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The genera of Galois closure curves for plane quartic curves

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Abstract. Let C be a smooth plane quartic curve defined over a field k and $k(C)$ the rational function field of C. Let π_P be the projection from C to a line ℓ with a center $P \in C$. Then π_P induces an extension of fields; $k(C)/k(\ell)$. Let $\tilde{C}$ be a nonsingular model of the Galois closure of the extension, which we call the Galois closure curve of $k(C)/k(\ell)$. We give an answer to the problem for the genus of the Galois closure curve of quartic curve [2].

1. Introduction

In [2] Miura and Yoshihara defined Galois points and several related notions for plane curves. We recall the definitions briefly. Let k be an algebraically closed field of characteristic zero. We fix k as a ground field of our discussions. Let C be a smooth plane curve of degree d (≥ 4) and $\pi_P : C \to \ell \cong \mathbf{P}^1$ be the projection from C to a line ℓ with a center $P \in C$. Then π_P induces the field extension $\pi_P^* : k(\ell) \cong k(\mathbf{P}^1) \hookrightarrow k(C)$. We put $K = k(C)$. The extension does not depend on ℓ but on P. So we denote the function field $k(\ell)$ by K_P.

Definition 1. The point $P \in \mathbf{P}^2$ is called a Galois point for C if K/K_P is a Galois extension.

Let L_P be the Galois closure for K/K_P and let $\tilde{C}_P$ be the nonsingular model of L_P. Moreover let $\tilde{\pi}_P : \tilde{C}_P \to C$ be the covering map induced by $k(C) \hookrightarrow L_P$. We call $\tilde{C}_P$ the Galois closure curve of $\pi_P : C \to \mathbf{P}^1$ ([3]). Let $g(P)$ denote the genus of $\tilde{C}_P$.

Miura and Yoshihara [2] obtained the following results for plane quartic curves.

Theorem 2. *For any smooth plane quartic curve C and any point $P \in C$, we have that $g(P) = 3, 6, 7, 8, 9,$ or 10. If P is a general point of C, then $g(P) = 10$. On the contrary, $g(P) = 3$ if and only if P is a Galois point.*

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For a point $Q \in C$ we denote by T_Q the tangent line to C at Q. Let $I_Q(C, T_Q)$ denote the intersection number of C and T_Q at Q.

DEFINITION 3. The point $Q \in C$ is called a 1-flex [resp. 2-flex], if $I_Q(C, T_Q) = 3$ [resp. 4].

As we have seen in Theorem 2 we have the examples with $g(P) = 3$ and 10. On the other hand, we can easily find examples with $g(P) = 9$ and 8 as follows.

REMARK 4. *If P satisfies the following condition* (i) *or* (ii)*, then $g(P) \leq 9$.*
(i) *P is a 2-flex.*
(ii) *$\{P\} = (C \cap T_Q) \backslash \{Q\}$, where T_Q is the tangent line to C at a 1-flex Q.*

This is easy to see from [2]. In fact, let C be the quartic Fermat curve $x^4 + y^4 = 1$. Then, for any 2-flex P we have $g(P) = 9$. But if P is not a 2-flex, then $g(P) = 10$. On the other hand, if C is the curve $x^4 - y^4 + x^2 + y^2 + y = 0$ and $P = (0, 0)$, then $g(P) = 8$.

The existence of quartic curves C with a point P such that $g(P) = 6$ or 7 has not been known for several years. In this note we will solve the above problems, *i.e.*, we will show the existence of such curves.

2. Statement of results

Let $(X : Y : Z)$ be the homogeneous coordinates on $\mathbf{P}^2$ and $(x, y) \in \mathbf{A}^2$ be coordinates of the affine part of $\mathbf{P}^2$ where $x = X/Z$, $y = Y/Z$. Let $f(x, y) = 0$ be the defining equation of (the affine part of) C, and suppose $P = (0, 0) \in C$.

THEOREM 5. *Suppose $g(P) = 7$. Then $f(x, y)$ can be expressed as one of the following equations* (a), (b), *or* (c) *by taking suitable projective transformations.*

$$f(x, y) = x(x + ay)(x^2 + bxy + cy^2 + y) + y. \tag{a}$$

$$f(x, y) = x(x^2 + axy + by^2)(cx + dy + 1) + y. \tag{b}$$

$$f(x, y) = (x^2 + axy + 3by^2)\left(cx^2 + exy + \frac{b^2}{3}y^2 + dx + by + 1\right) + y. \tag{c}$$

THEOREM 6. *Suppose $g(P) = 6$. Then $f(x, y)$ can be expressed as the following equation by taking suitable projective transformations.*

$$\begin{aligned} f(x, y) = (x^2 + axy + 3by^2)\Bigg\{ & cx^2 + \left(ac + \frac{2}{3}bd - 3cd - \frac{1}{3}ad^2 + d^3\right)xy \\ & + \frac{b^2}{3}y^2 + dx + by + 1 \Bigg\} + y. \end{aligned} \tag{d}$$

For concrete examples, we have the following.

Example 7. An example of quartic curve with $g(P) = 7$ is given as follows:

$$f(x, y) = x(x^2 - y^2)\left(\frac{1}{3}y + 1\right) + y.$$

Example 8. An example of quartic curve with $g(P) = 6$ is given as follows:

$$f(x, y) = (x^2 - 3y^2)\left(\frac{1}{3}xy + \frac{1}{3}y^2 + x - y + 1\right) + y.$$

By using projective transformations, we can produce many distinct defining equations for one curve. So it seems important to consider the equation such that the number of independent coefficients is the least one. Then we obtain the result.

Corollary 9. *If $g(P) = 6$, then P is not a flex and the number of independent coefficients of such curves is at most four.*

Consider the set

$$S(d) = \{g(P) \mid C \text{ is a smooth curve of degree } d \text{ and } P \in C\}.$$

We have proved $S(4) = \{3, 6, 7, 8, 9, 10\}$, however we have the following assertion.

Proposition 10. *For a quartic curve C, consider the set*

$$S(C) = \{g(P) \mid P \in C\}.$$

Then $S(C)$ cannot be equal to $\{3, 6, 7, 8, 9, 10\}$.

In the next section, we follow the method used in the proofs in [2]. By taking suitable coordinates, we can assume the following conditions, which are used only in this section.

(1) $P = (0, 0)$.

(2) The line $X = 0$ and C meet transversally.

(3) The line $y = 0$ is the tangent line to C at P.

Let ℓ_t be the line $y = tx$. Then we may assume that the projection is defined as $\pi_P(C \cap \ell_t) = t$. In the affine plane $(x, t) \in \mathbf{A}^2$, let $\hat{C}$ be the curve defined by

$$\hat{f}(x, t) = f(x, tx)/x.$$

Let $\psi(t)$ be the discriminant of $\hat{f}(x,t) \in k[t][x]$ with respect to x. Then it is easy to see that the degree of $\psi(t)$ is 10.

Thanks to Lemma 3.2, [2], we have the following lemma.

LEMMA 11. *If $(t-\alpha)^n$ is a factor of $\psi(t)$, then $n = 1$ or 2. Suppose that $\alpha \neq 0$. Then $n = 2$ [resp. $n = 1$] if and only if the line ℓ_α becomes a tangent line to C at a 1-flex [resp. a non-flex]. On the contrary, suppose that $\alpha = 0$. Then, $n = 2$ if and only if P is a 2-flex; $n = 1$ if and only if P is a 1-flex or ℓ_0 is a bitangent line.*

Let a and b be the numbers of simple and double factors of $\psi(t)$ respectively. Then we obtain the following lemma by the Riemann-Hurwitz formula for the covering $\tilde{\pi}_P : \tilde{C}_P \to C$ (Lemma 3.3, [2]).

LEMMA 12. *If P is not a Galois point, then $g(P) = 10 - b$, where $0 \leq b \leq 4$.*

Let $Q_1, \dots, Q_N$ denote the 1-flexes other than P such that the tangent line to C at the point passes through P. By Lemmas 11 and 12, we obtain the following results.

(a) Suppose $g(P) = 7$, then
  (a1) If P is a 2-flex, then $N = 2$.
  (a2) If P is not a 2-flex, then $N = 3$.

(b) Suppose $g(P) = 6$, then
  (b1) If P is a 2-flex, then $N = 3$.
  (b2) If P is not a 2-flex, then $N = 4$.

3. Proofs

PROOF OF THEOREM 5. The conditions (a1) and (a2) do not depend on the choice of coordinate systems. By using these facts, we write down the defining equation $f(x,y)$.

(i) Suppose P is a 2-flex. By taking suitable coordinates, we can assume that

(1′) $P = (0,0)$.
(2′) $Q_1 = (0:1:0)$.
(3′) The line $y = 0$ is the tangent line to C at P.
(4′) The point Q_2 is on the line at infinity.

Then $f(x,y)$ can be expressed as follows:

$$f(x,y) = x(ax+by)(px^2+qxy+ry^2+cy)+dy.$$

By dividing above equation by d, we can express $f(x,y)$ as follows:

$$f(x, y) = x(a'x + b'y)(p'x^2 + q'xy + r'y^2 + c'y) + y.$$

If $a' = 0$ or $p' = 0$, then $f(x, y)$ is reducible. Moreover, if $c' = 0$, then P is a Galois point (Prop. 3.7, [2]). Hence we can assume that $a' = p' = c' = 1$ by using the projective transformation

$$T(X, Y, Z) = ((a'p')^{-1/4}X, (c')^{-3/2}(p'/a')^{3/4}Y, (c')^{1/2}(a'/p')^{1/4}Z).$$

Rewriting the variables, we can express $f(x, y)$ as follows:

$$f(x, y) = x(x + ay)(x^2 + bxy + cy^2 + y) + y. \tag{a}$$

(ii) Suppose P is a 1-flex. By taking suitable coordinates, we can assume that

(1′) $P = (0, 0)$.
(2′) Q_1 is on the line $X = 0$.
(3′) The line $y = 0$ is the tangent line to C at P.
(4′) The points Q_2 and Q_3 are on the line at infinity.

Then $f(x, y)$ can be expressed as follows:

$$f(x, y) = (ax^2 + bxy + cy^2)(px^2 + qxy + ry^2 + dx + ey) + y.$$

By the conditions (2′) and (4′),

$$f(0, y) = cry^4 + cey^3 + y = y(\lambda y + \mu)^3, \qquad c \neq 0.$$

Then we obtain $e = r = 0$, and $f(x, y)$ can be expressed as follows:

$$f(x, y) = x(ax^2 + bxy + cy^2)(px + qy + d) + y.$$

If $a = 0$, then $f(x, y)$ is reducible. Moreover, if $d = 0$ then P is a Galois point. Hence we can assume that $a = d = 1$ by using the projective transformation $T(X, Y, Z) = (a^{-1/4}X, a^{3/4}d^3Y, a^{-1/4}Z/d)$. Rewriting the variables, we can express $f(x, y)$ as follows:

$$f(x, y) = x(x^2 + axy + by^2)(cx + dy + 1) + y. \tag{b}$$

From equation (b), if P is a 1-flex, then Q_1, Q_2, and Q_3 are collinear.

(iii) Suppose P is not a flex. By taking suitable coordinates, we can assume that

(1′) $P = (0, 0)$.
(2′) Q_1 is on the line $X = 0$.
(3′) The line $y = 0$ is the tangent line to C at P.
(4′) The points Q_2 and Q_3 are on the line at infinity.

Then $f(x, y)$ can be expressed as follows:

$$f(x, y) = (ax^2 + bxy + cy^2)(px^2 + qxy + ry^2 + sx + dy + e) + y.$$

If $a = 0$, then $f(x, y)$ is reducible. Moreover we can assume

$$f(0, y) = cry^4 + cdy^3 + cey^2 + y = y(\lambda y + 1)^3, \qquad c \neq 0$$

by the conditions ($2'$) and ($4'$). Since P is not a flex, then $e \neq 0$ and $\lambda \neq 0$. Therefore, we can assume $a = e = 1$ by using the projective transformation $T(X, Y, Z) = (a^{-1/4}X, a^{3/4}e^{3/2}Y, a^{-1/4}e^{-1/2}Z)$. Then $f(x, y)$ can be expressed as follows:

$$f(x, y) = (x^2 + b'xy + c'y^2)(p'x^2 + q'xy + r'y^2 + s'x + d'y + 1) + y.$$

$$f(0, y) = c'r'y^4 + c'd'y^3 + c'y^2 + y = y(\lambda' y + 1)^3, \qquad \lambda' \neq 0.$$

So we obtain $c' = 3\lambda'$, $r' = \lambda'^2/3$, and $d' = \lambda'$. Rewriting the variables, we can express $f(x, y)$ as follows:

$$f(x, y) = (x^2 + axy + 3by^2)\left(cx^2 + exy + \frac{b^2}{3}y^2 + dx + by + 1\right) + y. \qquad \text{(c)}$$

We infer from simple calculations that $g(P) = 7$ for general curves defined by the equations (a), (b), and (c).

We prove the following lemma before the proof of Theorem 6.

LEMMA 13. *If $N = 4$ and three points of $Q_1, \ldots, Q_4$ are collinear, then P is a Galois point.*

PROOF. Suppose Q_1, Q_2, and Q_3 are collinear. By taking suitable coordinates, we can assume that

($1'$) $P = (0, 0)$.

($2'$) Q_4 is on the line $X = 0$.

($3'$) The line $y = 0$ is the tangent line to C at P.

($4'$) The points Q_1, Q_2, and Q_3 are on the line at infinity.

Then $f(x, y)$ can be expressed as follows:

$$f(x, y) = (x^3 + ax^2y + bxy^2 + cy^3)(px + qy + r) + y.$$

By the conditions ($2'$) and ($4'$),

$$f(0, y) = cqy^4 + cry^3 + y = y(\lambda y + \mu)^3, \qquad c \neq 0.$$

Then we obtain $q = r = 0$. Therefore P is a Galois point. $\square$

PROOF OF THEOREM 6. This proof is done similarly as above, but for the sake of completeness we do not omit it. The conditions (b1) and (b2) do not depend on the choice of coordinate systems, too. By using these facts, we give the defining equation $f(x, y)$ of C.

If P is a 2-flex, then C satisfies the following conditions after taking suitable coordinates.

(1′) $P = (0,0)$.

(2′) Q_1 is on the line $X = 0$.

(3′) The line $y = 0$ is the tangent line to C at P.

(4′) The points Q_2 and Q_3 are on the line at infinity.

By the conditions (1′), (3′), and (4′), $f(x, y)$ must be expressed as follows:

$$f(x, y) = (x^2 + axy + by^2)(x^2 + cxy + dy^2 + ey) + y.$$

However we see that $f(x, y)$ does not satisfy the condition (2′) by simple calculations. Hence P is not a 2-flex.

If P is a 1-flex, then C satisfies the same conditions (1′), (2′), (3′), and (4′) as previous argument (ii) in our proof of Theorem 5. By the conditions (1′), (3′), and (4′), $f(x, y)$ must be expressed as follows:

$$f(x, y) = (x^3 + ax^2y + bxy^2 + cy^3)(px + qy + r) + y.$$

Moreover,

$$f(0, y) = cqy^4 + cry^3 + y = y(\lambda y + \mu)^3, \qquad c \neq 0$$

by the condition (2′). Then $q = r = 0$ and P is a Galois point by lemma 13. Hence P is not a 1-flex.

If P is not a flex, then $f(x, y)$ can be expressed as follows:

$$f(x, y) = (x^2 + axy + 3by^2)\left(cx^2 + exy + \frac{b^2}{3}y^2 + dx + by + 1\right) + y$$

by the previous argument (iii) in our proof of Theorem 5. Then the degree of $\psi(t)$ and the number of double factors of $\psi(t)$ become 8 and 3 respectively. Because P is not a flex, there exists a 1-flex Q_4 in $\mathbf{A}^2$ by Lemma 13 such that $y = \alpha x$ is a tangent line to C at Q_4 by Lemma 11. Let u and v be the coordinates with $Q_4 = (u, v)$. Since

$$\hat{f}(x, \alpha) = A(x - u)^3,$$

where A is a rational function of a, b, c, d, and e, we have the following equations:

$$\frac{\partial^2 \hat{f}}{\partial x^2}(u, \alpha) = \frac{\partial \hat{f}}{\partial x}(u, \alpha) = \hat{f}(u, \alpha) = 0.$$

Hence we obtain that

$$e = (3ac + 2bd - 9cd - ad^2 + 3d^3)/3,$$

and we obtain the following equation:

$$f(x,y) = (x^2 + axy + 3by^2)\left\{cx^2 + \left(ac + \frac{2}{3}bd - 3cd - \frac{1}{3}ad^2 + d^3\right)xy \right.$$
$$\left. + \frac{b^2}{3}y^2 + dx + by + 1\right\} + y. \tag{d}$$

It is easy to see from the above equation (d) that the assertion of Corollary 9 hold true.

If C is nonsingular, then $g(P) = 6$. However, there may be curves with singular points depending on the choices of a, b, c, and d. We infer from the defining equations $f(x,y)$ and $\psi(t)$ that the following conditions are necessary and sufficient by long and tedious calculations:

$$\begin{cases} \text{(i)} & a - 3d \neq 0, \quad b \neq 0, \\ \text{(ii)} & b + 9c - 3d^2 \neq 0, \\ \text{(iii)} & b - ad + 3d^2 \neq 0, \\ \text{(iv)} & b + 9c - ad \neq 0, \\ \text{(v)} & a^2 - 12b \neq 0, \quad 3c - d^2 \neq 0, \\ \text{(vi)} & b^2 - 3a^2c + 18bc + 81c^2 - abd + 9acd \\ & + a^2d^2 - 3bd^2 - 54cd^2 - 3ad^2 + 9d^4 \neq 0. \end{cases}$$

So we obtain that

$$(u, v, \alpha) = \left(\frac{-a + 3d}{-b + ad - 3d^2}, \frac{1}{-b + ad - 3d^2}, \frac{1}{-a + 3d}\right).$$

Finally we prove Proposition 10. We put

$$W(C) = \sum_{Q \in C} \{I_Q(C, T_Q) - 2\}.$$

Then we have the following lemma by [1].

LEMMA 14. $W(C) = 24$.

We fix a point $P \in C$. Let $\mathscr{L}$ be the set of lines ℓ passing through P. We put

$$\mathscr{L}' = \begin{cases} \mathscr{L} & \text{if } P \text{ is not a 1-flex.} \\ \mathscr{L} \backslash \{T_P\} & \text{if } P \text{ is a 1-flex.} \end{cases}$$

Now we define

$$I_{C \cap \ell}(C, \ell) = \max_{Q \in C \cap \ell} I_Q(C, \ell)$$

where $I_Q(C, \ell)$ is the intersection number at Q, and

$$W(C,P) = \sum_{\ell \in \mathscr{L}'} \max\{0, I_{C\cap\ell}(C,\ell) - 2\}.$$

Then we see that $\sum_{P\in C} W(C,P) = 24$. We obtain the following results by Theorem 6, Lemma 11 and Proposition 3.7 in [2].

Lemma 15.

$$\begin{cases} W(C,P) = 6 & \textit{if } g(P) = 3. \\ W(C,P) = 4 & \textit{if } g(P) = 6. \\ W(C,P) = 3 \textit{ or } 4 & \textit{if } g(P) = 7. \\ W(C,P) = 2 \textit{ or } 3 & \textit{if } g(P) = 8. \\ W(C,P) = 1 \textit{ or } 2 & \textit{if } g(P) = 9. \\ W(C,P) = 0 & \textit{if } g(P) = 10. \end{cases}$$

If there exist a Galois point in C, then C can be expressed as $f(x,y) = y + g(x,y)$, where $g(x,y)$ is a homogeneous polynomial of degree four with no multiple factor and $g(x,0) \neq 0$ (Prop. 3.7, [2]). Moreover the projective transformation $\Psi(x,y) = (\omega x, \omega y)$ induces an automorphism on C, where ω is a primitive cubic root of unity. Hence, if $Q \neq (0,0) \in C$, then we have

$$g(Q) = g(\Psi(Q)) = g(\Psi(\Psi(Q)))$$

and

$$W(C,Q) = W(C,\Psi(Q)) = W(C,\Psi(\Psi(Q))). \tag{e}$$

Suppose that $S(C) = \{3,6,7,8,9,10\}$. Then, from Lemma 15 and above equation (e), we obtain that

$$\sum_{P\in C} W(C,P) \geq 6 + 3(4+3+2+1) = 36 > W(C).$$

This is a contradiction. Therefore, $S(C)$ is a proper subset of $\{3,6,7,8,9,10\}$ for each quartic curve C. Thus we complete the proofs.

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References

[1] S. Iitaka, "Algebraic Geometry," Graduate Texts in Math., Vol. 76, Springer-Verlag, New York/Heidelberg/Berlin, 1982.

[2] K. Miura and H. Yoshihara, Field theory for function fields of plane quartic curves, J. Algebra **226** (2000), 283–294.
[3] H. Tokunaga, Triple coverings of algebraic surfaces according to the Cardano formula, J. Math. Kyoto Univ. **31** (1991), 359–375.
[4] H. Yoshihara, Function field theory of plane curves by dual curves, J. Algebra **239** (2001), 340–355.

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