, $p_2^{\log} : X^{\log} \times_k Y^{\log} \to Y^{\log}$) be the* 1*st (respectively,* 2*nd) projection. Then the following hold.*

(i) *$X^{\log} \times_k Y^{\log}$ is connected.*

(ii) *If $U_Y(k^{\text{sep}}) \neq \varnothing$ (for example, k is perfect, and Y is locally of finite type over k), then the natural morphism*

$$\pi_1(X^{\log} \times_k Y^{\log}) \to \pi_1(X^{\log}) \times_{\mathrm{Gal}(k^{\text{sep}}/k)} \pi_1(Y^{\log})$$

determined by $p_1^{\log}$ and $p_2^{\log}$ is an isomorphism.

Proof. First, we prove assertion (i). Since $\operatorname{Spec} k$ has the trivial log structure, the underlying scheme of $X^{\log} \times_k Y^{\log}$ is naturally isomorphic to $X \times_k Y$, which is *connected*. Assertion (i) follows from this fact.

Next, we prove assertion (ii). By the assumption, there exists a k^{sep}-rational point of U_Y. Thus, by Theorem 2, we obtain the following exact sequence:

$$\pi_1(X^{\log} \otimes_k k^{\text{sep}}) \longrightarrow \pi_1(X^{\log} \times_k Y^{\log}) \xrightarrow{\pi_1(p_2^{\log})} \pi_1(Y^{\log}) \longrightarrow 1.$$

Therefore, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_1(X^{\log} \otimes_k k^{\text{sep}}) & \longrightarrow & \pi_1(X^{\log} \times_k Y^{\log}) & \xrightarrow{\pi_1(p_2^{\log})} & \pi_1(Y^{\log}) & \longrightarrow & 1 \\
 & & \wr\downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(X^{\log} \otimes_k k^{\text{sep}}) & \longrightarrow & \pi_1(X^{\log}) \times_{\mathrm{Gal}(k^{\text{sep}}/k)} \pi_1(Y^{\log}) & \longrightarrow & \pi_1(Y^{\log}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X^{\log} \otimes_k k^{\text{sep}}) & \longrightarrow & \pi_1(X^{\log}) & \longrightarrow & \mathrm{Gal}(k^{\text{sep}}/k) & \longrightarrow & 1,
\end{array}$$

where all horizontal sequences are exact. Then it follows from the injectivity of the left-hand bottom horizontal arrow $\pi_1(X^{\log} \otimes_k k^{\text{sep}}) \to \pi_1(X^{\log})$ that the left-hand top horizontal arrow $\pi_1(X^{\log} \otimes_k k^{\text{sep}}) \to \pi_1(X^{\log} \times_k Y^{\log})$ is injective. Thus, assertion (ii) follows from the "Five lemma".

4. Log formal schemes and the algebraization

In this section, we define the notion of a log structure on a formal scheme and establish a theory of *algebraizations* of log formal schemes. First, we define the notion of a log structure on a locally noetherian formal scheme.

Definition 4. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally noetherian formal schemes.

(i) Let $\mathscr{M}_{\mathfrak{X}}$ be a sheaf of topological monoids on the étale site of $\mathfrak{X}$. (Concerning the étale site of a locally noetherian formal scheme, see [4], 6.1.) We shall refer to a continuous homomorphism of sheaves of topological monoids $\mathscr{M}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}$ (where we regard $\mathscr{O}_{\mathfrak{X}}$ as a sheaf of topological monoids via the monoid structure determined by the *multiplicative structure* on the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$) as a *pre-log structure* on $\mathfrak{X}$.

A morphism $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}})$ of locally noetherian formal schemes equipped with pre-log structures is defined to be a pair $(\mathfrak{f}, h)$ of a morphism of locally noetherian formal schemes $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ and a continuous homomorphism $h : \mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} \to \mathcal{M}_{\mathfrak{X}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} & \xrightarrow{h} & \mathcal{M}_{\mathfrak{X}} \\ \downarrow & & \downarrow \\ \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} & \xrightarrow[\text{via } \mathfrak{f}]{} & \mathcal{O}_{\mathfrak{X}}. \end{array}$$

(ii) We shall refer to a pre-log structure $\alpha : \mathcal{M}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}$ as a *log structure* on $\mathfrak{X}$ if the homomorphism α induces an isomorphism $\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}^*$.

We shall refer to a locally noetherian formal scheme equipped with a log structure as a *log locally noetherian formal scheme*. A morphism of log locally noetherian formal schemes is defined as a morphism of locally noetherian formal schemes equipped with pre-log structures.

For simplicity, we shall use the notation $\mathfrak{X}^{\log}$ to denote a log locally noetherian formal scheme whose underlying formal scheme is $\mathfrak{X}$. Then we shall denote by $\mathcal{M}_{\mathfrak{X}}$ the sheaf of monoids that determines the log structure of $\mathfrak{X}^{\log}$. Note that by a similar way to the way in which we regard the category of locally noetherian schemes as a full subcategory of the category of locally noetherian formal schemes (by regarding a scheme S as the formal scheme obtained by the completion of S along the closed subset S of S), we regard the category of locally noetherian schemes equipped with log structures as a full subcategory of the category of log locally noetherian formal schemes.

(iii) Let $\alpha : \mathcal{M}'_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ be a pre-log structure on $\mathfrak{X}$. We shall refer to the log structure determined by the push-out in the category of sheaves of topological monoids on the étale site of $\mathfrak{X}$ of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*) & \xrightarrow{\text{via } \alpha} & \mathcal{O}_{\mathfrak{X}}^* \\ \downarrow & & \\ \mathcal{M}'_{\mathfrak{X}} & & \end{array}$$

as the *log structure associated to the pre-log structure* $\alpha : \mathcal{M}'_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$.

(iv) Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes, and $\mathcal{M}_{\mathfrak{Y}}$ a log structure on $\mathfrak{Y}$. We shall refer to the log structure associated to the pre-log structure $\mathfrak{f}^{-1}\mathcal{M}_{\mathfrak{Y}} \to \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{X}}$ as the *pull-back of the log structure* $\mathcal{M}_{\mathfrak{Y}}$, or, alternatively, the *log structure on* $\mathfrak{X}$ *induced by* $\mathfrak{f}$.

Let $X^{\log}$ be a log scheme, and $F \subseteq X$ a closed subspace of the underlying topological space of X. Then we shall refer to the log formal scheme $\hat{X}^{\log}$ obtained by equipping the completion $\hat{X}$ of X along F with the pull-back of the log structure of $X^{\log}$ as the *log completion of* $X^{\log}$ *along* F.

(v) Let $\mathfrak{X}^{\log}$ be a log locally noetherian formal scheme. Then we shall say that $\mathfrak{X}^{\log}$ is an *fs log locally noetherian formal scheme* if étale locally on $\mathfrak{X}$, there exists a discrete fs monoid P and a homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (where $P_{\mathfrak{X}}$ is the constant sheaf on the étale site of $\mathfrak{X}$ determined by P) such that the log structure of $\mathfrak{X}^{\log}$ is isomorphic to the log structure associated to the homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$.

(vi) Let $\mathfrak{X}^{\log}$ be an fs log locally noetherian formal scheme, P a topological monoid (respectively, a discrete fs monoid), and $P_{\mathfrak{X}}$ the constant sheaf on the étale site of $\mathfrak{X}$ determined by P. Let $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ be a continuous homomorphism such that the log structure of $\mathfrak{X}^{\log}$ is isomorphic to the log structure associated to this homomorphism. Then we shall refer to this morphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ as a *chart* (respectively, an *fs chart*) of $\mathfrak{X}^{\log}$. By the definition of an fs log locally noetherian formal scheme, an fs chart always exists étale locally on $\mathfrak{X}^{\log}$.

Let $\bar{x} \to \mathfrak{X}$ be a geometric point of $\mathfrak{X}$ (i.e., $\bar{x} = \operatorname{Spec} k$ for some separably closed field k). We shall say that an fs chart $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ is *clean* at $\bar{x} \to \mathfrak{X}$ if the composite $P \to \mathcal{M}_{\mathfrak{X},\bar{x}} \to (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}}$ is an isomorphism. It follows immediately from a similar argument to the argument used to prove the existence of a clean chart for an fs log scheme (cf. Definition B.1, (ii)) that a clean chart of $\mathfrak{X}^{\log}$ always exists over an étale neighborhood of any given geometric point of $\mathfrak{X}$.

(vii) Let $\mathfrak{X}^{\log}$ and $\mathfrak{Y}^{\log}$ be fs log locally noetherian formal schemes, and $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ a morphism of log locally noetherian formal schemes. Let $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ be an fs chart of $\mathfrak{X}^{\log}$, $Q_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}$ an fs chart of $\mathfrak{Y}^{\log}$, and $Q \to P$ a morphism of monoids such that the diagram

$$\begin{array}{ccc} \mathfrak{f}^{-1}Q_{\mathfrak{Y}} = Q_{\mathfrak{X}} & \longrightarrow & P_{\mathfrak{X}} \\ \downarrow & & \downarrow \\ \mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}} & \longrightarrow & \mathcal{O}_{\mathfrak{X}} \end{array}$$

commutes. Then we shall refer to the collection of data consisting of $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$, $Q_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}$, and $Q \to P$ as an *fs chart of the morphism* $\mathfrak{f}^{\log}$. Moreover, in the above data, for a geometric point $\bar{x} \to \mathfrak{X}$, if $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (respectively, $Q_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}$) is a clean chart at $\bar{x} \to \mathfrak{X}$ (respectively, at the geometric point of $\mathfrak{Y}$ determined by the geometric point $\bar{x} \to \mathfrak{X}$), then we shall refer to such a chart as a *clean chart of the morphism* $\mathfrak{f}^{\log}$ *at* $\bar{x} \to \mathfrak{X}$.

Lemma 5. *Let A be an adic noetherian ring, I an ideal of definition of A, and $f : X \to \operatorname{Spec} A$ a proper morphism. If a subspace F of the underlying topological space of X contains the underlying topological space of $X \otimes_A (A/I)$ and is stable under generization, then F coincides with the underlying topological space of X.*

Proof. Assume that F does not coincide with the underlying topological space of X (and that X is non-empty). Then there exists an element x of $X \backslash F$. Since F is stable under generization, for any element a of F, there exists an open neighborhood U_a of a in X such that x does not belong to U_a. Thus, the open set $U \stackrel{\mathrm{def}}{=} \bigcup_{a \in F} U_a$ of the underlying topological space of X contains the underlying topological space of $X \otimes_A (A/I)$, and x does not belong to U. It thus follows from the properness of f that $f(X \backslash U)$ is a *non-empty closed subset* of the underlying topological space of $\operatorname{Spec} A$ and does not intersect the underlying topological space of $\operatorname{Spec}(A/I)$. However, since A is an adic noetherian ring, $\operatorname{Spec}(A/I)$ contains all closed points of $\operatorname{Spec} A$. Thus, there exists no such set; hence we obtain a contradiction.

Lemma 6. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

be a commutative diagram of commutative rings with unity. Suppose that the following conditions hold.

(i) *The morphism $A \to B$ is faithfully flat.*

(ii) *The morphisms $A \to A'$ and $B \to B'$ are injective. [Let us regard A (respectively, B) as a subring of A' (respectively, B').]*

(iii) *The natural morphism $B \otimes_A A' \to B'$ is injective.*

Then the natural morphism from A to the set-theoretic fiber product of

$$\begin{array}{ccc} & & A' \\ & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

is surjective.

Proof. By condition (iii), it is enough to show the assertion in the case where $B' = B \otimes_A A'$. Thus, assume that $B' = B \otimes_A A'$. Let $a' \in A'$ and $b \in B$ be elements such that the images in B' coincide. Now let us denote by ϕ the morphism of A-modules

$$\begin{aligned} A \oplus A &\to A' \\ (a_1, a_2) &\mapsto a_1 + a' \cdot a_2, \end{aligned}$$

and by I the image of ϕ. Then we obtain inclusions $A \subseteq I \subseteq A'$ (cf. condition (ii)). On the other hand, the fact that the images of $a' \in A'$ and $b \in B$ in B' coincide implies that the image of $\mathrm{id}_B \otimes_A \phi : B \oplus B \to B'$ is $B \subseteq B'$, i.e., $B = B \otimes_A I$. Thus,

$$0 = (B \otimes_A I)/B = B \otimes_A (I/A).$$

Since $A \to B$ is faithfully flat (cf. condition (i)), $I/A = 0$, i.e., $a' \in A$. This completes the proof of Lemma 6.

Lemma 7. *Let R be a henselian excellent reduced local ring, $\hat{R}$ the completion of R with respect to its maximal ideal $\mathfrak{m}$ of R, and $R \to \hat{R}$ the natural morphism. Then if a Kummer morphism $P \to Q$ of fs monoids (cf. Definition* B.3*) fits into a commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{\alpha_P} & R \\ \downarrow & & \downarrow \\ Q & \xrightarrow[\alpha_Q]{} & \hat{R} \end{array}$$

of monoids, then the morphism $\alpha_Q : Q \to \hat{R}$ factors through R.

Proof. Let q be an element of Q. Our claim is that the image $\alpha_Q(q)$ of q via α_Q is in R. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subseteq R$ be the associated primes of R. Then by the fact that R is reduced, the natural morphism $R \to R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_r$ is injective. We denote by K_i the field of fractions of $R/\mathfrak{p}_i$. Now since R is excellent, $R/\mathfrak{p}_i$ is excellent. Therefore, by [3], Corollaire 18.9.2, the completion $(\widehat{R/\mathfrak{p}_i})(\simeq R/\mathfrak{p}_i \otimes_R \hat{R})$ of $R/\mathfrak{p}_i$ with respect to its maximal ideal is an integral domain. We denote by $\hat{K}_i$ the field of fractions of $(\widehat{R/\mathfrak{p}_i})$. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_r & \longrightarrow & K_1 \oplus \cdots \oplus K_r \\ \downarrow & & \downarrow & & \downarrow \\ \hat{R} & \longrightarrow & (\widehat{R/\mathfrak{p}_1}) \oplus \cdots \oplus (\widehat{R/\mathfrak{p}_r}) & \longrightarrow & \hat{K}_1 \oplus \cdots \oplus \hat{K}_r, \end{array}$$

where all morphisms are injective.

Now the assumption on $P \to Q$ implies that $\alpha_Q(q)^n \in R$. Therefore, the image of $\alpha_Q(q)^n$ in $\hat{K}_i$ is in K_i. On the other hand, by the excellentness of $R/\mathfrak{p}_i$ and [3], Corollaire 18.9.3, K_i is algebraically closed in $\hat{K}_i$; it thus follows that the image of $\alpha_Q(q)$ in $\hat{K}_i$ is in K_i. Thus, by Lemma 6, $\alpha_Q(q) \in R$. This completes the proof of Lemma 7.

Definition 5. Let $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ be a morphism of fs log locally noetherian formal schemes.

(i) We shall refer to $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ as a *strictly Kummer morphism* if for any geometric point $\bar{x} \to \mathfrak{X}$ of $\mathfrak{X}$, there exists a positive integer n which is invertible in $\mathcal{O}_{\mathfrak{X},\bar{x}}$ such that the morphism of monoids $(\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}_{\mathfrak{Y}}^*)_{\mathfrak{f}(\bar{x})} \to (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}}$ induced by $\mathfrak{f}^{\log}$ is injective, and, moreover, the image of this morphism contains $n \cdot (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}}$, where the geometric point $\mathfrak{f}(\bar{x}) \to \mathfrak{Y}$ is the geometric point determined by the composite $\bar{x} \to \mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$.

(ii) We shall refer to $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ as an *exact morphism* if for any geometric point $\bar{x} \to \mathfrak{X}$ of $\mathfrak{X}$, the morphism of monoids $(\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}_{\mathfrak{Y}}^*)_{\mathfrak{f}(\bar{x})} \to (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}}$ induced by $\mathfrak{f}^{\log}$ is exact, i.e., for $a \in (\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}_{\mathfrak{Y}}^*)_{\mathfrak{f}(\bar{x})}^{\mathrm{gp}}$, if the image of a in $((\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}})^{\mathrm{gp}}$ satisfies that $a \in (\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}} \subseteq ((\mathcal{M}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^*)_{\bar{x}})^{\mathrm{gp}}$, then $a \in (\mathcal{M}_{\mathfrak{Y}}/\mathcal{O}_{\mathfrak{Y}}^*)_{\mathfrak{f}(\bar{x})}$.

Lemma 8. *Let $\mathfrak{X}^{\log}$ and $\mathfrak{Y}^{\log}$ be fs log locally noetherian formal schemes, and $\mathfrak{f}^{\log} : \mathfrak{X}^{\log} \to \mathfrak{Y}^{\log}$ a strictly Kummer morphism. Then strict étale locally on $\mathfrak{X}^{\log}$ and on $\mathfrak{Y}^{\log}$, the morphism $\mathfrak{f}^{\log}$ admits a clean chart.*

Proof. It follows immediately from definition that any Kummer morphisms of fs monoids are exact. Thus, the assertion follows from a similar argument to the argument used in the proof of [7], Corollary 2.3, together with the fact that the order of the stalk, which is a finite group, of the relative characteristic sheaf of $\mathfrak{f}^{\log}$ at any geometric point $\bar{x} \to \mathfrak{X}$ of $\mathfrak{X}$ is invertible in $\mathcal{O}_{\mathfrak{X},\bar{x}}$ (cf. also [7], Lemma 2.2).

The main result in this section is the following theorem.

Theorem 3. *Let A be an adic noetherian ring, and I an ideal of definition of A. Let $S^{\log}$ be an fs log scheme whose underlying scheme S is the spectrum of A, $X^{\log}$ a noetherian excellent fs log scheme, $X^{\log} \to S^{\log}$ a morphism that is separated and of finite type, and $\hat{X}^{\log}$ (respectively, $\hat{S}^{\log}$) the log completion of $X^{\log}$ (respectively, $S^{\log}$) along $X/I \overset{\mathrm{def}}{=} X \otimes_A (A/I)$ (respectively, $S/I \overset{\mathrm{def}}{=} \mathrm{Spec}(A/I)$). Let $\mathscr{C}_{X^{\log}}$ be the category of reduced fs log schemes that are finite and strictly Kummer over $X^{\log}$ and proper over $S^{\log}$, and $\mathscr{C}_{\hat{X}^{\log}}$ the category of reduced fs log formal schemes that are finite and strictly Kummer over $\hat{X}^{\log}$ and proper over $\hat{S}^{\log}$.*

Then the functor determined by the operation of taking the log completion along the fiber over S/I induces a natural equivalence between the category $\mathscr{C}_{X^{\log}}$ and the category $\mathscr{C}_{\hat{X}^{\log}}$.

Proof. Note that if $Y^{\log} \to X^{\log}$ is an object of the category $\mathscr{C}_{X^{\log}}$, then the excellentness of X implies that the completion $\hat{Y}$ of Y along $Y \otimes_A (A/I)$ is reduced. Therefore, the functor is well-defined. Moreover, it follows from definition that any morphisms in $\mathscr{C}_{X^{\log}}$ and $\mathscr{C}_{\hat{X}^{\log}}$ are strictly Kummer.

First, we prove that the functor is fully faithful. Let $Y_1^{\log} \to X^{\log}$ and $Y_2^{\log} \to X^{\log}$ be objects of the category $\mathscr{C}_{X^{\log}}$.

Let $f^{\log}, g^{\log} : Y_1^{\log} \to Y_2^{\log}$ be morphisms in the category $\mathscr{C}_{X^{\log}}$ such that $\hat{f}^{\log} = \hat{g}^{\log}$, where $\hat{f}^{\log}, \hat{g}^{\log} : \hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ are the morphisms induced by $f^{\log}$ and $g^{\log}$, respectively. Then since $\hat{f}^{\log} = \hat{g}^{\log}$, we obtain $\hat{f} = \hat{g}$. Thus, by [2], Théorème 5.4.1, we obtain $f = g$. To see that $f^{\log} = g^{\log}$, we take a geometric point $\bar{y}_1 \to Y_1$ of Y_1 whose image lies on $Y_1/I \stackrel{\text{def}}{=} Y_1 \otimes_A (A/I)$. Then it follows from the assumption that $\hat{f}^{\log} = \hat{g}^{\log}$ and a similar argument to the argument used in the proof of Proposition B.9 (note that $\mathscr{O}_{Y_1,\bar{y}_1} \to \hat{\mathscr{O}}_{Y_1,\bar{y}_1}$ is faithfully flat, where $\hat{\mathscr{O}}_{Y_1,\bar{y}_1}$ is the completion of $\mathscr{O}_{Y_1,\bar{y}_1}$ with respect to $I\mathscr{O}_{Y_1,\bar{y}_1}$) that the homomorphism $\mathscr{M}_{Y_2,\bar{y}_2} \to \mathscr{M}_{Y_1,\bar{y}_1}$ induced by $f^{\log}$ (where we denote by $\bar{y}_2 \to Y_2$ the geometric point determined by the composite $\bar{y}_1 \longrightarrow Y_1 \stackrel{f=g}{\longrightarrow} Y_2$) coincides with the homomorphism $\mathscr{M}_{Y_2,\bar{y}_2} \to \mathscr{M}_{Y_1,\bar{y}_1}$ induced by $g^{\log}$. Therefore, $f^{\log}$ coincides with $g^{\log}$ on an étale neighborhood of the geometric point $\bar{y}_1 \to Y_1$. Moreover, by Lemma 5, this implies that $f^{\log}$ coincides with $g^{\log}$ on $Y_1^{\log}$. This completes the proof that the functor in question is faithful.

Next, let $\mathfrak{f}^{\log} : \hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ be a morphism in the category $\mathscr{C}_{\hat{X}^{\log}}$. By [2], Théorème 5.4.1, there exists a unique morphism $f : Y_1 \to Y_2$ such that $\hat{f}$ coincides with the underlying morphism $\mathfrak{f}$ of formal schemes of $\mathfrak{f}^{\log}$. Note that it follows from the proof of the faithfulness of the functor in question that it is enough to show that an extension of f to a morphism of *log schemes* exists étale locally on $Y_1^{\log}$. Moreover, by Lemma 5, it is enough to show that for any geometric point of Y_1 whose image lies on Y_1/I, there exists such an extension of f on an étale neighborhood of the geometric point. To see this, let $\bar{y}_1 \to Y_1$ be a geometric point whose image lies on Y_1/I, $\bar{y}_2 \to Y_2$ the geometric point determined by the composite $\bar{y}_1 \to Y_1 \stackrel{f}{\to} Y_2$, and

$$\begin{array}{ccccc} P_2 & \longrightarrow & & & P_1 \\ \big\downarrow & & & & \big\downarrow \\ \mathscr{O}_{Y_2,\bar{y}_2} & \longrightarrow & (\mathscr{O}_{Y_1,\bar{y}_1} & \longrightarrow &)\tilde{\mathscr{O}}_{Y_1,\bar{y}_1} \end{array}$$

a clean chart of the natural morphism $(\mathrm{Spf}\ \tilde{\mathcal{O}}_{Y_1,\bar{y}_1})^{\log} \to Y_2^{\log}$ at $\bar{y}_1 \to \mathrm{Spf}\ \tilde{\mathcal{O}}_{Y_1,\bar{y}_1}$, where $\tilde{\mathcal{O}}_{Y_1,\bar{y}_1}$ is the completion of $\mathcal{O}_{Y_1,\bar{y}_1}$ with respect to its maximal ideal, and $(\mathrm{Spf}\ \tilde{\mathcal{O}}_{Y_1,\bar{y}_1})^{\log}$ is the log formal scheme obtained by equipping $\mathrm{Spf}\ \tilde{\mathcal{O}}_{Y_1,\bar{y}_1}$ with the log structure induced by the log structure of $\hat{Y}_1^{\log}$. (Indeed, by Lemma 8, the morphism $(\mathrm{Spf}\ \tilde{\mathcal{O}}_{Y_1,\bar{y}_1})^{\log} \to Y_2^{\log}$ admits a clean chart.) Then since $P_2 \to P_1$ is Kummer, by Lemma 7, the morphism $P_1 \to \tilde{\mathcal{O}}_{Y_1,\bar{y}_1}$ factors through $\mathcal{O}_{Y_1,\bar{y}_1}$; moreover, the resulting morphism $P_1 \to \mathcal{O}_{Y_1,\bar{y}_1}$ is a chart at $\bar{y}_1 \to Y_1$ of the log structure of $Y_1^{\log}$. In particular, the diagram

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_2,\bar{y}_2} & \longrightarrow & \mathcal{O}_{Y_1,\bar{y}_1} \end{array}$$

is a chart of a morphism from an étale neighborhood of $\bar{y}_1 \to Y_1^{\log}$ to $Y_2^{\log}$ for which the morphism $\hat{Y}_1^{\log} \to \hat{Y}_2^{\log}$ determined by this morphism coincides with $\mathfrak{f}^{\log}$. This completes the proof that the functor in question is full.

Finally, we prove that the functor is essentially surjective. Let $\mathfrak{Y}^{\log} \to \hat{X}^{\log}$ be an object of $\mathscr{C}_{\hat{X}^{\log}}$. By [2], Théorème 5.4.1 and Proposition 5.4.4, there exists a unique noetherian scheme Y that is finite over X, and proper over S such that the completion $\hat{Y}$ of Y along $Y/I \stackrel{\text{def}}{=} Y \otimes_A (A/I)$ is isomorphic to $\mathfrak{Y}$. (Note that then the reducedness of $\mathfrak{Y}$ implies that Y is reduced.) Now it follows from the proof of the full faithfulness of the functor in question that it is enough to show that an fs log structure of the desired type exists étale locally on Y. Moreover, by Lemma 5, it is enough to show that for any geometric point of Y for which the image lies on Y/I, there exists such an fs log structure on an étale neighborhood of the geometric point.

By replacing $X^{\log}$ by the log scheme obtained by equipping Y with the log structure induced by the log structure of $X^{\log}$ via the morphism $Y \to X$, we may assume that the morphism $Y \to X$ is the identity morphism of X; thus, we may assume that the underlying morphism of formal schemes of $\hat{Y}^{\log} \to \hat{X}^{\log}$ is the identity morphism of $\hat{X}$. Let $\bar{x} \to X$ be a geometric point of X whose image lies on X/I. Then we obtain a diagram

$$\begin{array}{ccc} \mathrm{Spf}\ \tilde{\mathcal{O}}_{X,\bar{x}} & \longrightarrow & \mathrm{Spec}\ \mathcal{O}_{X,\bar{x}} \\ \downarrow & & \downarrow \\ \hat{X} & \longrightarrow & X, \end{array}$$

where $\tilde{\mathcal{O}}_{X,\bar{x}}$ is the completion of $\mathcal{O}_{X,\bar{x}}$ with respect to its maximal ideal. Now we obtain a clean chart of the morphism $(\mathrm{Spf}\ \tilde{\mathcal{O}}_{X,\bar{x}})^{\log} \to X^{\log}$ (where the log structure of $(\mathrm{Spf}\ \tilde{\mathcal{O}}_{X,\bar{x}})^{\log}$ is induced by the log structure of $\hat{Y}^{\log}$)

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_{X,\bar{x}} & \longrightarrow & \tilde{\mathcal{O}}_{X,\bar{x}}. \end{array}$$

Thus, by Lemma 7, the chart $Q \to \tilde{\mathcal{O}}_{X,\bar{x}}$ factors through $\mathcal{O}_{X,\bar{x}}$. It thus follows that the log structure of $\hat{Y}^{\log}$ can be descended to an étale neighborhood of the geometric point $\bar{x} \to X$.

By applying Theorem 3, we obtain the following corollary. Note that the corollary generalizes [16], Théorème 2.2, (a). (In [16], Théorème 2.2, (a), the underlying scheme of the base log scheme is assumed to be the spectrum of a complete *discrete valuation ring*.)

Corollary 1. *Let $S^{\log}$ be an fs log scheme whose underlying scheme S is the spectrum of a noetherian complete local ring A whose maximal ideal (respectively, residue field) we denote by $\mathfrak{m}$ (respectively, k), $X^{\log}$ a log regular log scheme, and $X^{\log} \to S^{\log}$ a proper morphism. Then the strict closed immersion $X_0^{\log} \stackrel{\text{def}}{=} X^{\log} \times_{S^{\log}} s^{\log} \to X^{\log}$ induces a natural equivalence of the category of ket coverings over $X^{\log}$ and the category of ket coverings over $X_0^{\log}$, where $s^{\log}$ is the log scheme obtained by equipping* Spec k *with the log structure induced by the log structure of $S^{\log}$ via the closed immersion $s \to S$ induced by the natural projection $A \to A/\mathfrak{m} \simeq k$. In particular, if $X^{\log}$ is connected, then $X_0^{\log}$ is also connected, and $\pi_1(X_0^{\log}) \xrightarrow{\sim} \pi_1(X^{\log})$.*

Proof. We may assume that $X^{\log}$ is connected. Moreover, since the assertion is independent of the log structure of $S^{\log}$, we may assume that the log structure of $S^{\log}$ is trivial.

First, we prove that the functor is fully faithful. Let $Y^{\log} \to X^{\log}$ be a connected ket covering. Then if we denote by $Y \to S' \to S$ the Stein factorization of the underlying morphism of the composite $Y^{\log} \to X^{\log} \to S$, then the connectedness of Y and the surjectivity of $Y \to S'$ implies that S' is connected. Since S is the spectrum of the complete ring and $S' \to S$ is finite, it thus follows that $Y \times_S s$, hence also $Y^{\log} \times_S s$, is connected. Therefore, by the general theory of Galois categories, the functor in question is fully faithful.

Next, we prove that the functor is essentially surjective. Let $Y_0^{\log} \to X_0^{\log}$ be a connected ket covering. Then it follows from [17], Théorème 0.1 that there exists a unique connected ket covering $Y_n^{\log} \to X_n^{\log} \stackrel{\text{def}}{=} X^{\log} \times_S S_n$ such that $Y_n^{\log} \times_{S_n} s \simeq Y_0^{\log}$, where $S_n \stackrel{\text{def}}{=} \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. We shall denote by $\mathfrak{Y}$ the noetherian formal scheme obtained by the system $\{Y_n\}_n$. Now I *claim* that $\mathfrak{Y}$ admits an *fs* log structure, and there exists a natural isomorphism $\mathfrak{Y}^{\log} \times_S S_n \simeq Y_n^{\log}$, where $\mathfrak{Y}^{\log}$ is the resulting log formal scheme. Indeed, this claim is verified as follows. Let $\bar{x} \to X$ be a geometric point whose image lies on X_0, and $(\mathrm{Spec}\, R \to X, P \to R)$ a clean chart at the geometric point $\bar{x} \to X$ such that for any connected component $Z_0^{\log} \subseteq \mathrm{Spec}(R/\mathfrak{m}R)^{\log} \times_{X_0^{\log}} Y_0^{\log}$ (where $\mathrm{Spec}(R/\mathfrak{m}R)^{\log}$ is the log scheme obtained by equipping $\mathrm{Spec}(R/\mathfrak{m}R)$ with the log structure induced by the log structure of $X^{\log}$), the ket covering obtained as the composite

$$Z_0^{\log} \hookrightarrow \mathrm{Spec}(R/\mathfrak{m}R)^{\log} \times_{X_0^{\log}} Y_0^{\log} \to \mathrm{Spec}(R/\mathfrak{m}R)^{\log}$$

admits a chart

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ R/\mathfrak{m}R & \longrightarrow & \mathcal{O}_{Z_0}, \end{array}$$

and, moreover, this chart induces an isomorphism $(R/\mathfrak{m}R) \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \xrightarrow{\sim} \mathcal{O}_{Z_0}$ (cf. Proposition B.2). Then it follows from [17], Théorème 0.1, together with the fact that the ket coverings over $\mathrm{Spec}(R/\mathfrak{m}R)^{\log}$ obtained by the ket coverings

$$\mathrm{Spec}((R/\mathfrak{m}^n R) \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log} \to \mathrm{Spec}(R/\mathfrak{m}^n R)^{\log}$$

and

$$Z_n^{\log} \hookrightarrow \mathrm{Spec}(R/\mathfrak{m}^n R)^{\log} \times_{X_n^{\log}} Y_n^{\log} \to \mathrm{Spec}(R/\mathfrak{m}^n R)^{\log}$$

(where

$$Z_n^{\log} \subseteq \mathrm{Spec}(R/\mathfrak{m}^n R)^{\log} \times_{X_n^{\log}} Y_n^{\log}$$

is the connected component of $\mathrm{Spec}(R/\mathfrak{m}^n R)^{\log} \times_{X_n^{\log}} Y_n^{\log}$ corresponding to the connected component $Z_0^{\log} \subseteq \mathrm{Spec}(R/\mathfrak{m}R)^{\log} \times_{X_0^{\log}} Y_0^{\log}$) are isomorphic that $\mathrm{Spec}((R/\mathfrak{m}^n R) \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ is isomorphic to $Z_n^{\log}$; in particular, the log scheme $Z_n^{\log}$ admits a chart $Q \to \mathcal{O}_{Z_n}$ for any n. Thus, we obtain a morphism $Q \to \hat{R} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{Y}}$ (where $\hat{R}$ is the completion of R with respect to $\mathfrak{m}R \subseteq R$, and $\mathfrak{X}$ is the $\mathfrak{m}$-adic completion of X), i.e., we obtain an *fs log structure* on an étale neighborhood $(\mathrm{Spf}\, \hat{R}) \otimes_{\mathfrak{X}} \mathfrak{Y} \to \mathfrak{Y}$ of $\mathfrak{Y}$ such that $(\mathrm{Spf}\, \hat{R} \otimes_{\mathfrak{X}} \mathfrak{Y})^{\log} \times_S S_n$ is isomorphic to $(\mathrm{Spec}\, R)^{\log} \times_{X^{\log}} Y_n^{\log}$. Since these log structures on étale neighborhoods of $\mathfrak{Y}$ descend to a log structure of $\mathfrak{Y}$ by the construction, we obtain a log structure of $\mathfrak{Y}$ of the desired type. This completes the proof of the above *claim*.

We denote by $\mathfrak{X}^{\log}$ the log completion of $X^{\log}$ along X_0. Now it follows from the properness of $X \to S$ and the fact that A is complete that X is excellent. Now since $Y_0^{\log} \to X_0^{\log}$ is strictly Kummer, $\mathfrak{Y}^{\log} \to \mathfrak{X}^{\log}$ is also strictly Kummer; moreover, since $Y_n \to X_n$ is finite, $\mathfrak{Y} \to \mathfrak{X}$ is also finite. On the other hand, the reducedness of $\mathfrak{Y}^{\log}$ is verified as follows. Let $\bar{y} \to \mathfrak{Y}$ be a geometric point of $\mathfrak{Y}$, and $\bar{x} \to \mathfrak{X}$ the geometric point of $\mathfrak{X}$ determined by $\bar{y} \to \mathfrak{Y}$. Then by the construction of the log structure of $\mathfrak{Y}^{\log}$, there exists a clean chart $(\mathrm{Spec}\, R \to X, P \to R)$ at the geometric point $\bar{x} \to \mathfrak{X}$ such that the restriction of the morphism $(\mathrm{Spf}\, \hat{R})^{\log} \times_{\mathfrak{X}^{\log}} \mathfrak{Y}^{\log} \to (\mathrm{Spf}\, \hat{R})^{\log}$ to the connected component in which the image of $\bar{y} \to \mathfrak{Y}$ lies is isomorphic to the morphism which is of the form

$$\mathrm{Spf}(\hat{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log} \to (\mathrm{Spf}\, \hat{R})^{\log},$$

where $\mathrm{Spf}(\hat{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ (respectively, $(\mathrm{Spf}\, \hat{R})^{\log}$) is the log formal scheme obtained by equipping $\mathrm{Spf}(\hat{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])$ (respectively, $\mathrm{Spf}\, \hat{R}$) with the log structure induced by $Q \to \hat{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ (respectively, $P \to \hat{R}$), and $P \to Q$ is a Kummer morphism of fs monoids such that $n \cdot Q \subseteq \mathrm{Im}(P \to Q)$ for some integer n invertible in R, and, moreover, the chart $Q \to \hat{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ is clean at $\bar{y} \to \mathfrak{Y}$ (cf. the proof of the above *claim*). Let $\tilde{R}$ be the completion of the strict henselization of R at the geometric point of X determined by $\bar{x} \to \mathfrak{X}$ with respect to its maximal ideal, and $(\mathrm{Spec}\, \tilde{R})^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, \tilde{R}$ with the log structure induced by the morphism $P \to \tilde{R}$. Then since $X^{\log}$ is log regular, it follows from the definition of log regularity that $(\mathrm{Spec}\, \tilde{R})^{\log}$ is log regular at the geometric point of $\mathrm{Spec}\, \tilde{R}$ determined by $\bar{x} \to \mathfrak{X}$; thus, it follows from Proposition A.4 that $(\mathrm{Spec}\, \tilde{R})^{\log}$ is log regular. Therefore, since the natural morphism $\mathrm{Spec}(\tilde{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log} \to (\mathrm{Spec}\, \tilde{R})^{\log}$ is a ket covering by Proposition B.2, it follows from Proposition A.5 that $\mathrm{Spec}(\tilde{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ is also log regular. Thus, by Proposition A.3, $\mathrm{Spec}(\tilde{R} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])$ is normal; in particular, $\mathfrak{Y}$ is normal, hence reduced.

Therefore, by Theorem 3, there exists a unique finite strictly Kummer fs log scheme $Y^{\log}$ over $X^{\log}$ whose log completion along $Y \times_S s$ is naturally isomorphic to $\mathfrak{Y}^{\log}$. The assertion that the morphism $Y^{\log} \to X^{\log}$ is a ket covering is verified as follows. Since the property of being a ket covering is strict étale local on $X^{\log}$, it follows from Lemma 9 below that it is enough to show that the base-change of $Y^{\log} \to X^{\log}$ via the natural morphism $(\mathrm{Spec}\, \tilde{\mathcal{O}}_{X,\bar{x}})^{\log} \to X^{\log}$ is a ket covering, where $\bar{x} \to X$ is the geometric point determined by a geometric point $\bar{y} \to Y$ whose image lies on Y_0, $\tilde{\mathcal{O}}_{X,\bar{x}}$ is the completion of $\mathcal{O}_{X,\bar{x}}$ with respect to its maximal ideal $\mathfrak{m}_{X,\bar{x}}$, and $(\mathrm{Spec}\, \tilde{\mathcal{O}}_{X,\bar{x}})^{\log}$ is the log scheme obtained by equipping $\mathrm{Spec}\, \tilde{\mathcal{O}}_{X,\bar{x}}$ with the log structure induced by the log structure of $X^{\log}$. On the other hand, it follows from the proof of the above *claim* that there exist compatible charts with respect to n

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_{X,\bar{x}}/\mathfrak{m}^n\mathcal{O}_{X,\bar{x}} & \longrightarrow & \mathcal{O}_{Y,\bar{y}}/\mathfrak{m}^n\mathcal{O}_{Y,\bar{y}} \end{array}$$

(where $P \to Q$ is a morphism of clean monoids such that $n \cdot Q \subseteq \mathrm{Im}(P \to Q)$ for some integer n invertible in $\mathcal{O}_{X,\bar{x}}$) which induce compatible isomorphisms with respect to n

$$\mathrm{Spec}(\mathcal{O}_{Y,\bar{y}}/\mathfrak{m}^n\mathcal{O}_{Y,\bar{y}})^{\log} \simeq \mathrm{Spec}((\mathcal{O}_{X,\bar{x}}/\mathfrak{m}^n\mathcal{O}_{X,\bar{x}}) \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log};$$

in particular, we obtain compatible isomorphisms with respect to n

$$\mathrm{Spec}(\mathcal{O}_{Y,\bar{y}}/\mathfrak{m}^n_{X,\bar{x}}\mathcal{O}_{Y,\bar{y}})^{\log} \simeq \mathrm{Spec}((\mathcal{O}_{X,\bar{x}}/\mathfrak{m}^n_{X,\bar{x}}) \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}.$$

By taking the inductive limits, we obtain an isomorphism

$$\mathrm{Spec}(\mathcal{O}_{Y,\bar{y}} \otimes_{\mathcal{O}_{X,\bar{x}}} \tilde{\mathcal{O}}_{X,\bar{x}})^{\log} \simeq \mathrm{Spec}(\tilde{\mathcal{O}}_{X,\bar{x}} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}.$$

Therefore, it follows from Proposition B.2 that $Y^{\log} \to X^{\log}$ is a ket covering. This completes the proof of Corollary 1.

Lemma 9. *Let $X^{\log}$ be an fs log scheme whose underlying scheme is the spectrum of a strictly henselian local ring A, $\hat{X}^{\log}$ the log scheme obtained by equipping the spectrum of the completion $\hat{A}$ of A with respect to its maximal ideal with the log structure induced by the log structure of $X^{\log}$, $Y^{\log}$ an fs log scheme, and $Y^{\log} \to X^{\log}$ a finite strictly Kummer morphism. Then if $\hat{Y}^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_{X^{\log}} \hat{X}^{\log} \to \hat{X}^{\log}$ is a ket covering, then the morphism $Y^{\log} \to X^{\log}$ is a ket covering.*

Proof. It is immediate that we may assume that Y is connected; thus, assume that Y is connected. Let $B = \Gamma(Y, \mathcal{O}_Y)$, and

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & B \end{array}$$

a clean chart of the morphism $Y^{\log} \to X^{\log}$ (cf. Lemma 8). Then we obtain a morphism $Y^{\log} \to \mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ over $X^{\log}$, where $\mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ is the log scheme obtained by equipping $\mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])$ with the log structure induced by the natural morphism $Q \to A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$. Now by the assumption on the morphism $\hat{Y}^{\log} \to \hat{X}^{\log}$, together with the proof of Proposition B.2, the morphism $\hat{Y}^{\log} \to \mathrm{Spec}(\hat{A} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ is an isomorphism; thus, it follows from the faithful flatness of $A \to \hat{A}$ that $Y^{\log} \to \mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ is an isomorphism. Therefore, the assertion follows from Proposition B.2.

5. Morphisms of type $\mathbf{N}^{\oplus n}$

In this section, we define the notion of a morphism of type $\mathbf{N}^{\oplus n}$ and consider fundamental properties of such a morphism.

Definition 6. Let $X^{\log}$ and $Y^{\log}$ be fs log schemes, $f^{\log} : Y^{\log} \to X^{\log}$ a morphism of log schemes and n a natural number. Then we shall refer to $f^{\log} : Y^{\log} \to X^{\log}$ as a *morphism of type* $\mathbf{N}^{\oplus n}$ if étale locally on X, $f^{\log}$ is a morphism obtained as a base-change of the natural morphism $(\mathrm{Spec}\,\mathbf{Z})^{\log} \to \mathrm{Spec}\,\mathbf{Z}$, where $(\mathrm{Spec}\,\mathbf{Z})^{\log}$ is the log scheme obtained by equipping $\mathrm{Spec}\,\mathbf{Z}$ with the log structure induced by the chart

$$\begin{aligned} \mathbf{N}^{\oplus n} &\to \mathbf{Z} \\ (m_1, \ldots, m_n) &\mapsto 0^{m_1 + \cdots + m_n}. \end{aligned}$$

Remark 5. A typical example of a morphism of type $\mathbf{N}$ is as follows. Let X be a regular scheme, and $D \subseteq X$ an irreducible divisor of X such that the closed immersion $D \hookrightarrow X$ is regular immersion (of codimension 1). We denote by $X^{\log}$ the log scheme obtained by equipping X with the log structure associated to the divisor D, and by $D^{\log}$ the log scheme obtained by equipping D with the log structure induced by the log structure of $X^{\log}$ via $D \hookrightarrow X$. Then the morphism $D^{\log} \to D$ induced by the natural inclusion $\mathcal{O}_D^* \hookrightarrow \mathcal{M}_D$ is of type $\mathbf{N}$.

Remark 6. In this section, we often use the notation $\underline{X}^{\log} \to X^{\log}$ to denote a morphism of type $\mathbf{N}^{\oplus n}$. Moreover, we often identify the underlying scheme of $\underline{X}^{\log}$ with X via the underlying morphism of schemes of the morphism of type $\mathbf{N}^{\oplus n}$.

Remark 7. Let $f^{\log} : \underline{X}^{\log} \to X^{\log}$ be a morphism of type $\mathbf{N}^{\oplus n}$, $\bar{x} \to X$ a geometric point, and $\alpha : P \to \mathcal{O}_X$ a clean chart of $X^{\log}$ at the geometric point $\bar{x} \to X$. Then by the definition of morphisms of type $\mathbf{N}^{\oplus n}$, there exists a chart of $f^{\log}$ which is of the form

$$\begin{array}{ccc} P & \longrightarrow & Q \stackrel{\mathrm{def}}{=} P \oplus \mathbf{N}^{\oplus n} \\ {\scriptstyle \alpha}\downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow[f^* = \mathrm{id}]{} & \mathcal{O}_X, \end{array}$$

where the top horizontal arrow is the morphism given by mapping p to $(p, 0, \dots, 0)$, and the right-hand vertical arrow is the morphism given by mapping $(p, m_1, \dots, m_n)$ to $\alpha(p) \cdot 0^{m_1 + \cdots + m_n}$.

Now it is immediate that there exists a splitting $Q \xrightarrow{\sim} P \oplus (Q/P)$; moreover, it is *canonical*. Indeed, this is verified as follows. The quotient Q/P of Q by P is isomorphic to $\mathbf{N}^{\oplus n}$ non-canonically. We denote by e_i the element of Q/P that corresponds to $(0, \dots, \overset{i\text{-th}}{1}, \dots, 0)$ under the non-canonical isomorphism $Q/P \simeq \mathbf{N}^{\oplus n}$. Then, by the existence of the isomorphism $Q \xrightarrow{\sim} P \oplus \mathbf{N}^{\oplus n}$, there exists a unique element $\widetilde{e}_i$ of Q such that;

- $\widetilde{e}_i$ modulo P is e_i,
- $\widetilde{e}_i$ is an irreducible element of Q, i.e., if $q_1 + q_2 = \widetilde{e}_i$ (where $q_i \in Q$), then $q_1 = 0$ or $q_2 = 0$.

Thus, the section

$$\begin{aligned} Q/P &\to Q \\ e_i &\mapsto \widetilde{e}_i \end{aligned}$$

of the natural projection $Q \to Q/P$ induces a canonical splitting $Q \simeq P \oplus (Q/P)$. Moreover, the image of $\widetilde{e}_i$ via the morphism $Q \to \mathcal{O}_X$ which appears in the above chart is 0.

Lemma 10. *A morphism of type* $\mathbf{N}^{\oplus n}$ *is stable under base-change in the category of fs log schemes.*

Proof. This follows immediately from the definition of morphisms of type $\mathbf{N}^{\oplus n}$.

Definition 7. Let X be a scheme, and $\mathcal{M}_1 \to \mathcal{O}_X$ and $\mathcal{M}_2 \to \mathcal{O}_X$ fs log structures on X. Let $X_1^{\log}$ (respectively, $X_2^{\log}$) be the log scheme obtained by equipping X with the log structure $\mathcal{M}_1 \to \mathcal{O}_X$ (respectively, $\mathcal{M}_2 \to \mathcal{O}_X$). Then the natural morphism $X_1^{\log} \times_X X_2^{\log} \to X$ induces an isomorphism between the underlying schemes of $X_1^{\log} \times_X X_2^{\log}$ and X. We shall denote by $\mathcal{M}_1 + \mathcal{M}_2 \to \mathcal{O}_X$ the log structure of $X_1^{\log} \times_X X_2^{\log}$ on X. Note that by the definition, the log structure $\mathcal{M}_1 + \mathcal{M}_2$ on X is the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category of fs log structures on X.

Remark 8.

(i) In the notation of Definition 7, for any geometric point $\bar{x} \to X$, there exist an étale neighborhood $U \to X$ of $\bar{x} \to X$, fs monoids P_1 and P_2, and morphisms of monoids $\alpha_1 : P_1 \to \mathcal{O}_U$ and $\alpha_2 : P_2 \to \mathcal{O}_U$ such that $\alpha_1 : P_1 \to \mathcal{O}_U$ (respectively, $\alpha_2 : P_2 \to \mathcal{O}_U$) is an fs chart of $\mathcal{M}_1$ (respectively, $\mathcal{M}_2$) at $\bar{x} \to X$. Then there exists an fs chart of the log structure $\mathcal{M}_1 + \mathcal{M}_2 \to \mathcal{O}_X$ at $\bar{x} \to X$ that is of the form

$$\begin{aligned} &P_1 \oplus P_2 \to \mathcal{O}_U \\ &(p_1, p_2) \mapsto \alpha_1(p_1) \cdot \alpha_2(p_2). \end{aligned}$$

In particular, $(\mathcal{M}_1 + \mathcal{M}_2)/\mathcal{O}_X^* \simeq (\mathcal{M}_1/\mathcal{O}_X^*) \oplus (\mathcal{M}_2/\mathcal{O}_X^*)$.

(ii) In the notation of Definition 7, for any morphism of schemes $f : Y \to X$, $f^*(\mathcal{M}_1 + \mathcal{M}_2) = f^*(\mathcal{M}_1) + f^*(\mathcal{M}_2)$ (where f^* denotes the pull-back of log structures, not of sheaves).

(iii) Let X be a regular scheme, and $D = \sum_{i=1}^n D_i \subseteq X$ a divisor with normal crossings. If we denote by $\mathcal{M}(D)$ (respectively, $\mathcal{M}(D_i)$) the log structure of X defined by the divisor with normal crossings D (respectively, D_i), then $\mathcal{M}(D) = \sum_{i=1}^n \mathcal{M}(D_i)$.

(iv) Clearly, $(\mathcal{M}_1 + \mathcal{M}_2) + \mathcal{M}_3 = \mathcal{M}_1 + (\mathcal{M}_2 + \mathcal{M}_3)$.

Remark 9. Let $X^{\log}$ be an fs log scheme, and $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of type $\mathbf{N}^{\oplus n}$. Then we have a diagram

$$\begin{array}{ccccc} \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{M}_X/\mathcal{O}_X^* \\ \| & & \downarrow & & \downarrow \\ \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_{\underline{X}} & \longrightarrow & \mathcal{M}_{\underline{X}}/\mathcal{O}_X^* \\ & & \downarrow & & \downarrow \\ & & \mathcal{M}_{\underline{X}}/\mathcal{M}_X & \xrightarrow{\sim} & \mathcal{C}_{f^{\log}}, \end{array}$$

where $\mathcal{C}_{f^{\log}}$ is the quotient of $\mathcal{M}_{\underline{X}}/\mathcal{O}_X^*$ by the subsheaf $\mathcal{M}_X/\mathcal{O}_X^*$. Then, by the definition of a morphism of type $\mathbf{N}^{\oplus n}$, $\mathcal{C}_{f^{\log}}$ is locally constant, and the stalk at any geometric point of X is non-canonically isomorphic to $\mathbf{N}^{\oplus n}$. (Indeed, this follows from the existence of the chart in Remark 7.) Moreover, by Remark 7, the sheaf $\mathcal{M}_{\underline{X}}/\mathcal{O}_X^*$ admits a *canonical splitting* $(\mathcal{M}_X/\mathcal{O}_X^*) \oplus \mathcal{C}_{f^{\log}}$.

Now the group $\mathrm{Aut}(\mathbf{N}^{\oplus n})$ is isomorphic to the symmetric group on n letters, hence, in particular, is finite. (Indeed, this follows from the fact that any automorphism of $\mathbf{N}^{\oplus n}$ preserves the irreducible elements of $\mathbf{N}^{\oplus n}$, together with the fact that the irreducible elements of $\mathbf{N}^{\oplus n}$ are the e_i's [where $e_i = (0, \ldots, 0, \overset{i\text{-th}}{1}, 0, \ldots, 0)$].) Since $\mathcal{C}_{f^{\log}}$ is locally constant, and the stalk at any geometric point of X is isomorphic to $\mathbf{N}^{\oplus n}$, it thus follows that there exists a finite étale covering $X' \to X$ such that the pull-back of $\mathcal{C}_{f^{\log}}$ to X' is constant. (Indeed, this follows from the fact that since the sheaf of sets of isomorphisms between $\mathcal{C}_{f^{\log}}$ and $\mathbf{N}_X^{\oplus n}$ on the étale site of X is locally constant, and has finite stalks, there exists a finite étale covering $X' \to X$ such that the restriction of the sheaf to X' is constant.) Moreover, since $\mathrm{Aut}(\mathbf{N})$ is trivial, if $n = 1$, then $\mathcal{C}_{f^{\log}}$ is always constant.

On the other hand, in the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_X^{\mathrm{gp}} & \longrightarrow & \mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^* & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_{\underline{X}}^{\mathrm{gp}} & \longrightarrow & \mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{O}_X^* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{M}_X^{\mathrm{gp}} & \xrightarrow{\sim} & \mathscr{C}_{f^{\log}}^{\mathrm{gp}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

all vertical and horizontal sequences are exact. Now the sheaf $\mathscr{C}_{f^{\log}}^{\mathrm{gp}}$ is locally constant, and the stalk at any geometric point is non-canonically isomorphic to $\mathbf{Z}_X^{\oplus n}$. By Remark 7, the sheaf $\mathcal{M}_{\underline{X}}^{\mathrm{gp}}/\mathcal{O}_X^*$ admits a *canonical splitting* $(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*) \oplus \mathscr{C}_{f^{\log}}^{\mathrm{gp}}$.

Definition 8. Let $X^{\log}$ be a connected fs log scheme.

(i) Let $f^{\log} : \underline{X}^{\log} \to X^{\log}$ be a morphism of type $\mathbf{N}^{\oplus n}$. Then we shall refer to $f^{\log}$ as a *morphism of constant type* $\mathbf{N}^{\oplus n}$ if $\mathscr{C}_{f^{\log}}$ (in the notation of Remark 9) is constant. Let $f^{\log}$ be a morphism of constant type $\mathbf{N}^{\oplus n}$. Then we shall refer to an isomorphism $\tau : \mathbf{N}_X^{\oplus n} \xrightarrow{\sim} \mathscr{C}_{f^{\log}}$ as a *trivialization* of $f^{\log}$. Note that, by the portion of Remark 9 concerning the case "$n = 1$", any morphism of type $\mathbf{N}$ is of constant type $\mathbf{N}$; moreover, such a morphism has a canonical trivialization.

(ii) For pairs $(f_i^{\log}, \tau_i)$ $(i = 1, 2)$, where $f_i^{\log} : X_i^{\log} \to X^{\log}$ is a morphism of constant type $\mathbf{N}^{\oplus n}$ and τ_i is a trivialization of $f_i^{\log}$, we shall say that $(f_1^{\log}, \tau_1)$ is equivalent to $(f_2^{\log}, \tau_2)$ if there exists an isomorphism of fs log schemes $g^{\log} : X_1^{\log} \to X_2^{\log}$ over $X^{\log}$ such that the trivialization of $f_1^{\log}$ induced by the isomorphism $(g^{\log})^* : \mathcal{M}_{X_2} \xrightarrow{\sim} \mathcal{M}_{X_1}$ and τ_2 coincides with τ_1.

(iii) We shall denote by $\mathbf{M}_{X^{\log}}$ the set of pairs $(f^{\log}, \tau)$, where $f^{\log}$ is a morphism of constant type $\mathbf{N}^{\oplus n}$ to $X^{\log}$ and τ is a trivialization of $f^{\log}$ modulo the equivalence defined in (ii).

(iv) We shall denote by ι the morphism $\mathbf{M}_{X^{\log}} \to \mathrm{Pic}(X)^{\oplus n}$ defined as follows.

Let $(f^{\log} : \underline{X}^{\log} \to X^{\log}, \tau)$ be an element of $\mathbf{M}_{X^{\log}}$. Then the middle horizontal sequence in the second diagram in Remark 9 determines a connecting morphism

$$\mathrm{H}^0_{\text{ét}}(X, \mathscr{M}^{\text{gp}}_{\underline{X}}/\mathscr{O}^*_X) \to \mathrm{H}^1_{\text{ét}}(X, \mathscr{O}^*_X).$$

Now since one has a canonical splitting $\mathscr{M}^{\text{gp}}_{\underline{X}}/\mathscr{O}^*_X \simeq (\mathscr{M}^{\text{gp}}_X/\mathscr{O}^*_X) \oplus \mathscr{C}^{\text{gp}}_{f^{\log}}$ and a natural isomorphism $\mathrm{H}^1_{\text{ét}}(X, \mathscr{O}^*_X) \simeq \mathrm{Pic}(X)$, we obtain a morphism

$$\mathrm{H}^0_{\text{ét}}(X, \mathscr{M}^{\text{gp}}_X/\mathscr{O}^*_X) \oplus \mathrm{H}^0_{\text{ét}}(X, \mathscr{C}^{\text{gp}}_{f^{\log}}) \to \mathrm{Pic}(X).$$

For the element $e_i = (0, \ldots, \overset{i\text{-th}}{1}, \ldots, 0)$ of $\mathrm{H}^0_{\text{ét}}(\mathbf{Z}^{\oplus n}_X) = \mathbf{Z}^{\oplus n}$, let us denote by $\mathscr{L}_i$ the image of e_i via the composite

$$\begin{aligned}\mathrm{H}^0_{\text{ét}}(X, \mathbf{Z}^{\oplus n}_X) &\xrightarrow[\sim]{\text{via } \tau^{\text{gp}}} \mathrm{H}^0_{\text{ét}}(X, \mathscr{C}^{\text{gp}}_{f^{\log}}) \\ &\longrightarrow \mathrm{H}^0_{\text{ét}}(X, \mathscr{M}^{\text{gp}}_X/\mathscr{O}^*_X) \oplus \mathrm{H}^0_{\text{ét}}(X, \mathscr{C}^{\text{gp}}_{f^{\log}}) \longrightarrow \mathrm{Pic}(X),\end{aligned}$$

where the second arrow is $x \mapsto (0, x)$, and the third arrow is as above. Then we shall write $\imath(f^{\log}, \tau) = (\mathscr{L}_1, \ldots, \mathscr{L}_n)$.

(v) We shall denote by κ the morphism $\mathrm{Pic}(X)^{\oplus n} \to \mathbf{M}_{X^{\log}}$ defined as follows.

Let $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$ be an element of $\mathrm{Pic}(X)^{\oplus n}$. We denote by V_i the geometric line bundle defined by the invertible sheaf $\mathscr{L}_i^{\otimes(-1)}$ (i.e., the spectrum of the symmetric algebra of $\mathscr{L}_i$ over X), by $p_i : V_i \to X$ the natural morphism, by $s_i : X \to V_i$ the 0-section of p_i, by $p : V \stackrel{\text{def}}{=} V_1 \times_X \cdots \times_X V_n \to X$ the natural morphism, and by $s : X \to V$ the section $(s_1, \ldots, s_n)$ of p. Let $V^{\log}$ be the log scheme obtained by equipping V with the log structure $\mathscr{M}_V = p^*\mathscr{M}_X + \mathscr{M}(D_1) + \cdots + \mathscr{M}(D_n)$ (cf. Definition 7), where D_i is the divisor on V defined by the following cartesian diagram

$$\begin{array}{ccc} D_i & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \mathrm{pr}_i \\ X & \xrightarrow{s_i} & V_i, \end{array}$$

and $\mathscr{M}(D_i)$ is a log structure defined by the divisor D_i. (See Remark 10 below.) Then we obtain a natural morphism of log schemes $p^{\log} : V^{\log} \to X^{\log}$ whose underlying morphism of schemes is p. If we denote by $\underline{X}^{\log}$ the log scheme obtained by equipping X with the log structure $s^*\mathscr{M}_V$, then it is immediate that the composite

$f^{\log} : \underline{X}^{\log} \xrightarrow{s^{\log}} V^{\log} \xrightarrow{p^{\log}} X^{\log}$ is of type $\mathbf{N}^{\oplus n}$, where $s^{\log}$ is the strict morphism whose underlying morphism of schemes is s. On the other hand, since

$$\begin{aligned}\mathcal{M}_{\underline{X}} &= s^*(p^*\mathcal{M}_X + \mathcal{M}(D_1) + \cdots \mathcal{M}(D_n)) \\ &= \mathcal{M}_X + s^*\mathcal{M}(D_1) + \cdots + s^*\mathcal{M}(D_n),\end{aligned}$$

it follows that

$$\mathscr{C}_{f^{\log}} \simeq (s^*\mathcal{M}(D_1)/\mathcal{O}_{\underline{X}}^*) \oplus \cdots \oplus (s^*\mathcal{M}(D_n)/\mathcal{O}_{\underline{X}}^*)$$

(cf. Remark 8, (i)). Now, by the portion of Remark 9 concerning the case "$n = 1$", $s^*\mathcal{M}(D_i)/\mathcal{O}_{\underline{X}}^*$ is constant, i.e., there exists a canonical isomorphism $\tau_i : \mathbf{N}_{\underline{X}} \xrightarrow{\sim} s^*\mathcal{M}(D_i)/\mathcal{O}_{\underline{X}}^*$. Thus, $\mathscr{C}_{f^{\log}}$ is constant. Let us define a trivialization τ of $f^{\log} = p^{\log} \circ s^{\log}$ by

$$\begin{aligned}\mathbf{N}_{\underline{X}}^{\oplus n} &\xrightarrow{\tau} (s^*\mathcal{M}(D_1)/\mathcal{O}_{\underline{X}}^*) \oplus \cdots \oplus (s^*\mathcal{M}(D_n)/\mathcal{O}_{\underline{X}}^*) \\ (m_1, \ldots, m_n) &\mapsto (\tau_1(m_1), \ldots, \tau_n(m_n)).\end{aligned}$$

Then we shall write $\kappa(\mathscr{L}_1, \ldots, \mathscr{L}_n) = (p^{\log} \circ s^{\log}, \tau)$.

Remark 10. For a positive Cartier divisor D on a scheme X, we denote by $\mathcal{M}(D)$ the log structure on X that is defined as follows.

Let us denote by $\mathscr{G}_D \in \mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ the $\mathbf{G}_m$-torsor sheaf on (the étale site of) X that is determined by $-D$, and by $\mathscr{G}_D^i \in \mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ the $\mathbf{G}_m$-torsor sheaf on X that is obtained by applying a "change of structure of group" to $\mathscr{G}_D$ via the morphism

$$\begin{aligned}\mathbf{G}_m &\to \mathbf{G}_m \\ f &\mapsto f^i.\end{aligned}$$

Write $\mathcal{M}(D)' = \coprod_{i \in \mathbf{N}} \mathscr{G}_D^i$. Then the natural morphisms $\mathscr{G}_D^i \times \mathscr{G}_D^j \to \mathscr{G}_D^{i+j}$ determine a natural structure of sheaf of monoids on $\mathcal{M}(D)'$. Moreover, the composite $\mathscr{G}_D \hookrightarrow \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ (the first inclusion arises from the fact that the invertible sheaf determined by the $\mathbf{G}_m$-torsor sheaf $\mathscr{G}_D$ is naturally isomorphic to $\mathcal{O}_X(-D)$) induces a homomorphism $\mathcal{M}(D)' \to \mathcal{O}_X$ of sheaves of monoids. Then we define the log structure $\mathcal{M}(D)$ as the log structure associated to the above pre-log structure $\mathcal{M}(D)' \to \mathcal{O}_X$.

Note that if X is regular, and D is a smooth divisor, then this log structure $\mathcal{M}(D)$ coincides with the log structure defined in [8], 1.5, (1).

Remark 11. Let $X^{\log}$ be a connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbf{N}^{\oplus n}$, and $\tau : \mathbf{N}_{\underline{X}}^{\oplus n} \xrightarrow{\sim} \mathscr{C}_{f^{\log}}$ a trivialization. We write $\iota(f^{\log}, \tau) = (\mathscr{L}_1, \ldots, \mathscr{L}_n)$. If we denote by $\mathscr{G}_i$ the subsheaf of $\mathcal{M}_{\underline{X}}$ defined by the following cartesian diagram

$$\begin{array}{ccc}
\mathscr{G}_i & \longrightarrow & 0 \oplus \{e_{i,X}\} \\
\Big\downarrow & & \Big\downarrow \\
\mathscr{M}_{\underline{X}} & \longrightarrow & (\mathscr{M}_{\underline{X}}/\mathscr{O}_X^* \simeq)\,(\mathscr{M}_X/\mathscr{O}_X^*) \oplus \mathscr{C}_{f^{\log}}
\end{array}$$

(where $\{e_{i,X}\}$ is the subsheaf of $\mathbf{N}_X^{\oplus n}$ whose sections correspond to $e_i = (0, \dots, \overset{i\text{-th}}{1}, \dots, 0) \in \mathbf{N}^{\oplus n} \overset{\tau}{\underset{\sim}{\to}} \mathscr{C}_{f^{\log}}$), then $\mathscr{G}_i$ is a $\mathbf{G}_m$-torsor sheaf on X. Moreover, it is a tautology that the invertible sheaf determined by the $\mathbf{G}_m$-torsor sheaf $\mathscr{G}_i$ is naturally isomorphic to $\mathscr{L}_i$.

Lemma 11. *Let $X^{\log}$ be a connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of type $\mathbf{N}^{\oplus n}$. Then the following hold.*

(i) *There exists a unique morphism $g^{\log} : \underline{\underline{X}}^{\log} \to X$ of type $\mathbf{N}^{\oplus n}$ and a unique morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log}$ such that the resulting morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log} \times_X X^{\log}$ is an isomorphism, i.e., $\mathscr{M}_{\underline{X}} = \mathscr{M}_X + \mathscr{M}_{\underline{\underline{X}}}$.*

(ii) *Assume, moreover, that $f^{\log}$ is of constant type. Then the morphism $g^{\log} : \underline{\underline{X}}^{\log} \to X$ (obtained in assertion* (i)*) is also of constant type. Let τ be a trivialization of $g^{\log}$. Then there exist morphisms $g_i^{\log} : \underline{\underline{X_i}}^{\log} \to X$ of type $\mathbf{N}$ $(1 \le i \le n)$, whose canonical trivialization (see Definition* 8, (i)*) we denote by τ_i, such that the following hold.*

(1) *The morphism $\underline{\underline{X}}^{\log} \to X$ factors through $g_i^{\log} : \underline{\underline{X_i}}^{\log} \to X$, and the resulting morphism*

$$\underline{\underline{X}}^{\log} \to \underline{\underline{X_1}}^{\log} \times_X \cdots \times_X \underline{\underline{X_n}}^{\log}$$

is an isomorphism, i.e., $\mathscr{M}_{\underline{\underline{X}}} = \mathscr{M}_X + \sum_{i=1}^n \mathscr{M}_{\underline{\underline{X_i}}}$.

(2) *The composite*

$$\mathbf{N}^{\oplus n} \xrightarrow[\sim]{\tau_1 \oplus \cdots \oplus \tau_n} \mathscr{C}_{g_1^{\log}} \oplus \cdots \oplus \mathscr{C}_{g_n^{\log}} \xrightarrow[\sim]{\text{via}(1)} \mathscr{C}_{g^{\log}}$$

coincides with τ.

(3) $\iota(g^{\log}, \tau) = (\iota(g_1^{\log}, \tau_1), \dots, \iota(g_n^{\log}, \tau_n))$.

Proof. First, we prove assertion (i). By Remark 9, we have a canonical section $\mathscr{C}_{f^{\log}} \to \mathscr{M}_{\underline{X}}/\mathscr{O}_X^*$. We define the sheaf of monoids $\mathscr{M}_{\underline{\underline{X}}}$ by the following cartesian diagram:

$$\begin{array}{ccc}
\mathscr{M}_{\underline{\underline{X}}} & \longrightarrow & \mathscr{C}_{f^{\log}} \\
\Big\downarrow & & \Big\downarrow \\
\mathscr{M}_{\underline{X}} & \longrightarrow & \mathscr{M}_{\underline{X}}/\mathscr{O}_X^*.
\end{array}$$

Then since the inclusion $\mathscr{O}_X^* \hookrightarrow \mathscr{M}_{\underline{X}}$ factors through $\mathscr{M}_{\underline{\underline{X}}}$, the composite $\mathscr{M}_{\underline{\underline{X}}} \to \mathscr{M}_{\underline{X}} \to \mathscr{O}_X$ (where the second morphism $\mathscr{M}_{\underline{X}} \to \mathscr{O}_X$ is the log structure of $\underline{X}^{\log}$) is a log structure on X; moreover, the injection $\mathcal{M}_{\underline{X}} \to \mathcal{M}_{\underline{\underline{X}}}$ induces the morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log}$ (where $\underline{\underline{X}}^{\log}$ is the log scheme obtained by equipping X with the log structure $\mathcal{M}_{\underline{\underline{X}}} \to \mathcal{O}_X$). On the other hand, it follows from the fact that the stalk of $\mathcal{C}_{f^{\log}}$ at any geometric point of X is isomorphic to $\mathbf{N}^{\oplus n}$, together with the fact that the image of $\widetilde{e}_i$ via the morphism $Q \to \mathcal{O}_V$ is 0 in the notation of Remark 7 that the morphism $\underline{\underline{X}}^{\log} \to X$ induced by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_{\underline{\underline{X}}}$ is of type $\mathbf{N}^{\oplus n}$. Now, by construction and the fact that $f^{\log}$ is of type $\mathbf{N}^{\oplus n}$, the resulting morphism $\underline{X}^{\log} \to \underline{\underline{X}}^{\log} \times_X X^{\log}$ is an isomorphism.

Next, we prove assertion (ii). Let us denote by $\mathcal{M}_i$ the subsheaf of $\mathcal{M}_{\underline{\underline{X}}}$ defined by the following cartesian diagram

$$\begin{array}{ccc}
\mathcal{M}_i & \longrightarrow & 0 \oplus \mathbf{N}_X \\
\downarrow & & \downarrow \\
\mathcal{M}_{\underline{\underline{X}}} & \longrightarrow (\mathcal{M}_{\underline{\underline{X}}}/\mathcal{O}_X^* \xrightarrow{\sim} & (\mathcal{M}_X/\mathcal{O}_X^*) \oplus \mathcal{C}_{g^{\log}} \xleftarrow[\sim]{\mathrm{id}\oplus\tau} (\mathcal{M}_X/\mathcal{O}_X^*) \oplus \mathbf{N}_X^{\oplus n},
\end{array}$$

where the right-hand vertical arrow is

$$\begin{aligned}
0 \oplus \mathbf{N}_X &\to (\mathcal{M}_X/\mathcal{O}_X^*) \oplus \mathbf{N}_X^{\oplus n} \\
(0, m_X) &\mapsto (0, m \cdot e_{i,X}).
\end{aligned}$$

Then the composite $\mathcal{M}_i \to \mathcal{M}_{\underline{\underline{X}}} \to \mathcal{O}_X$ is a log structure. Moreover, if we denote by $\underline{\underline{X_i}}^{\log}$ the log scheme obtained by equipping X with the log structure $\mathcal{M}_i \to \mathcal{O}_X$ and by $g_i^{\log} : \underline{\underline{X_i}}^{\log} \to X$ the morphism determined by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_i$, then $g_i^{\log}$ satisfies conditions (1), (2), and (3) in the statement of Lemma 11, (ii).

Theorem 4. *Let $X^{\log}$ be a connected fs log scheme. Then ι is a bijection. The inverse of ι is κ.*

Proof. By Lemma 11, (i), the morphism $\mathbf{M}_X \to \mathbf{M}_{X^{\log}}$ induced by the morphism $X^{\log} \to X$ (determined by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$) is a bijection. Therefore, we may assume that the log structure of $X^{\log}$ is trivial. Moreover, by Lemma 11, (ii), we may assume $n = 1$.

First, we prove that $\kappa \circ \iota$ is the identity morphism. Let $f^{\log} : \underline{X}^{\log} \to X$ be a morphism of type $\mathbf{N}$. If we denote by $\mathcal{G}$ the $\mathbf{G}_m$-torsor sheaf defined in Remark 11, then it is a tautology that the restriction to X of the $\mathbf{G}_m$-torsor sheaf on V that corresponds to the invertible sheaf $\mathcal{O}_V(-X)$ (where we regard X as a Cartier divisor on V via the 0-section $X \to V$) is naturally isomorphic to the $\mathbf{G}_m$-torsor sheaf that corresponds to the conormal sheaf of X in V $(= \iota(f^{\log}))$, i.e., $\mathcal{G}$. Therefore, the pull-back to X of the log structure on V associated to the divisor X (cf. Remark 10) is naturally isomorphic to $\mathcal{M}_{\underline{X}}$.

Next, we prove that $\iota \circ \kappa$ is the identity morphism. Let $\mathscr{L}$ be an invertible sheaf on X. If we denote by $\mathscr{G}$ the $\mathbf{G}_m$-torsor sheaf that corresponds to $\mathscr{L}$, then it is a tautology that the restriction to X of the $\mathbf{G}_m$-torsor sheaf that corresponds to the invertible sheaf $\mathcal{O}_V(-X)$ (where we regard X as a Cartier divisor on V via the 0-section $X \to V$) is naturally isomorphic to the $\mathbf{G}_m$-torsor sheaf that corresponds to the conormal sheaf of X in V $(= \mathscr{L})$, i.e., $\mathscr{G}$. Thus, the assertion follows.

Remark 12. In the notation of Remark 5, the invertible sheaf on D which corresponds to the morphism $D^{\log} \to D$ of type $\mathbf{N}$ is the conormal sheaf $\mathscr{C}_{D/X}$ of D in X by the definition of ι.

Definition 9. Let $X^{\log}$ be a connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbf{N}^{\oplus n}$, $\tau : \mathbf{N}_X^{\oplus n} \xrightarrow{\sim} \mathscr{C}_{f^{\log}}$ a trivialization of $f^{\log}$, and $\iota(f^{\log}, \tau) = (\mathscr{L}_1, \dots \mathscr{L}_n)$. We shall denote by $\pi_i : P_i \to X$ the $\mathbf{P}^1$-bundle associated to the locally free sheaf $\mathscr{L}_i \oplus \mathcal{O}_X$, by $s_i^0 : X \to P_i$ (respectively, $s_i^{\infty} : X \to P_i$) the section of π_i induced by the projection $\mathscr{L}_i \oplus \mathcal{O}_X \to \mathcal{O}_X$ (respectively, $\mathscr{L}_i \oplus \mathcal{O}_X \to \mathscr{L}_i$) (see Remark 13 below), by $\pi : P \stackrel{\text{def}}{=} P_1 \times_X \cdots \times_X P_n \to X$ the natural morphism, and by $s^0 : X \to P$ the section $(s_1^0, \dots, s_n^0)$ of π. We shall denote by $P^{\log}$ the log scheme obtained by equipping P with the log structure $\mathscr{M}_P \stackrel{\text{def}}{=} \pi^*\mathscr{M}_X + \mathscr{M}(D_1^0) + \cdots + \mathscr{M}(D_n^0) + \mathscr{M}(D_1^{\infty}) + \cdots + \mathscr{M}(D_n^{\infty})$, where D_i^0 (respectively, D_i^{∞}) is the divisor on P defined by the following cartesian diagram

$$\begin{array}{ccc} D_i^0 & \longrightarrow & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{s_i^0} & P_i \end{array} \qquad \text{(respectively,} \quad \begin{array}{ccc} D_i^{\infty} & \longrightarrow & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{s_i^{\infty}} & P_i), \end{array}$$

and $\mathscr{M}(D_i^0)$ (respectively, $\mathscr{M}(D_i^{\infty})$) is the log structure defined by the divisor D_i^0 (respectively, D_i^{∞}). Then we obtain a natural morphism of log schemes $\pi^{\log} : P^{\log} \to X^{\log}$ whose underlying morphism of schemes is π; moreover, by Theorem 4, the log scheme obtained by equipping X with the log structure $(s^0)^*\mathscr{M}_P$ is isomorphic to $\underline{X}^{\log}$, and the composite $\underline{X}^{\log} \xrightarrow{(s^0)^{\log}} P^{\log} \xrightarrow{\pi^{\log}} X^{\log}$ is $f^{\log}$, where $(s^0)^{\log}$ is the strict morphism whose underlying morphism of schemes is s^0. We shall refer to $\pi^{\log} : P^{\log} \to X^{\log}$ as the *log* $\mathbf{G}_m^{\times n}$*-torsor associated to* $(f^{\log}, \tau)$ or, alternatively, *to* $(\mathscr{L}_1, \dots \mathscr{L}_n)$. Note that $\pi^{\log}$ is projective and log smooth.

Remark 13. Let $\mathscr{E}$ be a locally free sheaf of rank n on a scheme X, $V \to X$ the geometric vector bundle associated to $\mathscr{E}$, and $P \to X$ (respectively, $P' \to X$) the $\mathbf{P}^n$-bundle (respectively, the $\mathbf{P}^{n-1}$-bundle) associated to the locally free sheaf $\mathscr{E}^\vee \oplus \mathcal{O}_X$ (respectively, $\mathscr{E}^\vee$) (where $\mathscr{E}^\vee = \mathscr{H}om(\mathscr{E}, \mathcal{O}_X)$), and $P' \hookrightarrow P$ the closed immersion over X determined by the projection $\mathscr{E}^\vee \oplus \mathcal{O}_X \to \mathscr{E}^\vee$. Then V is naturally isomorphic to the complement of P' in P.

Indeed, it follows immediately from construction that $P \backslash P' \to X$ is a vector bundle of rank n over X. Moreover, for an open subscheme $U \hookrightarrow X$ of X, a section of $(P \backslash P')|_U \to U$ corresponds to the isomorphism class of the following data.

- An invertible sheaf $\mathscr{L}$ on U.
- A surjection $\pi : \mathscr{E}^\vee|_U \oplus \mathcal{O}_U \to \mathscr{L}$ such that the composite $\mathcal{O}_U \hookrightarrow \mathscr{E}^\vee|_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathscr{L}$ does not vanish on U. (We denote by $s \in \Gamma(U, \mathscr{L})$ the section of $\mathscr{L}$ determined by the above composite $\mathcal{O}_U \hookrightarrow \mathscr{E}^\vee|_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathscr{L}$.)

It is immediate that then $\mathcal{O}_U \xrightarrow{s} \mathscr{L}$ is an isomorphism, and if we denote by $\phi_U(s)$ the section of $\Gamma(U, \mathscr{E}|_U)$ determined by the composite $\mathscr{E}^\vee|_U \hookrightarrow \mathscr{E}^\vee|_U \oplus \mathcal{O}_U \xrightarrow{\pi} \mathscr{L} \xrightarrow{s^{-1}} \mathcal{O}_U$ for the above data, then the assignment

$$(\mathscr{L}, \pi : \mathscr{E}^\vee|_U \oplus \mathcal{O}_U \to \mathscr{L}) \mapsto \phi_U(s)$$

determines a bijection between the set of sections of $(P \backslash P')|_U \to U$ and $\Gamma(U, \mathscr{E}|_U)$; therefore, $P \backslash P' \to X$ is naturally isomorphic to $V \to X$. Moreover, by the above correspondence, $0 \in \Gamma(X, \mathscr{E})$ corresponds to the pair $(\mathcal{O}_X, \mathscr{E}^\vee \oplus \mathcal{O}_X \xrightarrow{\mathrm{pr}_2} \mathcal{O}_X)$.

The main result of this section is the following theorem.

Theorem 5. *Let $X^{\log}$ be a locally noetherian connected fs log scheme, $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbf{N}^{\oplus n}$, $\tau : \mathbf{N}_X^{\oplus n} \xrightarrow{\sim} \mathscr{C}_{f^{\log}}$ a trivialization of $f^{\log}$, and $\pi^{\log} : P^{\log} \to X^{\log}$ the log $\mathbf{G}_m^{\times n}$-torsor associated to $(f^{\log}, \tau)$. Then $(s^0)^{\log} : \underline{X}^{\log} \to P^{\log}$ induces a natural equivalence between the Galois category of ket coverings of $P^{\log}$ and the Galois category of ket coverings of $\underline{X}^{\log}$, i.e., $\pi_1((s^0)^{\log})$ is an isomorphism.*

Proof. (Step 1) *If X is the spectrum of a field k, and the log structure of $X^{\log}$ is trivial, then $\pi_1((s^0)^{\log})$ is an isomorphism.*

Since the horizontal sequences in the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\underline{X}^{\log} \otimes_k k^{\mathrm{sep}}) & \longrightarrow & \pi_1(\underline{X}^{\log}) & \longrightarrow & \mathrm{Gal}(k^{\mathrm{sep}}/k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(P^{\log} \otimes_k k^{\mathrm{sep}}) & \longrightarrow & \pi_1(P^{\log}) & \longrightarrow & \mathrm{Gal}(k^{\mathrm{sep}}/k) & \longrightarrow & 1 \end{array}$$

are exact, by base-changing, we may assume that k is separably closed. Moreover, by Proposition 3, together with Proposition B.5, we may assume $n = 1$. Then it follows from Lemma 12, (ii), below that $\pi_1((s^0)^{\log})$ is an isomorphism.

(Step 2) *If X is the spectrum of a strictly henselian local ring A whose residue field is k, and the log structure of $X^{\log}$ is trivial, then $\pi_1((s^0)^{\log})$ is injective.*

Let us write $\bar{x} \stackrel{\text{def}}{=} \operatorname{Spec} k \subseteq X$ for the closed subscheme of X determined by the natural surjection $A \to k$. Then we have a commutative diagram

$$\begin{array}{ccccc} \underline{\bar{x}}^{\log} \stackrel{\text{def}}{=} \underline{X}^{\log} \times_X \bar{x} & \longrightarrow & P^{\log}_{\bar{x}} \stackrel{\text{def}}{=} P^{\log} \times_X \bar{x} & \longrightarrow & \bar{x} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{X}^{\log} & \xrightarrow[(s^0)^{\log}]{} & P^{\log} & \longrightarrow & X. \end{array}$$

Now it follows immediately from the proof of Lemma 12 that any ket covering of $P^{\log}_{\bar{x}}$ extends to a ket covering of $P^{\log}$; thus, $\pi_1(P^{\log}_{\bar{x}}) \to \pi_1(P^{\log})$ is an injection. Therefore, the assertion follows from the fact that $\pi_1(\underline{\bar{x}}^{\log}) \to \pi_1(\underline{X}^{\log})$ and $\pi_1(\underline{\bar{x}}^{\log}) \to \pi_1(P^{\log}_{\bar{x}})$ are isomorphisms (cf. Proposition B.6 and Step 1).

(Step 3) *If X is the spectrum of a separably closed field k, then $\pi_1((s^0)^{\log})$ is surjective.*

We denote by $\alpha : M \to k$ a clean chart of $X^{\log}$. We write $R \stackrel{\text{def}}{=} k[[M]]$, and $S \stackrel{\text{def}}{=} \operatorname{Spec} R$. Let $S^{\log}$ be the log scheme obtained by equipping S with the log structure associated to the chart given by the natural morphism $M \to R$. Then, by Proposition A.2, $S^{\log}$ is log regular. Write $(\underline{S}^{\log} \to S^{\log}, \tau_S) \stackrel{\text{def}}{=} \kappa(\mathcal{O}_S, \ldots, \mathcal{O}_S)$, and denote by $P^{\log}_S \to S^{\log}$ the log $\mathbf{G}_m^{\times n}$-torsor associated to $(\mathcal{O}_S, \ldots, \mathcal{O}_S)$, and by $(s^0)^{\log}_S$ the closed immersion $\underline{S}^{\log} \to P^{\log}_S$. We denote by K the field of fractions of R, and by $\operatorname{Spec} K \to S^{\log}$ the strict morphism whose underlying morphism corresponds to the natural inclusion $R \hookrightarrow K$. Then we obtain a commutative diagram

$$\begin{array}{ccc} \underline{X}^{\log} & \xrightarrow{(s^0)^{\log}} & P^{\log} \\ \downarrow & & \downarrow \\ \underline{S}^{\log} & \xrightarrow[(s^0)^{\log}_S]{} & P^{\log}_S \\ \uparrow & & \uparrow \\ \underline{(\operatorname{Spec} K)}^{\log} \stackrel{\text{def}}{=} \underline{S}^{\log} \times_{S^{\log}} \operatorname{Spec} K & \xrightarrow{(s^0)^{\log}_K} & P^{\log}_K \stackrel{\text{def}}{=} P^{\log}_S \times_{S^{\log}} \operatorname{Spec} K, \end{array}$$

where the two squares are cartesian.

Now, in the above diagram, the following hold.

(i) $\pi_1(\underline{(\operatorname{Spec} K)}^{\log}) \to \pi_1(P^{\log}_K)$ is an isomorphism. (This follows from Step 1.)

(ii) $\pi_1(P_K^{\log}) \to \pi_1(P_S^{\log})$ is surjective. (This follows from the fact that if we denote by η_{P_S} the generic point of P_S [note that since $S^{\log}$ is log regular, $P_S^{\log}$ is also log regular], then $\pi_1(\eta_{P_S}) \to \pi_1(P_S^{\log})$ is surjective, together with the fact that $\eta_{P_S} \to P_S^{\log}$ factors through $P_K^{\log}$.)

(iii) $\pi_1(\underline{S}^{\log}) \to \pi_1(P_S^{\log})$ is surjective. (This follows from (i) and (ii).)

(iv) $\pi_1(\underline{X}^{\log}) \to \pi_1(\underline{S}^{\log})$ is an isomorphism. (This follows from Proposition B.6.)

(v) $\pi_1(P^{\log}) \to \pi_1(P_S^{\log})$ is an isomorphism. (This follows from Corollary 1 to Theorem 3.)

Therefore, by (iii), (iv), and (v), $\pi_1((s^0)^{\log})$ is surjective.

(Step 4) *If X is the spectrum of a strictly henselian local ring A whose residue field is k, then $\pi_1((s^0)^{\log})$ is an isomorphism.*

We denote by $\bar{x} \stackrel{\text{def}}{=} \operatorname{Spec} k \subseteq X$ the closed subscheme of X determined by the natural surjection $A \to k$, and by $\bar{x}^{\log}$ the log scheme obtained by equipping $\bar{x}$ with the log structure induced by the log structure of $X^{\log}$. First, we prove that $\pi_1((s^0)^{\log})$ is surjective. Let $Q^{\log} \to P^{\log}$ be a connected ket covering of $P^{\log}$. If we denote by $Q \to X' \to X$ the Stein factorization of the composite $Q \to P \to X$, then since Q is connected, and $Q \to X'$ is surjective, we obtain that X' is connected. Now since X is the spectrum of a strictly henselian local ring, and X' is finite over X, $X' \times_X \bar{x}$, hence also $Q \times_X \bar{x}$, is connected. Thus, by base-changing by $\bar{x}^{\log} \to X^{\log}$, we may assume that X is the spectrum of a separably closed field. Then the surjectivity in question follows from Step 3.

Next, we prove that $\pi_1((s^0)^{\log})$ is injective. Now it follows from Lemma 11 that there exists a morphism $\underline{\underline{X}}^{\log} \to X$ of constant type $\mathbf{N}^{\oplus n}$ with trivialization τ' such that the pair obtained as the base-change of $(\underline{\underline{X}}^{\log} \to X, \tau')$ via the natural morphism $X^{\log} \to X$ is isomorphic to $(\underline{X}^{\log} \to X^{\log}, \tau)$. Let $P_1^{\log} \to X$ be the log $\mathbf{G}_m^{\times n}$-torsor associated to the pair $(\underline{\underline{X}}^{\log} \to X, \tau')$, and $(s_1^0)^{\log} : \underline{\underline{X}}^{\log} \to P_1^{\log}$ the morphism "$(s^0)^{\log}$" for the log $\mathbf{G}_m^{\times n}$-torsor $P_1^{\log} \to X$. Thus, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \operatorname{Ker}(\alpha_1) & \longrightarrow & \pi_1(\underline{X}^{\log}) & \xrightarrow{\alpha_1} & \pi_1(\underline{\underline{X}}^{\log}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_1((s^0)^{\log})} & & \downarrow{\scriptstyle \pi_1((s_1^0)^{\log})} & & \\
 & & \operatorname{Ker}(\alpha_2) & \longrightarrow & \pi_1(P^{\log}) & \xrightarrow{\alpha_2} & \pi_1(P_1^{\log}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Ker}(\alpha_3) & \longrightarrow & \pi_1(X^{\log}) & \xrightarrow[\alpha_3]{} & \pi_1(X) & \longrightarrow & 1.
\end{array}$$

It follows from Proposition B.5 that the top sequence is exact; moreover, it follows from Step 2 that $\pi_1((s_1^0)^{\log})$ is *injective*. On the other hand it follows immediately from Proposition B.5 that the composite $\mathrm{Ker}(\alpha_1) \to \mathrm{Ker}(\alpha_2) \to \mathrm{Ker}(\alpha_3)$ is an isomorphism; in particular, $\mathrm{Ker}(\alpha_1) \to \mathrm{Ker}(\alpha_2)$ is *injective*. Now the assertion that $\pi_1((s^0)^{\log})$ is injective follows from the injectivity of $\pi_1((s_1^0)^{\log})$ and $\mathrm{Ker}(\alpha_1) \to \mathrm{Ker}(\alpha_2)$.

(Step 5) *The general case.*

We show that the functor $\mathrm{K\acute{e}t}(P^{\log}) \to \mathrm{K\acute{e}t}(\underline{X}^{\log})$ induced by the morphism $(s^0)^{\log} : \underline{X}^{\log} \to P^{\log}$ is an equivalence. First, we prove that the functor is fully faithful. It is immediate that the functor is faithful (indeed, this follows from the existence of a log geometric point of $P^{\log}$ that factors through $\underline{X}^{\log}$ and the general theory of Galois categories). Thus, it is enough to show that the functor is full. Let $Q_1^{\log} \to P^{\log}$ and $Q_2^{\log} \to P^{\log}$ be ket coverings over $P^{\log}$, and $g^{\log} : Y_1^{\log} \stackrel{\mathrm{def}}{=} Q_1^{\log} \times_{P^{\log}} \underline{X}^{\log} \to Y_2^{\log} \stackrel{\mathrm{def}}{=} Q_2^{\log} \times_{P^{\log}} \underline{X}^{\log}$. Then, by Step 4, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g'^{\log} : Y_1'^{\log} \stackrel{\mathrm{def}}{=} Y_1^{\log} \times_{X^{\log}} X'^{\log} \to Y_2'^{\log} \stackrel{\mathrm{def}}{=} Y_2^{\log} \times_{X^{\log}} X'^{\log}$ over $\underline{X}'^{\log} \stackrel{\mathrm{def}}{=} \underline{X}^{\log} \times_{X^{\log}} X'^{\log}$ obtained as the base-change of $g^{\log}$ by $X'^{\log} \to X^{\log}$ extends to a morphism $\tilde{g}'^{\log} : Q_1'^{\log} \stackrel{\mathrm{def}}{=} Q_1^{\log} \times_{X^{\log}} X'^{\log} \to Q_2'^{\log} \stackrel{\mathrm{def}}{=} Q_2^{\log} \times_{X^{\log}} X'^{\log}$ over $P'^{\log} \stackrel{\mathrm{def}}{=} P^{\log} \times_{X^{\log}} X'^{\log}$. (Indeed, by Step 4, for any geometric point of X, there exists an étale neighborhood $U \to X$ of the geometric point such that if we denote by $U^{\log} \to X^{\log}$ the strict morphism whose underlying morphism of schemes is the morphism $U \to X$, then the base-change of $g^{\log}$ by $U^{\log} \to X^{\log}$ extends to a morphism $Q_1^{\log} \times_{X^{\log}} U^{\log} \to Q_2^{\log} \times_{X^{\log}} U^{\log}$. Thus, if we denote by $X'^{\log}$ the disjoint union of such $U^{\log}$'s, then $X'^{\log} \to X^{\log}$ satisfies the above condition.) Let us denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1st (respectively, 2nd) projection $P'^{\log} \times_{P^{\log}} P'^{\log} \to P'^{\log}$. Now it follows immediately from the fact that the functor $\mathrm{K\acute{e}t}(P'^{\log} \times_{P^{\log}} P'^{\log}) \to \mathrm{K\acute{e}t}(\underline{X}'^{\log} \times_{\underline{X}^{\log}} \underline{X}'^{\log})$ induced by the morphism $(s^0)^{\log}$ is faithful that the following diagram commutes

$$\begin{array}{ccc} q_1^{\log *} Q_1'^{\log} & \xrightarrow{q_1^{\log *} \tilde{g}'^{\log}} & q_1^{\log *} Q_2'^{\log} \\ \big\downarrow & & \big\downarrow \\ q_2^{\log *} Q_1'^{\log} & \xrightarrow{q_2^{\log *} \tilde{g}'^{\log}} & q_2^{\log *} Q_2'^{\log}, \end{array}$$

where $q_i^{\log *}$ denotes the pull-back of each object over $P'^{\log}$ to an object over $P'^{\log} \times_{P^{\log}} P'^{\log}$ via $q_i^{\log}$, and the vertical arrows are the isomorphisms that arise from the fact that $Q_i'^{\log} \to P'^{\log}$ is induced by $Q_i^{\log} \to P^{\log}$. Thus, by Proposition B.8, $\tilde{g}'^{\log}$ extends to a morphism $\tilde{g}^{\log} : Q_1^{\log} \to Q_2^{\log}$. Since the base-change of $\tilde{g}^{\log}$ by $\underline{X}'^{\log} \to P^{\log}$ is $g'^{\log}$, we conclude that $\tilde{g}^{\log}$ is an extension of $g^{\log}$.

Next, we prove that the functor is essentially surjective. Let $Y^{\log} \to \underline{X}^{\log}$ be a ket covering over $\underline{X}^{\log}$. Then, by Step 4, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the ket covering $Y'^{\log} \overset{\text{def}}{=} Y^{\log} \times_{X^{\log}} X'^{\log} \to \underline{X}'^{\log} \overset{\text{def}}{=} \underline{X}^{\log} \times_{X^{\log}} X'^{\log}$ extends to a ket covering $Q'^{\log} \to P'^{\log} \overset{\text{def}}{=} P^{\log} \times_{X^{\log}} X'^{\log}$. Let us denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1st (respectively, 2nd) projection $P'^{\log} \times_{P^{\log}} P'^{\log} \to P'^{\log}$. Now it follows from the fact that the functor in question is *full* that the isomorphism over $\underline{X}'^{\log}$ (that arises from the fact that $Y'^{\log} \to \underline{X}'^{\log}$ is induced by $Y^{\log} \to \underline{X}^{\log}$) extends to an isomorphism $q_1^{\log *} Q'^{\log} \xrightarrow{\sim} q_2^{\log *} Q'^{\log}$; moreover, the fact that the functor in question is *faithful* implies that this isomorphism $q_1^{\log *} Q'^{\log} \xrightarrow{\sim} q_2^{\log *} Q'^{\log}$ satisfies the cocycle condition for being a descent datum. Thus, by Proposition B.8, the ket covering $Q'^{\log} \to P'^{\log}$ extends to a ket covering $Q^{\log} \to P^{\log}$. Moreover, it follows from the construction of $Q^{\log}$ that $Q^{\log} \times_{P^{\log}} \underline{X}^{\log}$ is naturally isomorphic to $Y^{\log}$ over $X^{\log}$.

Lemma 12. *Let k be a separably closed field, whose (not necessarily positive) characteristic we denote by p, $(\mathbf{P}_k^1)^{\log}$ the log scheme obtained by equipping the projective line $\mathbf{P}_k^1$ with the log structure associated to the divisor $\{0, \infty\} \subseteq \mathbf{P}_k^1$, $U \subseteq \mathbf{P}_k^1$ the interior of $(\mathbf{P}_k^1)^{\log}$ (so $U = \mathbf{G}_m$), and $(\operatorname{Spec} k)^{\log} \to (\mathbf{P}_k^1)^{\log}$ the strict morphism for which the image of the underlying morphism of schemes is $\{0\} \subseteq \mathbf{P}_k^1$. Then the following hold.*

(i) *The morphism $\pi_1(U) \to \pi_1((\mathbf{P}_k^1)^{\log})$ is an isomorphism.*

(ii) *The morphism $\pi_1((\operatorname{Spec} k)^{\log}) \to \pi_1((\mathbf{P}_k^1)^{\log})$ is an isomorphism.*

Proof. First, we prove assertion (i). If we denote by η the generic point of $\mathbf{P}_k^1$, then it follows from the fact that the natural morphism $\eta \to (\mathbf{P}_k^1)^{\log}$ induces a surjection $\pi_1(\eta) \to \pi_1((\mathbf{P}_k^1)^{\log})$, together with the fact that the natural morphism $\eta \to (\mathbf{P}_k^1)^{\log}$ factors through U, that $\pi_1(U) \to \pi_1((\mathbf{P}_k^1)^{\log})$ is surjective. Moreover, since any connected finite étale covering over U is of the form

$$\begin{gathered} U = \mathbf{G}_m \to \mathbf{G}_m = U \\ f \mapsto f^n \end{gathered}$$

for some positive integer n that is prime to p, it is easily seen that any finite étale covering over U extends to a ket covering over $(\mathbf{P}_k^1)^{\log}$; thus, $\pi_1(U) \to \pi_1((\mathbf{P}_k^1)^{\log})$ is injective. Therefore, $\pi_1(U) \to \pi_1((\mathbf{P}_k^1)^{\log})$ is an isomorphism.

Next, we prove assertion (ii). We denote by $(\mathbf{A}_k^1)^{\log} \to (\mathbf{P}_k^1)^{\log}$ the strict morphism whose underlying morphism of schemes is the natural open immersion $\mathbf{A}_k^1 \hookrightarrow \mathbf{P}_k^1$ (where we regard $\mathbf{A}_k^1$ as $\mathbf{P}_k^1 \backslash \{\infty\}$). By assertion (i), the restriction to $(\mathbf{A}_k^1)^{\log}$ of any connected ket covering over $(\mathbf{P}_k^1)^{\log}$ is of the form $(\mathbf{A}_k^1)^{\log} \to (\mathbf{A}_k^1)^{\log}$ whose underlying morphism of schemes is the morphism determined by the morphism

$$k[t] \to k[t]$$
$$t \mapsto t^n$$

for some positive integer n that is prime to p. It thus follows immediately from this fact and Proposition B.2 that $\pi_1((\mathrm{Spec}\,k)^{\log}) \to \pi_1((\mathbf{P}^1_k)^{\log})$ is an isomorphism.

The following corollary follows immediately from Theorems 2 and 5.

Corollary 2. *Let $X^{\log}$ be a connected log regular log scheme, and $f^{\log} : \underline{X}^{\log} \to X^{\log}$ a morphism of constant type $\mathbf{N}^{\oplus n}$. Then for any strict geometric point $\bar{x}^{\log} \to X^{\log}$ of $X^{\log}$, the following sequence is exact:*

$$\varprojlim \pi_1(\underline{X}^{\log} \times_{X^{\log}} \bar{x}^{\log}_\lambda) \xrightarrow{s} \pi_1(\underline{X}^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1.$$

Here the projective limit is over all reduced covering points $\bar{x}^{\log}_\lambda \to \bar{x}^{\log}$, and s is induced by the natural projections $\underline{X}^{\log} \times_{X^{\log}} \bar{x}^{\log}_\lambda \to \underline{X}^{\log}$. In particular, by means of a natural isomorphism

$$\varprojlim \pi_1(\underline{X}^{\log} \times_{X^{\log}} \bar{x}^{\log}_\lambda) \xrightarrow{\sim} \hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$$

obtained in Remark 14 *below, we obtain the following exact sequence*

$$\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n} \longrightarrow \pi_1(\underline{X}^{\log}) \xrightarrow{\pi_1(f^{\log})} \pi_1(X^{\log}) \longrightarrow 1,$$

where p is the characteristic of the residue field of the image of the underlying morphism of schemes of the strict geometric point $\bar{x}^{\log} \to X^{\log}$, and $\hat{\mathbf{Z}}^{(p')}(1)$ is the pro-prime to p quotient of $\hat{\mathbf{Z}}(1)$.

Remark 14. Let k be a separably closed field, whose (not necessarily positive) characteristic we denote by p, and $S^{\log}$ an fs log scheme whose underlying scheme S is the spectrum of k. Let $f^{\log} : \underline{S}^{\log} \to S^{\log}$ be a morphism of constant type $\mathbf{N}^{\oplus n}$, and τ a trivialization of $f^{\log}$.

Let $P \to k$, $Q \to k$ be respective clean charts of $S^{\log}$, $\underline{S}^{\log}$ given in Remark 7. Then it follows from Proposition B.5 that the log fundamental group $\pi_1(S^{\log})$ (respectively, $\pi_1(\underline{S}^{\log})$) is naturally isomorphic to $\mathrm{Hom}(P^{\mathrm{gp}}, \hat{\mathbf{Z}}^{(p')}(1))$ (respectively, $\mathrm{Hom}(Q^{\mathrm{gp}}, \hat{\mathbf{Z}}^{(p')}(1))$), where $\hat{\mathbf{Z}}^{(p')}(1)$ is the maximal pro-prime to p quotient of $\hat{\mathbf{Z}}(1)$. Moreover, the morphism $\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})$ induced by $f^{\log}$ is the morphism

$$\mathrm{Hom}(Q^{\mathrm{gp}}, \hat{\mathbf{Z}}^{(p')}(1)) \to \mathrm{Hom}(P^{\mathrm{gp}}, \hat{\mathbf{Z}}^{(p')}(1))$$

induced by $P \to Q$ in Remark 7. In particular, the kernel of $\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})$ is naturally isomorphic to $\mathrm{Hom}(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, \hat{\mathbf{Z}}^{(p')}(1))$. Now the trivialization τ induces a natural isomorphism $\mathbf{Z}^{\oplus n} \xrightarrow{\sim} Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. Therefore, we obtain a *natural isomorphism*

$$(\varprojlim \pi_1(\underline{S}^{\log} \times_{S^{\log}} S_\lambda^{\log})) \xrightarrow{\sim} \mathrm{Ker}(\pi_1(\underline{S}^{\log}) \to \pi_1(S^{\log})) \xrightarrow{\sim} \hat{\mathbf{Z}}^{(p')}(1)^{\oplus n},$$

where the projective limit is over all reduced covering points $S_\lambda^{\log} \to S^{\log}$.

Proposition 4. *Let $X^{\log}$ be a connected log regular log scheme over a field k, whose (not necessarily positive) characteristic we denote by p, $U_X \subseteq X$ the interior of $X^{\log}$, and $\mathcal{L}_1, \ldots, \mathcal{L}_n$ invertible sheaves on X. Let $\pi^{\log} : P^{\log} \to X^{\log}$ be the log $\mathbf{G}_m^{\times n}$-torsor associated to $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$. If the condition $(*)$ below is satisfied, then, in the following exact sequence obtained in Corollary 2 to Theorem 5*

$$(\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n} \simeq)\ \varprojlim \pi_1(P^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \xrightarrow{\ s\ } \pi_1(P^{\log}) \xrightarrow{\pi_1(\pi^{\log})} \pi_1(X^{\log}) \longrightarrow 1,$$

the first morphism is injective.

$(*)$ *For any integer i such that $1 \leq i \leq n$ and any positive integer N that is prime to p, there exists a covering $V \to U_X$ tamely ramified along $X \backslash U_X$ and an invertible sheaf $\mathcal{N}$ such that $\mathcal{N}^{\otimes N} \xrightarrow{\sim} \mathcal{L}_i|_V$.*

Proof. First, observe that it is enough to prove the assertion in the case where the image of the underlying morphism of schemes of the strict geometric point $\bar{x}^{\log} \to X^{\log}$ lies on U_X. Indeed, this follows from the fact that a surjective endomorphism of $\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$ is an *automorphism*. Assume that the image of the underlying morphism of schemes of the strict geometric point $\bar{x}^{\log} \to X^{\log}$ lies on U_X.

If we denote by $P_i^{\log} \to X^{\log}$ the log $\mathbf{G}_m$-torsor associated to $\mathcal{L}_i$ $(1 \leq i \leq n)$, then there exists a natural isomorphism $P^{\log} \xrightarrow{\sim} P_1^{\log} \times_{X^{\log}} \cdots \times_{X^{\log}} P_n^{\log}$ over $X^{\log}$. Thus, if the assertion in the case where $n = 1$ is verified, then the composite

$$\pi_1(P_i^{\log} \times_{X^{\log}} \bar{x}) \longrightarrow \prod_{\nu=1}^{n} \pi_1(P_\nu^{\log} \times_{X^{\log}} \bar{x}) \xleftarrow[\sim]{\prod_{\nu=1}^{n} \mathrm{pr}_\nu} \pi_1(P^{\log} \times_{X^{\log}} \bar{x})$$
$$\longrightarrow \pi_1(P^{\log}) \xrightarrow{\pi_1(\mathrm{pr}_j)} \pi_1(P_j^{\log})$$

is injective (respectively, zero) if $i = j$ (respectively, if $i \neq j$). Therefore, to complete the proof of Proposition 4, we may assume that $n = 1$. Write $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{L}_1$. Let N be a positive integer that is prime to p. Note that it is enough to show that the N-th (cyclic) ket covering over $P^{\log} \times_{X^{\log}} \bar{x}$ (cf. Lemma 12) lifts to a ket covering $Q^{\log} \to P^{\log}$ over $P^{\log}$ to complete the proof of Proposition 4.

We denote by $Q_V^{\log} \to V$ the log $\mathbf{G}_m$-torsor associated to $\mathcal{N}$ (in the condition $(*)$), and by $Q_V \to P \times_X V$ the morphism determined by the following composite:

$$
\begin{aligned}
\mathscr{N} &\to \mathscr{N}^{\otimes N} \xrightarrow{\sim} \mathscr{L}|_V \\
f &\mapsto f^{\otimes N}.
\end{aligned}
$$

Then it follows from the definition of a log $\mathbf{G}_m$-torsor associated to an invertible sheaf that the morphism $Q_V \to P \times_X V$ extends to a morphism of log schemes $Q_V^{\log} \to P^{\log} \times_{X^{\log}} V$; thus, we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
Q_V^{\log} \times_{P^{\log}} U_P & \longrightarrow & & & U_P & & \\
\downarrow & & & & \downarrow & & \\
Q_V^{\log} & \longrightarrow & P^{\log} \times_{X^{\log}} V & \longrightarrow & P^{\log} \times_{X^{\log}} U_X & \longrightarrow & P^{\log} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & V & \longrightarrow & U_X & \longrightarrow & X^{\log},
\end{array}
$$

where U_P is the interior of $P^{\log}$, and the three squares are cartesian. It follows immediately from the construction of $Q_V^{\log}$ that the log structure of $Q_V^{\log} \times_{P^{\log}} U_P$ is trivial, and that the top horizontal arrow $Q_V^{\log} \times_{P^{\log}} U_P = Q_V \times_P U_P \to U_P$ is finite étale.

Now I claim that the normalization Q of U_P in $Q_V \times_P U_P$ is tamely ramified over P along $P \backslash U_P$. Indeed, this claim may be verified as follows. Every point a of $P \backslash U_P$ with $\dim \mathcal{O}_{P,a} = 1$ is either

(i) the generic point of a (reduced) divisor on P determined by s^0 or s^∞ (see Definition 9), or

(ii) the generic point of a (reduced) divisor on P which is the pull-back of a reduced divisor on X whose generic point x is a point of $X \backslash U_X$ with $\dim \mathcal{O}_{X,x} = 1$.

Thus, it is easily verified that the claim holds. Therefore, by the log purity theorem (cf. Remark B.2), the covering extends to a ket covering $Q^{\log} \to P^{\log}$. Moreover, by the construction of the morphism $Q_V \to P \times_X V$, the restriction of the ket covering $Q^{\log} \times_{X^{\log}} \bar{x} \to P^{\log} \times_{X^{\log}} \bar{x}$ to any of the connected components of $Q^{\log} \times_{X^{\log}} \bar{x}$ is the N-th (cyclic) covering over $P^{\log} \times_{X^{\log}} \bar{x}$.

Definition 10. In the notation of Proposition 4, we shall refer to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$

$$
1 \longrightarrow \varprojlim \pi_1(P^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \longrightarrow \pi_1(P^{\log}) \xrightarrow{\pi_1(\pi^{\log})} \pi_1(X^{\log}) \longrightarrow 1
$$

as the *extension of* $\pi_1(X^{\log})$ *by* $\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$ *associated to* $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$. More generally, let Σ be a set of prime numbers which does not contain p, and N the kernel of the composite of the natural isomorphism

$$\varprojlim \pi_1(P^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log}) \xrightarrow{\sim} \hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$$

and the surjection $\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n} \to \hat{\mathbf{Z}}^{(\Sigma)}(1)^{\oplus n}$ induced by the natural projection $\hat{\mathbf{Z}}^{(p')}(1) \to \hat{\mathbf{Z}}^{(\Sigma)}(1)$. Then we shall refer to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(\Sigma)}(1)^{\oplus n}$

$$1 \longrightarrow \varprojlim \pi_1(P^{\log} \times_{X^{\log}} \bar{x}_\lambda^{\log})/N \longrightarrow \pi_1(P^{\log})/N \xrightarrow{\text{via } \pi_1(\pi^{\log})} \pi_1(X^{\log}) \longrightarrow 1$$

naturally obtained from the extension of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(p')}(1)^{\oplus n}$ associated to $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$ as the *extension of* $\pi_1(X^{\log})$ *by* $\hat{\mathbf{Z}}^{(\Sigma)}(1)^{\oplus n}$ *associated to* $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$.

Remark 15. If we denote by $\mathscr{S}(\pi_1(U_X))$ (respectively, $(U_X)_{\text{ét}}$) the classifying site of $\pi_1(U_X)$ (i.e., the site defined by considering the category of finite sets equipped with a continuous action of $\pi_1(U_X)$ [and coverings given by surjections of such sets]) (respectively, the étale site of U_X), then the natural morphism of sites

$$(U_X)_{\text{ét}} \to \mathscr{S}(\pi_1(U_X))$$

induces a natural morphism

$$\mathrm{H}^n(\pi_1(U_X), \hat{\mathbf{Z}}^{(p')}(1)) \to \mathrm{H}^n_{\text{ét}}(U_X, \hat{\mathbf{Z}}^{(p')}(1)).$$

If the morphism $\mathrm{H}^2(\pi_1(U_X), \hat{\mathbf{Z}}^{(p')}(1)) \to \mathrm{H}^2_{\text{ét}}(U_X, \hat{\mathbf{Z}}^{(p')}(1))$ is an *isomorphism*, then, by a similar argument to the argument used in the proof of [11], Lemma 4.3, any invertible sheaf on X satisfies the condition $(*)$ in Proposition 3. Moreover, if the morphism

$$\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbf{Z}}^{(p')}(1)) \to \mathrm{H}^2(\pi_1(U_X), \hat{\mathbf{Z}}^{(p')}(1))$$

induced by the natural surjection $\pi_1(U_X) \to \pi_1(X^{\log})$ is an *isomorphism*, then, by a similar argument to the argument used in the proof of [11], Lemma 4.4, the extension of $\pi_1(X^{\log})$ associated to $\mathscr{L}$ is isomorphic to the extension of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(p')}(1)$ determined by the (étale-theoretic) first Chern class (cf. [11], Definition 4.1) of the invertible sheaf $\mathscr{L}$ via the isomorphisms

$$\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbf{Z}}^{(p')}(1)) \xrightarrow{\sim} \mathrm{H}^2(\pi_1(U_X), \hat{\mathbf{Z}}^{(p')}(1)) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(U_X, \hat{\mathbf{Z}}^{(p')}(1)).$$

(Now, by means of the natural bijection in [13], Theorem 1.2.5, we identify the set of equivalence classes of extensions of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(p')}(1)$ with $\mathrm{H}^2(\pi_1(X^{\log}), \hat{\mathbf{Z}}^{(p')}(1))$.) Moreover, then the extension of $\pi_1(X^{\log})$ associated to $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$ is isomorphic to the fiber product of the extensions of $\pi_1(X^{\log})$ by $\hat{\mathbf{Z}}^{(p')}(1)$ determined by the (étale-theoretic) first Chern classes of the invertible sheaves $\mathscr{L}_i$ $(1 \le i \le n)$.

Appendix A. Étale analogues of the results in [9]

In this section, we prove étale analogues of the results in [9].

Definition A.1 (cf. [9], Definition 2.1). Let $X^{\log}$ be a locally noetherian fs log scheme.

(i) Let $\bar{x} \to X$ be a geometric point of X, and $I_{X,\bar{x}} \subseteq \mathcal{O}_{X,\bar{x}}$ the ideal of $\mathcal{O}_{X,\bar{x}}$ generated by the image of $\mathcal{M}_{X,\bar{x}} \backslash \mathcal{O}^*_{X,\bar{x}}$. Then we shall say that $X^{\log}$ is *log regular at* $\bar{x} \to X$ if the following hold.

(1) $\mathcal{O}_{X,\bar{x}}/I_{X,\bar{x}}$ is a regular local ring.

(2) $\dim(\mathcal{O}_{X,\bar{x}}) = \dim(\mathcal{O}_{X,\bar{x}}/I_{X,\bar{x}}) + \mathrm{rank}_{\mathbf{Z}}(\mathcal{M}^{\mathrm{gp}}_{X,\bar{x}}/\mathcal{O}^*_{X,\bar{x}})$.

(ii) We shall say that $X^{\log}$ is *log regular* if for any $x \in X$, there exists a geometric point $\bar{x} \to X$ for which the image of the underlying morphism of schemes is $x \in X$ such that $X^{\log}$ is log regular at $\bar{x} \to X$.

Proposition A.1. *Let $X^{\log}$, $U^{\log}$ be locally noetherian fs log schemes, $U^{\log} \to X^{\log}$ a strict étale morphism, and $\bar{x} \to U$ a geometric point of U. Then $X^{\log}$ is log regular at the geometric point obtained as the composite $\bar{x} \to U \to X$ if and only if $U^{\log}$ is log regular at $\bar{x} \to U$.*

Proof. This follows from the definition of log regularity.

Proposition A.2 (cf. [9], Theorem 3.2, (1)). *Let $X^{\log}$ be a locally noetherian fs log scheme, $\bar{x} \to X$ a geometric point of X, and $P \to \mathcal{M}_X$ a clean chart of $X^{\log}$ at $\bar{x} \to X$ (cf. Definition* B.1, (ii)*). Assume that $\mathcal{O}_{X,\bar{x}}/I_{X,\bar{x}}$ is a regular local ring, and the natural surjection $\mathcal{O}_{X,\bar{x}} \to k(\bar{x})$ admits a section. Let $t_1, \ldots, t_r \in \mathcal{O}_{X,\bar{x}}$ be elements whose images in $\mathcal{O}_{X,\bar{x}}/I_{X,\bar{x}}$ form a regular system of parameters of the regular local ring $\mathcal{O}_{X,\bar{x}}/I_{X,\bar{x}}$. Then $X^{\log}$ is log regular at $\bar{x} \to X$ if and only if the surjection*

$$k(\bar{x})[[P]][[T_1, \ldots, T_r]] \to \hat{\mathcal{O}}_{X,\bar{x}}$$

given by $T_i \mapsto t_i$ $(i = 1, \ldots, r)$ is an isomorphism.

Proof. This follows from a similar argument to the argument used in the proof of [9], Theorem 3.2, (1).

Proposition A.3 (cf. [9], Theorem 4.1). *A log regular log scheme is Cohen-Macaulay and normal.*

Proof. This follows from Proposition A.1; [9], Theorem 4.1, and [14], Lemma 2.3.

Proposition A.4 (cf. [9], Proposition 7.1). *Let $X^{\log}$ be a locally noetherian fs log scheme, and $\bar{x}, \bar{y} \to X$ geometric points of X such that the image of $\bar{x} \to X$*

is contained in the closure of the image of $\bar{y} \to X$. *Then if* $X^{\log}$ *is log regular at* $\bar{x} \to X$, *then* $X^{\log}$ *is log regular at* $\bar{y} \to X$.

Proof. This follows from Proposition A.1; [9], Proposition 7.1, and [14], Lemma 2.3.

Proposition A.5 (cf. [9], Theorem 8.2). *Let* $X^{\log}$, $Y^{\log}$ *be fs log schemes, and* $f^{\log} : Y^{\log} \to X^{\log}$ *a log smooth morphism. Then if* $X^{\log}$ *is log regular, then* $Y^{\log}$ *is also log regular.*

Proof. This follows from Proposition A.1; [9], Theorem 8.2; [8], Proposition 3.8, and [14], Lemma 2.3.

Proposition A.6 (cf. [9], Theorem 11.6). *Let* $X^{\log}$ *be a log regular log scheme, and* $U_X \subseteq X^{\log}$ *the interior of* $X^{\log}$. *Then the log structure of* $X^{\log}$ *is isomorphic to the log structure which is of the form*

$$\mathcal{O}_X \cap (U_X \hookrightarrow X)_* \mathcal{O}_{U_X}^* \hookrightarrow \mathcal{O}_X.$$

Proof. This follows from [14], Proposition 2.6.

Appendix B. Existence of log fundamental groups

In this section, we prove the well-known fact that the category of ket coverings of a connected fs log scheme is *a Galois category*; this implies, in particular, the existence of log fundamental groups. The *assertion* that the category of ket coverings is Galois essentially follows from the *assertion* in the case where the underlying scheme of the base log scheme is the spectrum of a strictly henselian local ring (cf. Proposition B.5), together with the descent theory for strict étale surjections (cf. Proposition B.8).

Definition B.1.

(i) Let P be a monoid. We shall say that P is *clean* if P is an fs monoid and $P^* = \{0\}$ (where P^* is the set of invertible elements of P).

(ii) Let $X^{\log}$ be an fs log scheme, $\bar{x} \to X$ a geometric point of X, and $P \to \mathcal{O}_X$ an fs chart of $X^{\log}$. Then we shall say that the chart $P \to \mathcal{O}_X$ is *clean at* $\bar{x} \to X$ if the composite $P \to \mathcal{M}_{X,\bar{x}} \to (\mathcal{M}_X/\mathcal{O}_X^*)_{\bar{x}}$ is an isomorphism. Note that a clean chart of $X^{\log}$ always exists over an étale neighborhood of any given geometric point of X. (See the discussion following [10], Definition 1.3.)

Definition B.2. Let P be a torsion-free fs monoid. We shall denote by $(1/n)P$ the monoid $\{p \in P^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{Q} \mid np \in \mathrm{Im}(P \hookrightarrow P^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{Q})\}$. Note that the natural inclusion $P \hookrightarrow P^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{Q}$ factors through $(1/n)P$. Thus, we always assume that $(1/n)P$ is a P-monoid via the natural inclusion $P \hookrightarrow (1/n)P$. Moreover, the morphism

$$\begin{aligned}(1/n)P &\to (1/n)P \\ p &\mapsto np\end{aligned}$$

factors through P ($\subseteq (1/n)P$). On the other hand, the resulting morphism $(1/n)P \to P$ is an isomorphism. We shall denote by $(1/n)_P$ the inverse isomorphism $P \to (1/n)P$.

Definition B.3. Let $f : P \to Q$ be a morphism of monoids. Then we shall say that f is *Kummer* if f is injective, and there exists a positive integer n such that $n \cdot Q \subseteq \mathrm{Im}(f)$.

Proposition B.1.

(i) *Let P, Q be clean monoids. Then for any Kummer morphism $f : P \to Q$, there exists a positive integer n such that the natural inclusion $P \hookrightarrow (1/n)P$ uniquely factors as a composite $P \xrightarrow{f} Q \xrightarrow{g} (1/n)P$, and, moreover, g is Kummer such that $n \cdot (1/n)P \subseteq \mathrm{Im}(g)$.*

(ii) *Let P be a clean monoid, n a natural number, and $G \subseteq ((1/n)P)^{\mathrm{gp}}/P^{\mathrm{gp}}$ a subgroup of $((1/n)P)^{\mathrm{gp}}/P^{\mathrm{gp}}$. Then the submonoid $Q \subseteq (1/n)P$ obtained by pulling back the subgroup $G \subseteq ((1/n)P)^{\mathrm{gp}}/P^{\mathrm{gp}}$ via the natural morphism $(1/n)P \to ((1/n)P)^{\mathrm{gp}}/P^{\mathrm{gp}}$ is fs.*

Proof. First, we prove assertion (i). Since f is Kummer, there exists a positive integer n such that $n \cdot Q \subseteq \mathrm{Im}(f)$. Thus, it follows from the injectivity of f that for any $q \in Q$, there exists a *unique* element $p_q \in P$ such that $nq = f(p_q)$. Now define $g : Q \to (1/n)P$ by $q \mapsto (1/n)_P(p_q)$. It is immediate that g is a homomorphism of monoids and $g \circ f(p) = p$ for any $p \in P$. Moreover, for any $(1/n)_P(p) \in (1/n)P$, $n((1/n)_P(p)) = p = g \circ f(p)$; hence $n((1/n)_P(p)) \in \mathrm{Im}(g)$. It remains to show that g is injective. If $g(q) = g(q')$, then $nq = nq'$. Since Q is integral and torsion-free, $q = q'$; thus, g is injective.

Next, we prove assertion (ii). Since Q is a submonoid of $(1/n)P$, Q is integral; moreover, since P is finitely generated, and G is a finite group, Q is finitely generated. Thus, it remains to show that Q is saturated. To prove the saturatedness of Q, it follows from the saturatedness of $(1/n)P$ that it is enough to show that the natural inclusion $Q \hookrightarrow ((1/n)P) \cap Q^{\mathrm{gp}}$ (in $((1/n)P)^{\mathrm{gp}}$) is surjective. On the other hand, the surjectivity of the inclusion $Q \hookrightarrow ((1/n)P) \cap Q^{\mathrm{gp}}$ follows from the construction of Q, together with the fact that the natural morphism $Q \to G$ factors through $Q \hookrightarrow Q^{\mathrm{gp}}$.

Proposition B.2. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring A. Let us fix a global* *clean chart* $P \to \mathcal{O}_X$. *Then any connected ket covering of* $X^{\log}$ *is of the form* $(X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log} \to X^{\log}$, *where* $P \to Q$ *is a Kummer morphism of fs monoids such that* $n \cdot Q \subseteq \mathrm{Im}(P \to Q)$ *for some integer* n *invertible on* X, *and the log structure of* $(X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$ *is induced by the natural morphism* $Q \to A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$. *Conversely, if a morphism of log schemes* $Y^{\log} \to X^{\log}$ *has this form, then it is a ket covering.*

Proof. The last assertion is immediate from the definition of ket covering. Let $Y^{\log} \to X^{\log}$ be a connected ket covering. Then since $Y \to X$ is finite, Y is affine. Let us write $Y = \mathrm{Spec}\, B$. Since $A \to B$ is finite, and Y is connected, B is a strictly henselian local ring. By [8], Theorem 3.5, there exists an fs chart $Q \to B$ of $Y^{\log}$ and a chart

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

of $Y^{\log} \to X^{\log}$ such that the following conditions hold.

(i) $P \to Q$ is injective, and the cokernel of $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is finite and of order n invertible in A.

(ii) $\mathrm{Spec}\, B \to \mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])$ is étale.

(iii) $P \to Q/(Q \to B)^{-1}(B^*)$ is Kummer.

Since $\mathbf{Z}[P] \to \mathbf{Z}[Q]$ is finite, $A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ is a strictly henselian local ring. Thus, since the morphism $A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \to B$ is finite and étale, this morphism is an isomorphism. Moreover, since it is immediate that the chart $Q \to A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \simeq B$ is clean (i.e., $(Q \to B)^{-1}(B^*) = \{0\}$), it follows from conditions (i) and (iii) that $P \to Q$ is Kummer and satisfies $n \cdot Q \subseteq \mathrm{Im}(P \to Q)$.

Proposition B.3. *A ket covering is an open and closed map. In particular, a non-empty ket covering over a connected fs log scheme is a surjection.*

Proof. This follows from Proposition B.2 and [6], Proposition 3.2.

Proposition B.4. *Let* $X^{\log}$, $Y^{\log}$, *and* $Z^{\log}$ *be fs log schemes, and* $f^{\log} : X^{\log} \to Y^{\log}$ *and* $g^{\log} : Y^{\log} \to Z^{\log}$ *morphisms. Then if* $g^{\log}$ *and* $g^{\log} \circ f^{\log}$ *are ket coverings, then so is* $f^{\log}$.

Proof. The finiteness of f is classical; moreover, the log étaleness of $f^{\log}$ is formally showed as in the *non-log case*. On the other hand, the Kummerness of $f^{\log}$ follows from the definition of the Kummerness.

Definition B.4. Let $X^{\log}$ be an fs log scheme, and $\tilde{x}^{\log} \to X^{\log}$ a log geometric point of $X^{\log}$.

(i) We shall denote by $\text{Két}(X^{\log})$ the category of ket coverings of $X^{\log}$ and morphisms over $X^{\log}$. Note that it follows from Proposition B.4 that any morphisms in the category $\text{Két}(X^{\log})$ are ket coverings.

(ii) We shall denote by $F_{\tilde{x}^{\log}}$ the functor defined by

$$\begin{gathered}\text{Két}(X^{\log}) \to (\text{The category of finite sets}) \\ Y^{\log} \to X^{\log} \mapsto \text{Hom}_{X^{\log}}(\tilde{x}^{\log}, Y^{\log}).\end{gathered}$$

(Note that it follows from Proposition B.2 that the set $\text{Hom}_{X^{\log}}(\tilde{x}^{\log}, Y^{\log})$ is finite.)

Proposition B.5. *Let $X^{\log}$ be an fs log scheme whose underlying scheme X is the spectrum of a strictly henselian local ring A whose residue field we denote by k, p the characteristic of k, $\alpha : P \to A$ a (global) clean chart of $X^{\log}$,*

$$\tilde{P} \stackrel{\text{def}}{=} \varinjlim (1/n)P,$$

where the inductive limit is over all natural numbers n prime to p, $\tilde{x}^{\log}$ a log scheme obtained by equipping $\tilde{x} \stackrel{\text{def}}{=} \operatorname{Spec} k$ with the log structure induced by the morphism $\tilde{P} \to k$ given by mapping $a \in \tilde{P}\backslash\{0\}$ to $0 \in k$, and $\tilde{x}^{\log} \to X^{\log}$ the log geometric point obtained by the natural morphisms $A \to k$ and $P \to \tilde{P}$. Then the functor $F_{\tilde{x}^{\log}}$ induces an equivalence between the category $\text{Két}(X^{\log})$ and the category of finite sets equipped with continuous actions of the profinite group

$$\pi \stackrel{\text{def}}{=} \text{Hom}(\tilde{P}^{\text{gp}}/P^{\text{gp}}, A^*) \simeq \varprojlim \text{Hom}(((1/n)P)^{\text{gp}}/P^{\text{gp}}, A^*) \simeq \text{Hom}(P^{\text{gp}}, \hat{\mathbf{Z}}(1)(k)),$$

where the projective limit is over all natural numbers n prime to p. In particular, the pair $(\text{Két}(X^{\log}), F_{\tilde{x}^{\log}})$ forms a Galois category with a fundamental functor.

Proof. First, we verify that π acts on the finite set $F_{\tilde{x}^{\log}}(Y^{\log})$ for a ket covering $Y^{\log} \to X^{\log}$. Let $Y_1^{\log}$ be a connected component of $Y^{\log}$. Then it follows from Proposition B.2 that $Y_1^{\log}$ is of the form $\operatorname{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log}$, where $P \to Q$ is a Kummer morphism of fs monoids which satisfies the condition in the statement of Proposition B.2. Now it is easily verified that the group $\text{Aut}_{X^{\log}}(Y_1^{\log})$ is naturally isomorphic to $\text{Hom}(Q^{\text{gp}}/P^{\text{gp}}, A^*)$. Since $\text{Aut}_{X^{\log}}(Y_1^{\log})$ naturally acts on $F_{\tilde{x}^{\log}}(Y_1^{\log})$, and the inclusion $Q \hookrightarrow \tilde{P}$ obtained by Proposition B.1, (i), induces a continuous morphism $\pi \to \text{Hom}(Q^{\text{gp}}/P^{\text{gp}}, A^*)$, we obtain an action of π on $F_{\tilde{x}^{\log}}(Y_1^{\log})$; in particular, we obtain an action of π on $F_{\tilde{x}^{\log}}(Y^{\log})$.

Next, we prove the full faithfulness of the functor in question. Let $Y_i^{\log} \to X^{\log}$ be a connected ket covering of $X^{\log}$ (where $i = 1, 2$). Then it follows from Proposition B.2 that $Y_i^{\log}$ is of the form $\operatorname{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q_i])^{\log}$, where $P \to Q_i$ is a Kummer morphism of fs monoids which satisfies the condition in the statement of Proposition B.2. Our claim is that the natural morphism

$$\mathrm{Hom}_{X^{\log}}(Y_1^{\log}, Y_2^{\log}) \to \mathrm{Hom}_{\pi}(F_{\tilde{x}^{\log}}(Y_1^{\log}), F_{\tilde{x}^{\log}}(Y_2^{\log}))$$

is bijective. Now it is immediate that if $Q_2 \not\subseteq Q_1$ (in $\tilde{P}$), then the both sides are empty; thus, we may assume that $Q_2 \subseteq Q_1$. Then it is easily verified that the both sides are $\mathrm{Hom}(Q_2^{\mathrm{gp}}/P^{\mathrm{gp}}, A^*)$-*torsors*. Thus, it follows that the above morphism is bijective.

Finally, we prove the essential surjectivity of the functor in question. Let S be a finite set equipped with a continuous action of π. By taking a "connected component" of S (i.e., the orbit of an element of S), we may assume that the action on S is transitive. Let $s_0 \in S$, and $\mathrm{Stab}(s_0) \subseteq \pi$ the stabilizer of s_0. Then since π is *abelian*, the subgroup $\mathrm{Stab}(s_0) \subseteq \pi$ is *normal*; moreover, the morphism $\pi \to S$ given by mapping g to $g(s_0)$ determines a bijection $\pi/\mathrm{Stab}(s_0) \xrightarrow{\sim} S$ of $(\pi/\mathrm{Stab}(s_0))$-torsors. Let $Q \subseteq \tilde{P}$ be the submonoid obtained by pulling back the subgroup $\mathrm{Hom}(\pi/\mathrm{Stab}(s_0), A^*) \subseteq \mathrm{Hom}(\pi, A^*) \simeq \tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}}$ via $\tilde{P} \to \tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}}$. Then it follows from the continuity of the action of π on S that there exists a natural number n which is prime to p such that $Q \subseteq (1/n)P$ $(\subseteq \tilde{P})$. Moreover, by the construction of Q, together with Proposition B.1, (ii), the monoid Q is an fs monoid; on the other hand, since $Q \subseteq (1/n)P$, the natural morphism $P \to Q$ satisfies the condition in the statement of Proposition B.2. Therefore, by Proposition B.2, $Y^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])^{\log} \to X^{\log}$ is a ket covering. Moreover, again by the construction of Q, $F_{\tilde{x}^{\log}}(Y^{\log})$ is isomorphic to S. This completes the proof of the essential surjectivity of the functor in question.

Remark B.1. The assertion proven in Proposition B.5 (i.e., the assertion that the category $\mathrm{K\acute{e}t}(X^{\log})$ is a Galois category for an fs log scheme $X^{\log}$ whose underlying scheme X is the spectrum of a strictly henselian local ring) can be also proven by means of the *log purity theorem* (cf. Proposition B.7 below). Indeed, it follows from Proposition B.6 below that we may assume that X is the spectrum of a separably closed field k. Let $P \to k$ be a clean chart of $X^{\log}$, and $\hat{X}^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k[[P]]$ with the log structure induced by the natural morphism $P \to k[[P]]$. Then since $\hat{X}^{\log}$ is log regular (cf. Propositions A.2 and A.4), again by Proposition B.6 below, by replacing $X^{\log}$ by $\hat{X}^{\log}$, we may assume that $X^{\log}$ is log regular. Then it follows from Proposition B.7 below that $\mathrm{K\acute{e}t}(X^{\log})$ is naturally equivalent to the category of coverings of the interior U of $X^{\log}$ tamely ramified along $D \stackrel{\mathrm{def}}{=} X \backslash U$; in particular, $\mathrm{K\acute{e}t}(X^{\log})$ is a Galois category.

The following two propositions (i.e., Propositions B.6 and B.7) and one remark (i.e., Remark B.2) are not logically necessary for the proof of the assertion that the category of ket coverings is Galois, but these were used in the argument of Remark B.1.

Proposition B.6. *Let $X^{\log}$ be an fs log scheme whose underlying scheme is the spectrum of a strictly henselian local ring A, whose residue field we denote by k, and $\bar{x}^{\log} \to X^{\log}$ the strict morphism whose underlying morphism of schemes is the morphism obtained by the natural surjection $A \to k$. Then $\bar{x}^{\log} \to X^{\log}$ induces an equivalence between the category of ket coverings of $X^{\log}$ and the category of ket coverings of $\bar{x}^{\log}$.*

Proof. It follows immediately from Proposition B.2 that the functor in question is essentially surjective and full. Moreover, by considering the graphs of morphisms, the faithfulness of the functor in question follows from Proposition B.3. (Note that any morphisms in "Két$(-)$" are ket coverings by Proposition B.4.)

Proposition B.7. *Let $X^{\log}$ be a log regular log scheme, and $U_X \subseteq X$ the interior of $X^{\log}$. Then the morphism $U_X \to X^{\log}$ induces an equivalence of the category of ket coverings of $X^{\log}$ and the category of coverings of U_X tamely ramified along $D_X = X \backslash U_X$.*

Proof. Note that the assertion that the morphism $U_X \hookrightarrow X^{\log}$ induces a functor from the category of ket coverings of $X^{\log}$ to the category of coverings of U_X tamely ramified along D_X follows immediately from the definition of ket coveringness. Moreover, the essential surjectivity of this functor follows immediately from the log purity theorem in [10] (cf. also Remark B.2 below).

Finally, we show that this functor is fully faithful. Let $Y_i^{\log} \to X^{\log}$ be ket coverings (where $i = 1, 2$), and U_{Y_i} the interior of $Y_i^{\log}$. Then since it is immediate that the natural strict open immersion $U_{Y_i} \hookrightarrow Y_i^{\log}$ induces an isomorphism $U_{Y_i} \xrightarrow{\sim} Y_i^{\log} \times_{X^{\log}} U_X$, our claim is that the natural morphism

$$\begin{aligned}\mathrm{Hom}_{X^{\log}}(Y_1^{\log}, Y_2^{\log}) &\xrightarrow{\phi} \mathrm{Hom}_{U_X}(Y_1^{\log} \times_{X^{\log}} U_X, Y_2^{\log} \times_{X^{\log}} U_X) \\ &= \mathrm{Hom}_{U_X}(U_{Y_1}, U_{Y_2})\end{aligned}$$

is bijective. To show the injectivity of ϕ, let $f^{\log}, g^{\log} : Y_1^{\log} \to Y_2^{\log}$ be morphisms of ket coverings over $X^{\log}$ such that $f^{\log}|_{U_{Y_1}} = g^{\log}|_{U_{Y_1}} : U_{Y_1} \to U_{Y_2}$. Now since $X^{\log}$ is log regular, and $Y_i^{\log} \to X^{\log}$ is log étale, $Y_i^{\log}$ is log regular (cf. Proposition A.5); therefore, $U_{Y_i} \subseteq Y_i$ is a dense open subset of Y_i (cf. Proposition A.3). Thus, $f^{\log}|_{U_{Y_1}} = g^{\log}|_{U_{Y_1}}$ implies $f = g$. Moreover, since $Y_i^{\log}$ is log regular, the log structure of $Y_i^{\log}$ is $\mathcal{O}_{Y_i} \cap (U_{Y_i} \hookrightarrow Y_i)_* \mathcal{O}^*_{U_{Y_i}} \hookrightarrow \mathcal{O}_{Y_i}$ (cf. Proposition A.6); therefore, a morphism of log schemes from $Y_1^{\log}$ to $Y_2^{\log}$ is determined by the underlying morphism of schemes. In other words, $f = g$ implies $f^{\log} = g^{\log}$; we thus conclude that ϕ is injective. Next, we prove the surjectivity of ϕ. Let $f_U : U_{Y_1} \to U_{Y_2}$ be a morphism over U_X. Since the normalization of X in U_{Y_i} is isomorphic to Y_i, the morphism f_U extends to a morphism $f : Y_1 \to Y_2$. By a similar argument to the argument used to prove the injectivity of ϕ, a morphism of log schemes from $Y_1^{\log}$ to $Y_2^{\log}$ is given by the underlying morphism of schemes. Therefore, $f : Y_1 \to Y_2$ extends to a morphism $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ of log schemes. We thus conclude that ϕ is surjective.

Remark B.2. In [10], Theorem 3.3, it is only stated that

Let $X^{\log}$ be a log regular log scheme, and $U_X \subseteq X$ the interior of $X^{\log}$. Let $V \to U_X$ be a finite étale morphism which is tamely ramified over the generic points of $X \backslash U_X$. Let Y be the normalization of X in V, and $Y^{\log}$ the log scheme obtained by equipping Y with the log structure $\mathcal{O}_Y \cap (V \hookrightarrow Y)_* \mathcal{O}_V^* \to \mathcal{O}_Y$. Then the following hold.

- *$Y^{\log}$ is log regular.*
- *The finite étale morphism $V \to U_X$ extends uniquely to a log étale morphism $Y^{\log} \to X^{\log}$.*

However, in fact, this log étale morphism $Y^{\log} \to X^{\log}$ is Kummer by the proof of the log purity theorem in *loc. cit.* (More precisely, in the notation of *loc. cit.*, the inclusions $P \subseteq P_Y \subseteq (1/n)P$ imply this fact.) Moreover, since $V \to U_X$ is finite étale, it follows that the normalization $Y \to X$ is finite, i.e., $Y^{\log} \to X^{\log}$ is a ket covering.

We return to the proof of the assertion that the category of ket coverings is Galois.

Proposition B.8. *Let $X^{\log}$ be an fs log scheme, and $f^{\log} : Y^{\log} \to X^{\log}$ a strict étale surjection. Then $f^{\log}$ induces a natural equivalence between the category of ket coverings of $X^{\log}$ and the category of ket coverings of $Y^{\log}$ equipped with descent data with respect to $f^{\log}$.*

Proof. This follows immediately from the fact that the property of being a ket covering is étale local, together with [17], Proposition 4.4.

Proposition B.9. *Let $X^{\log}$ and $Y^{\log}$ be fine log schemes, and $f^{\log}, g^{\log} : X^{\log} \to Y^{\log}$ morphisms of log schemes such that $f = g$. Then if there exist a fine log scheme $X'^{\log}$, a morphism $h^{\log} : X'^{\log} \to X^{\log}$, and a geometric point $\bar{x}' \to X'$ (we denote the image by $x' \in X'$) such that the following conditions hold, then $f^{\log}$ coincides with $g^{\log}$ on an étale neighborhood of the geometric point $\bar{x} \to X$ determined by the geometric point $\bar{x}' \to X'$.*

(i) *h is flat at $x' \in X'$.*
(ii) *The homomorphism $(\mathscr{M}_X/\mathscr{O}_X^*)_{\bar{x}} \to (\mathscr{M}_{X'}/\mathscr{O}_{X'}^*)_{\bar{x}'}$ induced by $h^{\log}$ is injective.*
(iii) *$f^{\log} \circ h^{\log}$ coincides with $g^{\log} \circ h^{\log}$ on an étale neighborhood of $\bar{x}' \to X'$.*

Proof. We denote by $\bar{y} \to Y$ the geometric point determined by the composite $\bar{x} \longrightarrow X \overset{f=g}{\longrightarrow} Y$. Then the fact that the log structures are *fine* implies that it is enough to show that the homomorphism $\mathscr{M}_{Y,\bar{y}} \to \mathscr{M}_{X,\bar{x}}$ induced by $f^{\log}$ coincides with the homomorphism $\mathscr{M}_{Y,\bar{y}} \to \mathscr{M}_{X,\bar{x}}$ induced by $g^{\log}$. Now, in the diagram induced by $h^{\log}$

$$\begin{array}{ccccc}
\mathscr{O}_{X,\bar{x}}^* & \longrightarrow & \mathscr{M}_{X,\bar{x}} & \longrightarrow & (\mathscr{M}_X/\mathscr{O}_X^*)_{\bar{x}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}_{X',\bar{x}'}^* & \longrightarrow & \mathscr{M}_{X',\bar{x}'} & \longrightarrow & (\mathscr{M}_{X'}/\mathscr{O}_{X'}^*)_{\bar{x}'},
\end{array}$$

since the left-hand vertical arrow is injective (by assumption (i)), and the right-hand vertical arrow is injective (by assumption (ii)), we conclude that the homomorphism $\mathscr{M}_{X,\bar{x}} \to \mathscr{M}_{X',\bar{x}'}$ is injective. Therefore, by assumption (iii), the homomorphism $\mathscr{M}_{Y,\bar{y}} \to \mathscr{M}_{X,\bar{x}}$ induced by $f^{\log}$ coincides with the homomorphism $\mathscr{M}_{Y,\bar{y}} \to \mathscr{M}_{X,\bar{x}}$ induced by $g^{\log}$.

Proposition B.10. *A strict étale surjection is a strict epimorphism in the category of fine log schemes.*

Proof. Let $X^{\log}$, $Y^{\log}$, and $Z^{\log}$ be fine log schemes, $f^{\log} : Y^{\log} \to X^{\log}$ a strict étale surjection, and $p_1^{\log}$ (respectively, $p_2^{\log}$) the 1st (respectively, 2nd) projection $Y^{\log} \times_{X^{\log}} Y^{\log} \to Y^{\log}$. Note that our claims are

(i) the morphism $\mathrm{Hom}(X^{\log}, Z^{\log}) \to \mathrm{Hom}(Y^{\log}, Z^{\log})$ induced by $f^{\log}$ is injective; and
(ii) if a morphism $g^{\log} : Y^{\log} \to Z^{\log}$ satisfies the equality $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, then $g^{\log}$ descends to a morphism $X^{\log} \to Z^{\log}$.

Assertion (i) follows immediately from Proposition B.9. Assertion (ii) may be verified as follows. Since $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, we obtain that $g \circ p_1 = g \circ p_2$. Since a surjective étale morphism is a strict epimorphism in the category of schemes, it thus follows that there exists a morphism $\tilde{g} : X \to Z$ such that $\tilde{g} \circ f = g$. Moreover, since $\mathscr{M}_X$ is *a sheaf on the étale site of X*, and $Y^{\log} \to X^{\log}$ strict étale surjection, it thus follows from the fact that the morphism $(g \circ p_1)^{-1}\mathscr{M}_Z \to \mathscr{M}$ (where $\mathscr{M}$ is the sheaf of monoids which determines the log structure of $Y^{\log} \times_{X^{\log}} Y^{\log}$) coincides with the morphism $(g \circ p_2)^{-1}\mathscr{M}_Z \to \mathscr{M}$ that the morphism $g^{-1}\mathscr{M}_Z \to \mathscr{M}_Y$ descends to a morphism $\tilde{g}^{-1}\mathscr{M}_Z \to \mathscr{M}_X$. This completes the proof of assertion (ii).

Proposition B.11. *Let $X^{\log}$ be an fs log scheme. Then, for a morphism $f^{\log}$ in the category of ket coverings of $X^{\log}$, $f^{\log}$ is a strict epimorphism in the category of ket coverings of $X^{\log}$ if and only if $f^{\log}$ is a surjection.*

Proof. Let $Y_1^{\log} \to X^{\log}$, $Y_2^{\log} \to X^{\log}$, and $Z^{\log} \to X^{\log}$ be ket coverings, and $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ a morphism over $X^{\log}$. Now it is immediate that if $f^{\log}$ is not surjective, then $f^{\log}$ is not a strict epimorphism. Thus, assume that $f^{\log}$ is surjective. Let $p_1^{\log}$ (respectively, $p_2^{\log}$) be the 1st (respectively, 2nd) projection $Y_1^{\log} \times_{Y_2^{\log}} Y_1^{\log} \to Y_1^{\log}$. Note that our claims are

(i) the morphism $\mathrm{Hom}_{X^{\log}}(Y_2^{\log}, Z^{\log}) \to \mathrm{Hom}_{X^{\log}}(Y_1^{\log}, Z^{\log})$ induced by $f^{\log}$ is injective;

(ii) if a morphism $g^{\log} : Y_1^{\log} \to Z^{\log}$ satisfies the equality $g^{\log} \circ p_1^{\log} = g^{\log} \circ p_2^{\log}$, then $g^{\log}$ extends to a morphism $Y_2^{\log} \to Z^{\log}$.

First, we prove assertion (i). Let $g_1^{\log}$ and $g_2^{\log} : Y_2^{\log} \to Z^{\log}$ be morphisms over $X^{\log}$ such that $g_1^{\log} \circ f^{\log} = g_2^{\log} \circ f^{\log}$. Then, by Proposition B.5, together with the definition of Galois categories, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g_1'^{\log}$ obtained by base-changing of $g_1^{\log}$ by $X'^{\log} \to X^{\log}$ coincides with the morphism $g_2'^{\log}$ obtained by base-changing of $g_2^{\log}$ by $X'^{\log} \to X^{\log}$. On the other hand, since a strict étale surjection is a strict epimorphism by Proposition B.10, we conclude that $g_1^{\log} = g_2^{\log}$. This completes the proof of assertion (i).

Next, we prove assertion (ii). By Proposition B.5, together with the definition of Galois categories, there exists a strict étale surjection $X'^{\log} \to X^{\log}$ such that the morphism $g'^{\log}$ obtained by base-changing of $g^{\log}$ by $X'^{\log} \to X^{\log}$ extends to a morphism $\widetilde{g'}^{\log} : Y_2'^{\log} \stackrel{\mathrm{def}}{=} Y_2^{\log} \times_{X^{\log}} X'^{\log} \to Z'^{\log} \stackrel{\mathrm{def}}{=} Z^{\log} \times_{X^{\log}} X'^{\log}$. Now if we denote by $q_1^{\log}$ (respectively, $q_2^{\log}$) the 1st (respectively, 2nd) projection $Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \to Y_2'^{\log}$, then the composite

$$Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \xrightarrow{q_1^{\log}} Y_2'^{\log} \xrightarrow{\widetilde{g'}^{\log}} Z'^{\log} \longrightarrow Z^{\log}$$

coincides with the composite

$$Y_2'^{\log} \times_{Y_2^{\log}} Y_2'^{\log} \xrightarrow{q_2^{\log}} Y_2'^{\log} \xrightarrow{\widetilde{g'}^{\log}} Z'^{\log} \longrightarrow Z^{\log}.$$

Therefore, by Proposition B.10, the composite $Y_2'^{\log} \xrightarrow{\widetilde{g'}^{\log}} Z'^{\log} \longrightarrow Z^{\log}$ extends to a morphism $\tilde{g}^{\log} : Y_2^{\log} \to Z^{\log}$. (Note that $Y_2'^{\log} \to Y_2^{\log}$ is a strict étale surjection.) This completes the proof of assertion (ii).

Definition B.5. Let $\mathscr{C}$ be a category with fiber products and finite sums, A_1 and A_2 classes of morphisms in $\mathscr{C}$, X an object of $\mathscr{C}$, Y and Q objects of $\mathscr{C}$ over X, G a finite group of automorphisms of Y over X, and $f : Y \to Q$ a G-equivariant morphism in $\mathscr{C}$ over X with respect to the action of G on Y and the trivial action of G on Q.

(i) We shall say that f is a *quotient in* $\mathscr{C}$ *via the action of* G *over* X *for* A_1 *which is universal for* A_2 if for any morphism $Z \to X$ belonging to A_2, any morphism $W \to Z$ belonging to A_1, and any G-equivariant morphism $Y \times_X Z \to W$ over Z with respect to the natural action of G on $Y \times_X Z$ and the trivial action of G on W, there exists a unique morphism $Q \times_X Z \to W$ over Z such that the morphism $Y \times_X Z \to W$ factors as the composite $Y \times_X Z \to Q \times_X Z \to W$.

(ii) If A_2 consists of the identity morphism of X (respectively, all morphisms in $\mathscr{C}$), then we shall refer to a quotient in $\mathscr{C}$ via the action of G over X for A_1 which is universal for A_2 as a *quotient* (respectively, *universal quotient*) *in* $\mathscr{C}$ *via the action of* G *over* X *for* A_1. Moreover, if A_1 consists of all morphisms in $\mathscr{C}$, and A_2 consists of the identity morphism of X, then we shall refer to a quotient in $\mathscr{C}$ via G over X for A_1 which is universal for A_2 as a *quotient in* $\mathscr{C}$ *via the action of* G.

(iii) We shall say that f is *Galois with Galois group* G if f is an epimorphism, and the top horizontal arrow in the commutative diagram

$$\begin{array}{ccc} \coprod_G Y & \xrightarrow{\coprod_{g \in G}(g, \mathrm{id})} & Y \times_Q Y \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ Y & = & Y, \end{array}$$

where the right-hand vertical arrow is the 2nd projection, is an isomorphism.

Remark B.3. Note that it is immediate that if f is Galois with Galois group G, then the action of G on Y is faithful. Moreover, it is also immediate that if $\mathscr{C}$ is a Galois category, then f is Galois with Galois group G in the classical sense (i.e., the action of G is faithful, and f is a quotient in $\mathscr{C}$ via the action of G) if and only if f is Galois with Galois group G in the sense of Definition B.5, (iii).

Proposition B.12. *Let* $X^{\log}$ *be a connected fs log scheme, and* $Y^{\log}$ *a ket covering of* $X^{\log}$ *equipped with an action over* $X^{\log}$ *of a finite group* G.

(i) *Let* $\tilde{x}^{\log} \to X^{\log}$ *be a log geometric point of* $X^{\log}$. *Then the action of* G *on* $Y^{\log}$ *is faithful if and only if the natural action of* G *on* $F_{\tilde{x}^{\log}}(Y^{\log})$ *is faithful.*

(ii) *Let* $Q^{\log} \to X^{\log}$ *be a ket covering of* $X^{\log}$ *equipped with the trivial action of* G, *and* $f^{\log} : Y^{\log} \to Q^{\log}$ *a* G-*equivariant morphism over* $X^{\log}$. *Assume that the action of* G *on* $Y^{\log}$ *is faithful. Then* $f^{\log}$ *is*

Galois with Galois group G if and only if $f^{\log}$ is a universal quotient in the category of fs log schemes over $X^{\log}$ via the action of G over $X^{\log}$ for ket coverings, i.e., for any morphism $Z^{\log} \to X^{\log}$ of fs log schemes, any ket covering $W^{\log} \to Z^{\log}$, and any G-equivariant morphism $Y^{\log} \times_{X^{\log}} Z^{\log} \to W^{\log}$ over $Z^{\log}$ with respect to the trivial action on $W^{\log}$, there exists a unique morphism $Q^{\log} \times_{X^{\log}} Z^{\log} \to W^{\log}$ over $Z^{\log}$ such that the morphism $Y^{\log} \times_{X^{\log}} Z^{\log} \to W^{\log}$ factors as the composite $Y^{\log} \times_{X^{\log}} Z^{\log} \to Q^{\log} \times_{X^{\log}} Z^{\log} \to W^{\log}$.

(iii) *There exists a ket covering $Q^{\log} \to X^{\log}$ of $X^{\log}$ and a morphism $Y^{\log} \to Q^{\log}$ over $X^{\log}$ such that the morphism $Y^{\log} \to Q^{\log}$ is a universal quotient in the category of fs log schemes over $X^{\log}$ via the action of G over $X^{\log}$ for ket coverings.*

Proof. First, we prove assertion (i). The "if part" of the assertion is immediate; thus, we prove the "only if part" of the assertion. Let $g_0 \in G$. Then it is enough to show that if the action of g_0 on $Y^{\log}$ is not trivial, then the action of g_0 on $F_{\tilde{x}^{\log}}(Y^{\log})$ is not trivial. By replacing G by the subgroup of G generated by g_0, we may assume that G is generated by g_0. Let $N \subseteq G$ be the kernel of the composite $G \to \mathrm{Aut}(Y^{\log}) \to \mathrm{Aut}(\pi_0(Y^{\log}))$, where $\pi_0(Y^{\log})$ is the set of the connected components of $Y^{\log}$. Then it is immediate that if $g_0 \notin N$, then the action of g_0 on $F_{\tilde{x}^{\log}}(Y^{\log})$ is not trivial; thus, we may assume that $g_0 \in N$. Moreover, since the action of g_0 on $Y^{\log}$ is not trivial, there exists a connected component of $Y^{\log}$ on which the action of g_0 is not trivial. By taking such a connected component, we may assume that $Y^{\log}$ is *connected*. Let $\tilde{y}^{\log} \to Y^{\log}$ be a log geometric point of $Y^{\log}$ which belongs to $F_{\tilde{x}^{\log}}(Y^{\log})$. Then it is immediate that there exists a natural G-equivariant isomorphism $F_{\tilde{y}^{\log}}(Y^{\log} \times_{X^{\log}} Y^{\log}) \xrightarrow{\sim} F_{\tilde{x}^{\log}}(Y^{\log})$ with respect to the action of G on $F_{\tilde{y}^{\log}}(Y^{\log} \times_{X^{\log}} Y^{\log})$ induced by the action of G on the 1st factor of $Y^{\log} \times_{X^{\log}} Y^{\log}$ and the natural action of G on $F_{\tilde{x}^{\log}}(Y^{\log})$. On the other hand, it follows from Propositions B.3 and B.4 that the top horizontal arrow $\phi^{\log}$ in the commutative diagram

$$\begin{array}{ccc} \coprod_G Y^{\log} & \xrightarrow{\phi^{\log} \stackrel{\mathrm{def}}{=} \coprod_{g \in G}(g, \mathrm{id})} & Y^{\log} \times_{X^{\log}} Y^{\log} \\ \downarrow & & \downarrow p_2 \\ Y^{\log} & = & Y^{\log}, \end{array}$$

induces an isomorphism $Y^{\log} \times_{X^{\log}} Y^{\log} \simeq \mathrm{Im}(\phi^{\log}) \sqcup Z^{\log}$ over $Y^{\log}$, where $\mathrm{Im}(\phi^{\log}) \subseteq Y^{\log} \times_{X^{\log}} Y^{\log}$ is the open and closed log subscheme of $Y^{\log} \times_{X^{\log}} Y^{\log}$ obtained as the image of $\phi^{\log}$, and $Z^{\log} \to Y^{\log}$ is a ket covering of $Y^{\log}$. Now I *claim* that $\phi^{\log}$ induces an isomorphism $\coprod_G Y^{\log} \xrightarrow{\sim} \mathrm{Im}(\phi^{\log})$. Note that it follows from this *claim* that the action of G on $F_{\tilde{x}^{\log}}(Y^{\log}) \xleftarrow{\sim} F_{\tilde{y}^{\log}}(Y^{\log} \times_{X^{\log}} Y^{\log}) \simeq \coprod_G F_{\tilde{y}^{\log}}(Y^{\log}) \sqcup F_{\tilde{y}^{\log}}(Z^{\log})$ is *faithful*.

The *claim* of the preceding paragraph may be verified as follows. Let $\psi^{\log} : Y^{\log} \sqcup Y^{\log} \to Y^{\log} \times_{X^{\log}} Y^{\log}$ be the morphism over $Y^{\log}$ induced by $(\mathrm{id}, \mathrm{id}) : Y^{\log} = Y^{\log} \sqcup \varnothing \to Y^{\log} \times_{X^{\log}} Y^{\log}$ and $(g_0, \mathrm{id}) : Y^{\log} = \varnothing \sqcup Y^{\log} \to Y^{\log} \times_{X^{\log}} Y^{\log}$. Then to prove the above *claim*, it is easily verified that it is enough to show that the morphism $\psi^{\log}$ induces an isomorphism $Y^{\log} \sqcup Y^{\log} \xrightarrow{\sim} \mathrm{Im}(\psi^{\log})$. Now it follows from Propositions B.3 and B.4, the surjectivity of $Y^{\log} \sqcup Y^{\log} \to \mathrm{Im}(\psi^{\log})$, together with the connectedness of $Y^{\log}$, that if the morphism $Y^{\log} \sqcup Y^{\log} \to \mathrm{Im}(\psi^{\log})$ induced by $\psi^{\log}$ is *not* an isomorphism, then the structure morphism $\mathrm{Im}(\psi^{\log}) \to Y^{\log}$ is an isomorphism. Assume that the structure morphism $\mathrm{Im}(\psi^{\log}) \to Y^{\log}$ is an isomorphism. Then by Proposition B.2, it is verified that for any geometric point $\bar{x} \to X$, there exists an étale neighborhood $U \to X$ of $\bar{x} \to X$ such that the action of g_0 on $Y^{\log} \times_{X^{\log}} U^{\log}$ is trivial, where $U^{\log}$ is the log scheme obtained by equipping U with the log structure induced by the log structure of $X^{\log}$; thus, it follows that the action of g_0 on $Y^{\log}$ is trivial. Therefore, we obtain a contradiction. This completes the proof of assertion (i).

Next, we prove assertion (ii). First, assume that $f^{\log}$ is a universal quotient in the category of fs log schemes over $X^{\log}$ via the action of G over $X^{\log}$ for ket coverings. Then it follows from Propositions B.5, B.11, and Remark B.3, together with assertion (i), that there exists a strict étale surjection $U^{\log} \to X^{\log}$ such that $Y^{\log} \times_{X^{\log}} U^{\log} \to Q^{\log} \times_{X^{\log}} U^{\log}$ is Galois with Galois group G; thus, it follows that $f^{\log}$ is also Galois with Galois group G. Next, assume that $f^{\log}$ is Galois with Galois group G. Since Galoisness is stable under base-change by the definition of Galoisness, together with Proposition B.11, to prove that $f^{\log}$ is a universal quotient in the category of fs log schemes over $X^{\log}$ via the action of G over $X^{\log}$ for ket coverings, by base-changing, it is enough to show that for any ket covering $W^{\log} \to X^{\log}$ of $X^{\log}$ and any G-equivariant morphism $g^{\log} : Y^{\log} \to W^{\log}$ over $X^{\log}$ with respect to the action of G on $Y^{\log}$ and the trivial action of G on $W^{\log}$, there exists a unique G-equivariant morphism $h^{\log} : Q^{\log} \to W^{\log}$ over $X^{\log}$ such that $h^{\log} \circ f^{\log} = g^{\log}$. Then it follows from Proposition B.5, Remark B.3, together with assertion (i), that there exist a strict étale surjection $U^{\log} \to X^{\log}$ and a G-equivariant morphism $\tilde{h}^{\log} : Q^{\log} \times_{X^{\log}} U^{\log} \to W^{\log} \times_{X^{\log}} U^{\log}$ over $U^{\log}$ such that $\tilde{h}^{\log} \circ (f^{\log} \times \mathrm{id}_{U^{\log}}) = g^{\log} \times \mathrm{id}_{U^{\log}}$; moreover, since a morphism which is Galois with Galois group G is an *epimorphism* by the definition of Galoisness, such a morphism "$\tilde{h}^{\log}$" is unique. Therefore, it follows from Proposition B.10 that there exists a unique G-equivariant morphism $h^{\log} : Q^{\log} \to W^{\log}$ over $X^{\log}$ of the desired type.

Finally, we prove assertion (iii). To prove assertion (iii), it is immediate that we may assume that the action of G on $Y^{\log}$ is faithful. Then it follows from Propositions B.5, B.11, and Remark B.3, together with assertion (i), that there exists a strict étale surjection $U^{\log} \to X^{\log}$ and a ket covering $Y^{\log} \times_{X^{\log}} U^{\log} \to \tilde{Q}^{\log}$ over $U^{\log}$ which is Galois with Galois group G. Then it follows from assertion (ii) that this ket covering $Y^{\log} \times_{X^{\log}} U^{\log} \to \tilde{Q}^{\log}$ is a universal quotient in the category of fs log schemes over $U^{\log}$ via the action of G over $U^{\log}$ for ket coverings. Thus, by the definition of universal quotients, assertion (ii), together with Proposition B.8, there exists a ket covering $Y^{\log} \to Q^{\log}$ over $X^{\log}$ such that $Q^{\log} \times_{X^{\log}} U^{\log}$ is isomorphic to $\tilde{Q}^{\log}$ over $U^{\log}$; in particular, this ket covering $Y^{\log} \to Q^{\log}$ is Galois with Galois group G. Therefore, it follows from assertion (ii) that $Y^{\log} \to Q^{\log}$ is a universal quotient in the category of fs log schemes over $X^{\log}$ via the action of G over $X^{\log}$ for ket coverings.

Theorem B.1. *Let $X^{\log}$ be a connected fs log scheme, and $\tilde{x}^{\log} \to X^{\log}$ a log geometric point of $X^{\log}$. Then the pair* (Két($X^{\log}$), $F \stackrel{\text{def}}{=} F_{\tilde{x}^{\log}}$) *forms a Galois category with a fundamental functor.*

We must verify that (Két($X^{\log}$), F) satisfies the conditions $(\mathcal{G}_1), \ldots, (\mathcal{G}_5)$, and $(\mathcal{G}_6)$ in the definition of Galois categories in [5], Exposé V, 4.

$(\mathcal{G}_1)$ Két($X^{\log}$) has a final object and there exist fiber products in Két($X^{\log}$).

Proof. It is immediate that $X^{\log} \xrightarrow{\mathrm{id}_{X^{\log}}} X^{\log}$ is a final object of Két($X^{\log}$). Next, we prove the existence of fiber products. Since any object $Y^{\log}$ of Két($X^{\log}$) is an fs log scheme, for the existence of fiber products, by Proposition B.4, it is enough to show that finiteness, log étaleness, and Kummerness are stable under composition and base-change. The assertion for finiteness is classical; moreover, the assertion for log étaleness and Kummerness follows immediately from definitions.

$(\mathcal{G}_2)$ There exist finite sums in Két($X^{\log}$). Moreover, if $f^{\log} : Y^{\log} \to X^{\log}$ is an object of Két($X^{\log}$) and G is a finite group of automorphisms of $Y^{\log}$ in Két($X^{\log}$), then there exists a quotient $Y^{\log}/G$ of $Y^{\log}$ by G in Két($X^{\log}$).

Proof. The existence of finite sums (respectively, quotients) follows immediately from the definition of a ket covering (respectively, Proposition B.12, (iii)).

$(\mathcal{G}_3)$ Any morphism $f^{\log} : Y_1^{\log} \to Y_2^{\log}$ in Két($X^{\log}$) admits a factorization $Y_1^{\log} \xrightarrow{f'^{\log}} Y_2'^{\log} \xrightarrow{g^{\log}} Y_2^{\log}$, where $f'^{\log}$ is a strict epimorphism and $g^{\log}$ is a monomorphism. Moreover, then $Y_2^{\log} = Y_2'^{\log} \sqcup Z^{\log}$ (disjoint union) for some object $Z^{\log}$ of Két($X^{\log}$).

PROOF. This follows immediately from Propositions B.3 and B.11.

($\mathscr{G}_4$) F is left exact.

PROOF. This follows immediately from Proposition B.5, together with the definition of Galois categories.

($\mathscr{G}_5$) F commutes with the operation of taking a finite sum and the quotient by an action of a finite group (cf. ($\mathscr{G}_2$)). Moreover, if $f^{\log}$ is a strict epimorphism, then $F_{\tilde{x}^{\log}}(f^{\log})$ is surjective.

PROOF. The assertion for finite sums is immediate. The assertion for quotients follows from the fact that a quotient in $\mathrm{K\acute{e}t}(X^{\log})$ is universal (cf. Proposition B.12, (iii)), together with Proposition B.5 and the definition of Galois categories. The assertion for strict epimorphisms follows from Proposition B.11 and the definition of a log geometric point.

($\mathscr{G}_6$) If $f^{\log}$ is a morphism in $\mathrm{K\acute{e}t}(X^{\log})$, then $f^{\log}$ is an isomorphism if and only if $F_{\tilde{x}^{\log}}(f^{\log})$ is an isomorphism.

PROOF. The "only if part" of the assertion is immediate; thus, we prove the "if part" of the assertion. To prove this assertion, it is immediate that it is enough to show the following *assertion*.

Let $f^{\log} : Y^{\log} \to X^{\log}$ be a ket covering such that $F_{\tilde{x}^{\log}}(Y^{\log})$ is of cardinality 1, *then $f^{\log}$ is an isomorphism.*

Moreover, let $\tilde{y}^{\log} \in F_{\tilde{x}^{\log}}(Y^{\log})$, and $Y_1^{\log}$ the connected component of $Y^{\log}$ in which the image of the underlying morphism of schemes of $\tilde{y}^{\log}$ lies. Then since surjective ket coverings are strict epimorphisms by Proposition B.11, and there exists a natural bijection $F_{\tilde{y}^{\log}}(Y^{\log} \times_{X^{\log}} Y_1^{\log}(\xrightarrow{p_2} Y_1^{\log})) \xrightarrow{\sim} F_{\tilde{x}^{\log}}(Y^{\log})$, by replacing $Y^{\log}$ (respectively, $X^{\log}$; respectively, $f^{\log}$) by $Y^{\log} \times_{X^{\log}} Y_1^{\log}$ (respectively, $Y_1^{\log}$; respectively, the 2nd projection), we may assume that the ket covering $f^{\log}$ in the statement of the above *assertion* admits a section.

Then it follows from Propositions B.3 and B.4 that the section $X^{\log} \to Y^{\log}$ of $f^{\log}$ induces an isomorphism $Y^{\log} \xrightarrow{\sim} X^{\log} \sqcup Z^{\log}$, where $Z^{\log} \to X^{\log}$ is a ket covering of $X^{\log}$. Thus, we obtain a bijection $F_{\tilde{x}^{\log}}(Y^{\log}) \simeq F_{\tilde{x}^{\log}}(X^{\log}) \sqcup F_{\tilde{x}^{\log}}(Z^{\log})$. On the other hand, since $F_{\tilde{x}^{\log}}(Y^{\log})$ and $F_{\tilde{x}^{\log}}(X^{\log})$ are of cardinality 1, we obtain that $F_{\tilde{x}^{\log}}(Z^{\log})$ is empty; in particular, $Z^{\log}$ is empty by Propositions B.2 and B.3. This completes the proof of the above *assertion*.

THEOREM B.2. *Let $X^{\log}$ and $Y^{\log}$ be connected fs log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then the functor*

$$\begin{aligned} \mathrm{K\acute{e}t}(Y^{\log}) &\xrightarrow{(f^{\log})^*} \mathrm{K\acute{e}t}(X^{\log}) \\ (Y'^{\log} \to Y^{\log}) &\longmapsto (Y'^{\log} \times_{Y^{\log}} X^{\log} \to X^{\log}) \end{aligned}$$

induced by $f^{\log}$ is exact. In particular, (by [5], *Exposé V, Corollaire* 6.2*) for any log geometric point $\tilde{x}^{\log} \to X^{\log}$ of $X^{\log}$, the functor $(f^{\log})^*$ induces a continuous homomorphism*

$$\pi_1(f^{\log}) : \pi_1(X^{\log}, \tilde{x}^{\log}) \to \pi_1(Y^{\log}, f^{\log}(\tilde{x}^{\log})),$$

where $f^{\log}(\tilde{x}^{\log}) \to Y^{\log}$ is the log geometric point obtained as the composite $\tilde{x}^{\log} \longrightarrow X^{\log} \xrightarrow{f^{\log}} Y^{\log}$.

Proof. Let $\tilde{x}^{\log} \to X^{\log}$ be a log geometric point of $X^{\log}$. Then, by [5], Exposé V, Proposition 6.1, it is enough to show that the composite of functors

$$\text{Két}(Y^{\log}) \xrightarrow{(f^{\log})^*} \text{Két}(X^{\log}) \xrightarrow{F_{\tilde{x}^{\log}}} (\text{the category of finite sets})$$

is a fundamental functor on $\text{Két}(Y^{\log})$. Now, by the definitions of $(f^{\log})^*$ and $F_{\tilde{x}^{\log}}$, for any ket covering $Y'^{\log} \to Y^{\log}$, $F_{\tilde{x}^{\log}} \circ (f^{\log})^*(Y'^{\log} \to Y^{\log}) = F_{f^{\log}(\tilde{x}^{\log})}(Y'^{\log} \to Y^{\log})$, i.e., $F_{\tilde{x}^{\log}} \circ (f^{\log})^* = F_{f^{\log}(\tilde{x}^{\log})}$. By Theorem B.1, the functor $F_{f^{\log}(\tilde{x}^{\log})}$ is a fundamental functor on $\text{Két}(Y^{\log})$. This completes the proof of Theorem B.2.

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