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The law of small numbers and the limit theorem for symmetric statistics with mixing conditions

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§1. Introduction

There has been considerable and theoretical interest in how well the Poisson distribution approximates the distribution of the sums of arbitrary indicator (zero-one) variables. Results of this type, either limit theorems or quantitative estimates of the distance to a Poisson distribution, have been shown under various conditions by many authors. Janson [14] gave a sufficient condition (not of mixing type) for convergence to Poisson distribution of a sequence of sums of dependent indicator (zero-one) random variables. Chen [5] gave a general method of obtaining and bounding the error in approximating the distribution of the sums of dependent Bernoulli random variables by the Poisson distribution. Dobrushin and Sukhov [9], gave necessary and sufficient conditions for convergence to a Poisson process of infinite particle systems under the action of free dynamic (see also Willms [22] and Zessin [23]). The other investigations in this direction were conducted within the rapidly developing field of symmetric statistics. Silverman and Brown [20] have obtained Poisson limit theorems for certain sequences of symmetric statistics

$$\sum h_k(X_{i_1}, \ldots, X_{i_k}), \tag{1.1}$$

based on a sample of identically distributed independent random variables $X_1, \ldots, X_n$, where h_k is a symmetric zero-one function and the summation is extended over all sets $\{i_1, \ldots, i_k\}$ of distinct integers drawn from $\{1, \ldots, n\}$. Barbour and Eagleson [2], [3] gave a general Poisson approximation theorem for symmetric statistics (1.1) from a sample of independent but not necessarily identically distributed random variables and with a symmetric zero-one function of k variables.

The Poisson limit theorems in the more general setting of symmetric statistics have been obtained by Mustafid and Kubo [18]. They have obtained the asymptotic distribution of the sums of symmetric statistics

$$\sum_{1 \le s_1 < \cdots < s_k \le n} h_k(X_{n,s_1}, \ldots, X_{n,s_k}), \tag{1.2}$$

in terms of multiple Poisson-Wiener-Ito integrals, where the symmetric statistics (1.2) is based on samples of identically distributed independent random elements $X_{n,1}, \ldots, X_{n,n}$, and h_k is a symmetric function. The results also still hold, even if random elements are not identically distributed, provided that random elements are infinitesimal (see Mustafid [17]). An alternative approach to limiting distribution due to Avram and Taqqu [1] in a simple case when the symmetric statistics (1.2) is symmetric polynomials. They have expressed the asymptotic distribution of symmetric polynomials in terms of a multiple integral with respect to a Lévy process. In the case of central limit theorems, the limiting distributions of symmetric statistics (1.2) have been obtained by several authors. See Dynkin and Mandelbaum [10], Mandelbaum and Taqqu [16], Dehling [6], Dehling, Denker and Philipp [7], Denker and Keller [8] and Teicher [21].

The aim of this paper is the following. First we establish a method of the Poisson approximation for sequences of dependent p-dimensional Bernoulli arrays, while secondly we extend the Poisson limit theorems in [17] to dependent random elements case. In Section 2, we will discuss convergence of Radon measures and a mixing condition of sequences of random elements. The results are stated in Sections 3 and 4.

§2. Dependent random elements

Let $\mathfrak{X}$ be a locally compact second countable Hausdorff space. Let $\mathscr{A}$ denote the topological Borel field in $\mathfrak{X}$ and $\mathscr{B}$ the ring of all bounded (i.e. relatively compact) sets in $\mathscr{A}$. Let $\mathscr{M}(\mathfrak{X})$ be the family of all Radon measures on $(\mathfrak{X}, \mathscr{A})$ with vague topology. For a given $\lambda \in \mathscr{M}(\mathfrak{X})$, a random measure $\{P_\lambda(B) = P_\lambda(\omega, B), B \in \mathscr{B}\}$ is called a *Poisson random measure with intensity* λ if for any natural number p, any disjoint sets $B_1, \ldots, B_p \in \mathscr{B}$ and any non-negative integers $q_1, \ldots, q_p$,

$$\Pr(P_\lambda(B_1) = q_1, \ldots, P_\lambda(B_p) = q_p)$$
$$= \frac{1}{q_1! \cdots q_p!} \lambda(B_1)^{q_1} \cdots \lambda(B_p)^{q_p} \exp\left[-\lambda(B_1) - \cdots - \lambda(B_p)\right].$$

We define a class of bounded sets $\mathscr{B}_\lambda$ by

$$\mathscr{B}_\lambda \equiv \{B \in \mathscr{B}\,; \lambda(\partial B) = 0\},$$

where ∂B denotes the boundary of B.

By a random element X in the space $\mathfrak{X}$ we mean a measurable mapping from some fixed probability space $(\Omega, \mathscr{F}, Pr)$ into $(\mathfrak{X}, \mathscr{A})$. A sequence of random elements X_n converges to X in distribution sense and is denoted as $X_n \xrightarrow{d} X$, if the distribution ν_n of X_n converges weakly to the distribution ν of X as $n \to \infty$.

Let $X_{n,1}, \dots, X_{n,k_n}$ ($1 \le k_n \le \infty$), $n = 1, 2, \dots$, be sequences of dependent random elements on $\mathfrak{X}$ with marginal distributions $\nu_{n,1}, \dots, \nu_{n,k_n} \in \mathcal{M}(\mathfrak{X})$, respectively. Denote by $\mathscr{B}_{ab}^{(n)}$ the σ-algebra of events generated by $\{X_{n,j}; a \le j \le b\}$, $1 \le a \le b \le k_n$. We assume the following:

(A.1) $\lambda_n \equiv \sum_{i=1}^{k_n} \nu_{n,i}$ converges vaguely to a $\lambda \in \mathcal{M}(\mathfrak{X})$ without atoms as $n \to \infty$,

(A.2) $\lim_{n\to\infty} \max_i \nu_{n,i}(K) = 0$ for any compact set K,

(A.3) for any events $A \in \mathscr{B}_{1r}^{(n)}$ and $B \in \mathscr{B}_{r+m,k_n}^{(n)}$, $n \ge 1$,

$$|\Pr(AB) - \Pr(A)\Pr(B)| \le \varphi(m)\Pr(A)\Pr(B),$$

with $\varphi(m) \downarrow 0$ and $\varphi(1) < \infty$.

We denote $\alpha = \varphi(1) + 1$. We refer to Philipp [19] for a detailed treatment of such mixing condition (A.3).

LEMMA 2.1 ([19]). *If the condition* (A.3) *is satisfied, and if X and Y are bounded measurable over $\mathscr{B}_{1r}^{(n)}$ and $\mathscr{B}_{r+m,k_n}^{(n)}$ respectively. Then*

$$|E(XY) - E(X)E(Y)| \le \varphi(m)E|X|E|Y|.$$

LEMMA 2.2. *Suppose that the family of σ-fields $\{\mathscr{B}_{ij}^{(n)}, 1 \le i < j \le k_n\}$ satisfies the mixing condition* (A.3). *Let M_0 be a natural number such that $\varphi(M_0) < 1$. Then there exists a constant ρ such that for any $m > M_0$ and any bounded random variables X, Y, Z measurable over $\mathscr{B}_{ab}^{(n)}$, $\mathscr{B}_{cd}^{(n)}$, $\mathscr{B}_{ef}^{(n)}$ respectively, with $c - b \ge m$, $e - d \ge m$ and $c < d$,*

$$|E(XZY) - E(XZ)E(Y)| \le \rho\varphi(m)E|XZ|E|Y|.$$

PROOF. Let $A \in \mathscr{B}_{ab}^{(n)}$, $B \in \mathscr{B}_{cd}^{(n)}$ and $C \in \mathscr{B}_{ef}^{(n)}$. By (A.3), we have inequalities

$$\Pr(ABC) \le \{1 + \varphi(m)\}\Pr(A)\Pr(BC) \le \{1 + \varphi(m)\}^2\Pr(A)\Pr(B)\Pr(C),$$

$$\{1 - \varphi(m)\}\Pr(A)\Pr(C) \le \Pr(AC).$$

Therefore, we see

$$\Pr(ABC) - \Pr(B)\Pr(AC) \le \left\{3 + \frac{4\varphi(m)}{1 - \varphi(m)}\right\}\varphi(m)\Pr(B)\Pr(AC).$$

Similarly, we see

$$\Pr(ABC) - \Pr(B)\Pr(AC) \ge -\left\{3 + \frac{4\varphi(m)}{1 - \varphi(m)}\right\}\varphi(m)\Pr(B)\Pr(AC).$$

Take $\rho = 3 + \dfrac{4\varphi(M_0)}{1 - \varphi(M_0)}$. Then

$$|\Pr(ABC) - \Pr(B)\Pr(AC)| \le \rho\varphi(m)\Pr(B)\Pr(AC), \qquad \text{for any} \quad m > M_0.$$

Therefore, for any $D \in \mathscr{B}_{ab}^{(n)} \vee \mathscr{B}_{ef}^{(n)}$ and $B \in \mathscr{B}_{cd}^{(n)}$, we have

$$|\Pr(BD) - \Pr(B)\Pr(D)| \leq \rho\varphi(m)\Pr(B)\Pr(D)\,, \qquad \text{for any} \quad m > M_0\,.$$

The assertion of the lemma follows from Lemma 2.1.

§3. Poisson approximation for dependent Bernoulli arrays

In this section, we discuss the case when $\{X_{n,i}\}_{i=1}^{k_n}$ is a sequence of dependent p-dimensional Bernoulli arrays, i.e. for each n and i, $X_{n,i}$ is a random p-vector of the form $X_{n,i} = (X_{n,i}^{(1)}, \ldots, X_{n,i}^{(p)})$, where $X_{n,i}^{(j)} = 0$ or 1 and such that $X_{n,i}^{(j)} = 0$ except for at most one, $1 \leq j \leq p$. In other words,

$$\Pr(X_{n,i}^{(1)} = 0, \ldots, X_{n,i}^{(j-1)} = 0, X_{n,i}^{(j)} = 1, X_{n,i}^{(j+1)} = 0, \ldots, X_{n,i}^{(p)} = 0)$$
$$= \Pr(X_{n,i}^{(j)} = 1) = p_{n,i}^{(j)}\,,$$

where $\sum_{j=1}^{p} p_{n,i}^{(j)} + \Pr(X_{n,i}^{(1)} = 0, \ldots, X_{n,i}^{(p)} = 0) = 1$.

We will prove the convergence of the distribution of $\sum_i X_{n,i}$ to a p-dimensional Poisson distribution under the assumptions:

(A.1)′ $\lim_{n\to\infty} \sum_{i=1}^{k_n} p_{n,i}^{(j)} = \lambda_j$, $j = 1, 2, \ldots, p$,

(A.2)′ $\lim_{n\to\infty} \max_{j,i} p_{n,i}^{(j)} = 0$

and the mixing condition (A.3).

We denote $W_n^{(j)} \equiv \sum_{i=1}^{k_n} X_{n,i}^{(j)}$, $\lambda_{n,j} \equiv \sum_{i=1}^{k_n} p_{n,i}^{(j)}$, $1 \leq j \leq p$ and

$$\Lambda \equiv \sup_{j,n} \sum_{i=1}^{k_n} p_{n,i}^{(j)}\,.$$

We extend Chen's method (cf. [5]) to prove the following lemma.

LEMMA 3.1. *Under the mixing condition* (A.3), *let* M_0 *and* ρ *be as in Lemma* 2.2. *Then for any* $q \leq p$ *and* $m > M_0$, *there exist constants* $C_1(m, p, q)$ *and* $C_2(q)$ *such that*

$$(3.1) \qquad |E\{(W_n^{(1)} \ldots W_n^{(q)} - e^{it_1}\lambda_{n,1} W_n^{(2)} \ldots W_n^{(q)}) \exp[i \textstyle\sum_{j=1}^{p} t_j W_n^{(j)}]\}|$$
$$\leq C_1(m, p, q) \sup_{j,i} p_{n,i}^{(j)} + C_2(q)\varphi(m+1)\,.$$

PROOF. We have

$$(3.2) \quad E\{(W_n^{(1)} W_n^{(2)} \ldots W_n^{(q)} - e^{it_1}\lambda_{n,1} W_n^{(2)} W_n^{(3)} \ldots W_n^{(q)}) \exp[i \textstyle\sum_{j=1}^{p} t_j W_n^{(j)}]\}$$
$$= \left(\sum_{i_1,\ldots,i_q}^{*} + \sum_{i_1,\ldots,i_q}^{**} \right)$$
$$\times (E\{(X_{n,i_1}^{(1)} \ldots X_{n,i_q}^{(q)} - p_{n,i_1}^{(1)} e^{it_1} X_{n,i_2}^{(2)} \ldots X_{n,i_q}^{(q)}) \exp[i \textstyle\sum_{j=1}^{p} t_j W_n^{(j)}]\})\,,$$

where the sum $\sum^*$ is extended over $i_1, \ldots, i_q$ with $|i_k - i_l| > 2m$, for any k and l, $1 \le k$, $l \le q$, $k \ne l$, and the sum $\sum^{**}$ is extended over $i_1, \ldots, i_q$ with $|i_k - i_l| \le 2m$, for some k, l, $1 \le k$, $l \le q$, $k \ne l$.

Let $V_{n,j}^{(i_1,\ldots,i_q)} \equiv \sum' X_{n,k}^{(j)}$, where the sum $\sum'$ is extended over all k with $|k - i_z| > m$, $z = 1, \ldots, q$. Let

$$H(i_1, \ldots, i_q) \equiv \exp\,[i \textstyle\sum_{j=1}^q t_j] \exp\,[i \sum_{j=1}^p t_j V_{n,j}^{(i_1,\ldots,i_q)}]\,.$$

The first sum on the right-hand side of (3.2) can be rewritten as

$$(3.3)\quad \sum_{i_1,\ldots,i_q}^* E\{(X_{n,i_1}^{(1)} \cdots X_{n,i_q}^{(q)} - p_{n,i_1}^{(1)} e^{it_1} X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)}) \exp\,[i \textstyle\sum_{j=1}^p t_j W_n^{(j)}]\}$$

$$= \sum_{i_1,\ldots,i_q}^* E(X_{n,i_1}^{(1)} \cdots X_{n,i_q}^{(q)}$$

$$\times \{\exp\,[i \textstyle\sum_{j=1}^q t_j(\sum_{k\ne i_j} X_{n,k}^{(j)} + 1) + i \sum_{j=q+1}^p t_j W_n^{(j)}] - H(i_1, \ldots, i_q)\})$$

$$- \sum_{i_1,\ldots,i_q}^* p_{n,i_1}^{(1)} E(X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)}$$

$$\times \{\exp\,[it_1(W_n^{(1)} + 1) + i \textstyle\sum_{j=2}^q t_j(\sum_{k\ne i_j} X_{n,k}^{(j)} + 1) + i \sum_{j=q+1}^p t_j W_n^{(j)}]$$

$$- H(i_1, \ldots, i_q)\})$$

$$+ \sum_{i_1,\ldots,i_q}^* E\{(X_{n,i_1}^{(1)} - p_{n,i_1}^{(1)}) X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} H(i_1, \ldots, i_q)\}\,.$$

We know that $|H(i_1, \ldots, i_q)| \le 1$. By Lemma 2.1 and Lemma 2.2, the third sum on the right-hand side of (3.3) can be estimated as

$$\sum_{i_1,\ldots,i_q}^* |E\{(X_{n,i_1}^{(1)} - p_{n,i_1}^{(1)}) X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} H(i_1, \ldots, i_q)\}|$$

$$\le \rho \sum_{i_1,\ldots,i_q}^* E|(X_{n,i_1}^{(1)} - p_{n,i_1}^{(1)})| E|X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)}| \varphi(m+1)$$

$$\le 2\rho \sum_{i_1,\ldots,i_q}^* p_{n,i_1}^{(1)} \cdots p_{n,i_q}^{(q)} \alpha^{q-2} \varphi(m+1)$$

$$\le 2\rho \alpha^{q-2} \Lambda^q \varphi(m+1)\,,$$

for any $m > M_0$.

Take $X_{n,i}^{(j)}$ to be identically zero when $i \le 0$ or $i > k_n$. For $a = 1, \ldots, q$, $s = 1, \ldots, p$, we define

$$Y(a, s, l) \equiv Y(i_1, \ldots, i_q, a; s, l)$$

$$\equiv \exp[i \textstyle\sum_{j=1}^{s-1} t_j(V_{n,j}^{(i_1,\ldots,i_q)} + \sum_{z=1}^a \sum_{k=i_z-m,\, k\ne i_z}^{i_z+m} X_{n,k}^{(j)})$$

$$+ it_s(V_{n,s}^{(i_1,\ldots,i_q)} + \textstyle\sum_{z=1}^{a-1} \sum_{k=i_z-m,\, k\ne i_z}^{i_z+m} X_{n,k}^{(s)} + \sum_{k=i_a-m,\, k\ne i_a}^{l} X_{n,k}^{(s)})$$

$$+ i \textstyle\sum_{j=s+1}^p t_j(V_{n,j}^{(i_1,\ldots,i_q)} + \sum_{z=1}^{a-1} \sum_{k=i_z-m,\, k\ne i_z}^{i_z+m} X_{n,k}^{(j)})]\,,$$

for $i_a - m \le l \le i_a + m$ and

$$Y(a, s, i_a - m - 1) \equiv \exp\Big[i \textstyle\sum_{j=1}^{s-1} t_j\big(V_{n,j}^{(i_1,\dots,i_q)} + \sum_{z=1}^{a} \sum_{\substack{k=i_z-m\\ k\ne i_z}}^{i_z+m} X_{n,k}^{(j)}\big) + i \sum_{j=s}^{p} t_j\big(V_{n,j}^{(i_1,\dots,i_q)} + \sum_{z=1}^{a-1} \sum_{\substack{k=i_z-m\\ k\ne i_z}}^{i_z+m} X_{n,k}^{(j)}\big)\Big],$$

for $l = i_a - m - 1$. We write $Y'(a, s, l)$ and $Y'(a, s, i_a - m - 1)$ for the sums of $X_{n,k}^{(j)}$ over $k \ne i_a$, $a = 2, \dots, q$, in the above definition. Furthermore, we define

$$\Delta Y(a, s, l) \equiv \exp[i \textstyle\sum_{j=1}^{q} t_j]\{Y(a, s, l) - Y(a, s, l-1)\},$$

$$\Delta Y'(a, s, l) \equiv \exp[i \textstyle\sum_{j=1}^{q} t_j]\{Y'(a, s, l) - Y'(a, s, l-1)\}.$$

Since each $X_{n,i}^{(j)}$, $1 \le j \le p$, $1 \le i \le k_n$, is 0 or 1, and $|\Delta Y(a, s, l)|$ and $|\Delta Y'(a, s, l)| \le 2$, (3.3) can be estimated as

$$\Big|\sum_{i_1,\dots,i_q}^{*} E(\{X_{n,i_1}^{(1)} \cdots X_{n,i_q}^{(q)} - p_{n,i_1}^{(1)} e^{it_1} X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)}\} \exp[i \textstyle\sum_{j=1}^{p} t_j W_n^{(j)}])\Big| \tag{3.4}$$

$$= \Big|\sum_{i_1,\dots,i_q}^{*} \sum_{z=1}^{p} \sum_{a=1}^{q} \sum_{\substack{l=i_a-m\\ l\ne i_a}}^{i_a+m} E\{X_{n,i_1}^{(1)} \cdots X_{n,i_q}^{(q)} X_{n,l}^{(z)} \Delta Y(a, z, l)\}$$

$$- \sum_{i_1,\dots,i_q}^{*} \sum_{z=1}^{p} \sum_{a=2}^{q} \sum_{\substack{l=i_a-m\\ l\ne i_a}}^{i_a+m} p_{n,i_1}^{(1)} E\{X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} X_{n,l}^{(z)} \Delta Y'(a, z, l)\}$$

$$+ \sum_{i_1,\dots,i_q}^{*} E\{(X_{n,i_1}^{(1)} - p_{n,i_1}^{(1)}) X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} H(i_1, \dots, i_q)\}\Big|$$

$$\le 2 \sum_{i_1,\dots,i_q}^{*} \sum_{z=1}^{p} \sum_{a=1}^{q} \sum_{\substack{l=i_a-m\\ l\ne i_a}}^{i_a+m} E(X_{n,i_1}^{(1)} \cdots X_{n,i_q}^{(q)} X_{n,l}^{(z)})$$

$$+ 2 \sum_{i_1,\dots,i_q}^{*} \sum_{z=1}^{p} \sum_{a=2}^{q} \sum_{\substack{l=i_a-m\\ l\ne i_a}}^{i_a+m} p_{n,i_1}^{(1)} E(X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} X_{n,l}^{(z)})$$

$$+ \sum_{i_1,\dots,i_q}^{*} |E\{(X_{n,i_1}^{(1)} - p_{n,i_1}^{(1)}) X_{n,i_2}^{(2)} \cdots X_{n,i_q}^{(q)} H(i_1, \dots, i_q)\}|$$

$$\le 2q\alpha^{q-1}(1+\alpha)(2m+1) \sum_{i_1,\dots,i_q}^{*} p_{n,i_1}^{(1)} \cdots p_{n,i_q}^{(q)} \Big\{\sum_{j=1}^{p} \sup_{j,i} p_{n,i}^{(j)}\Big\}$$

$$+ 2\rho\alpha^{q-2}\Lambda^q \varphi(m+1)$$

$$\le 2pq(2m+1)\alpha^{q-1}(\alpha+1)\Lambda^q \sup_{j,i} p_{n,i}^{(j)} + 2\rho\alpha^{q-2}\Lambda^q\varphi(m+1),$$

by using Lemma 2.1. Furthermore, since for $r \ne s$, $1 \le r, s \le p$,

$$EX_{n,i}^{(r)} X_{n,i}^{(s)} = Pr(X_{n,i}^{(r)} = 1, X_{n,i}^{(s)} = 1) = 0,$$

the second sum on the right-hand side of (3.2) can be estimated as

$$(3.5)\quad \left|\sum\nolimits^{**}_{i_1,\dots,i_q} E(\{X^{(1)}_{n,i_1}\cdots X^{(q)}_{n,i_q} - p^{(1)}_{n,i_1}e^{it_1}X^{(2)}_{n,i_2}\cdots X^{(q)}_{n,i_q}\}\exp[i\textstyle\sum_{j=1}^{p} t_j W_n^{(j)}])\right|$$

$$\le \left|\sum\nolimits^{**}_{i_1,\dots,i_q} E\{X^{(1)}_{n,i_1}\cdots X^{(q)}_{n,i_q} - p^{(1)}_{n,i_1}e^{it_1}X^{(2)}_{n,i_2}\cdots X^{(q)}_{n,i_q}\}\right|$$

$$\le \alpha^{q-2}(1+\alpha)\sum\nolimits^{**}_{i_1,\dots,i_q} p^{(1)}_{n,i_1}\cdots p^{(q)}_{n,i_q}$$

$$\le \alpha^{q-2}(1+\alpha)(2m+1)\left\{\lambda_{n,2}\lambda_{n,3}\cdots\lambda_{n,q}\sup_i p^{(1)}_{n,i} + \cdots\right.$$

$$\left. + \lambda_{n,1}\lambda_{n,2}\cdots\lambda_{n,q-1}\sup_i p^{(q)}_{n,i}\right\}$$

$$\le \alpha^{q-2}(1+\alpha)\frac{q(q-1)}{2}(2m+1)\Lambda^{q-1}\sup_{j,i} p^{(j)}_{n,i}\,,$$

again by Lemma 2.1. Summarizing (3.4) and (3.5) in $\sup_{j,i} p^{(j)}_{n,i}$ and $\varphi(m+1)$, we have (3.1), and the proof of lemma is completed.

THEOREM 3.2. *Under the assumptions* (A.1)′, (A.2)′ *and* (A.3),

$$(3.6)\qquad (\textstyle\sum_{i=1}^{k_n} X^{(1)}_{n,i}, \dots, \sum_{i=1}^{k_n} X^{(p)}_{n,i}) \xrightarrow{d} (P_{\lambda_1}, \dots, P_{\lambda_p})$$

as $n \to \infty$, *where* P_{λ_j} *are independent Poisson distributed random variables with means* λ_j, $j = 1, \dots, p$, *respectively.*

PROOF. The proof of the theorem bases on Lemma 3.1. To prove (3.6), we shall show the convergence of the characteristic functions of $\{W_n^{(j)}\}$;

$$(3.7)\qquad E\exp[i\textstyle\sum_{j=1}^{p} t_j W_n^{(j)}] \to \exp[\sum_{j=1}^{p}\lambda_j(e^{it_j}-1)] \qquad \text{as } n\to\infty\,,$$

by induction. From the assumptions (A.2)′ and (A.3), for any $\varepsilon > 0$, there exists an $m_0 > M_0$ such that

$$C_2(q)\varphi(m_0+1) < \varepsilon/2\,, \qquad \text{for any } 1 \le q \le p\,,$$

and for the fixed m_0, we can choose N_ε such that

$$C_1(m_0, r, q)\max_{j,i} p^{(j)}_{n,i} < \varepsilon/2\,, \qquad \text{for any } n \ge N_\varepsilon \text{ and any } r, q,\ 1 \le q \le r \le p\,.$$

Then we have

$$(3.8)\qquad C_1(m_0, r, q)\max_{j,i} p^{(j)}_{n,i} + C_2(q)\varphi(m_0+1) < \varepsilon\,,$$

for any $n \ge N_\varepsilon$ and any r, q, $1 \le q \le r \le p$.

First, we shall prove for $p = 1$, i.e.,

$$E\exp[itW_n^{(1)}] \to \exp[\lambda_1(e^{it} - 1)] \tag{3.9}$$

as $n \to \infty$. By Lemma 3.1 for $p = q = 1$ and (3.8), we have

$$\begin{aligned}&\left|\frac{d}{dt}E\{\exp[itW_n^{(1)}]\}\exp[\lambda_{n,1}(1 - e^{it})]\right| \\ &\quad = |E\{(W_n^{(1)} - \lambda_{n,1}e^{it})\exp[itW_n^{(1)}]\}i\exp[\lambda_{n,1}(1 - e^{it})]| \\ &\quad \le C_1(m, 1, 1)\max_i p_{n,i}^{(1)} + C_2(1)\varphi(m + 1) < \varepsilon .\end{aligned}$$

Whence by integration

$$\lim_{n\to\infty} E\{\exp[itW_n^{(1)}]\}\exp[\lambda_{n,1}(1 - e^{it})] - 1 = 0 . \tag{3.10}$$

Consequently, by (A.1)′, we have (3.9).

Furthermore, we assume that (3.7) is valid for $p - 1$. Then

$$E\{\exp[i\textstyle\sum_{j=1}^{p-1} st_j W_n^{(j)}]\}\exp[\sum_{j=1}^{p-1}\lambda_j(1 - e^{ist_j})] \to 1 \tag{3.11}$$

as $n \to \infty$, for any s and t_j, $j = 1, \ldots, p - 1$. Let

$$\varphi_n(t, s) \equiv E\{\exp[itW_n^{(p)} + is\textstyle\sum_{j=1}^{p-1} t_j W_n^{(j)}]\} .$$

Note that we can apply Lemma 3.1 for

$$\begin{aligned}&E\{(W_n^{(p)}W_n^{(r)} - e^{it_p}\lambda_{n,p}W_n^{(r)})\exp[i\textstyle\sum_{j=1}^{p} t_j W_n^{(j)}]\} , \\ &E\{(W_n^{(p)} - e^{it_p}\lambda_{n,p})\exp[i\textstyle\sum_{j=1}^{p} t_j W_n^{(j)}]\} .\end{aligned}$$

Then by Lemma 3.1 and (3.8),

$$\begin{aligned}&\left|\frac{\partial^2}{\partial t\,\partial s}\varphi_n(t, s)\exp[\lambda_{n,p}(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1}\lambda_j(1 - e^{ist_j})]\right| \\ &\quad = |(\textstyle\sum_{j=1}^{p-1} t_j E\{(W_n^{(p)}W_n^{(j)} - \lambda_{n,p}e^{it_p}W_n^{(j)})\exp[itW_n^{(p)} + is\sum_{j=1}^{p-1} t_j W_n^{(j)}]\} \\ &\qquad - \textstyle\sum_{j=1}^{p-1} t_j\lambda_j e^{ist_j}E\{(W_n^{(p)} - \lambda_{n,p}e^{it_p})\exp[itW_n^{(p)} + is\sum_{j=1}^{p-1} t_j W_n^{(j)}]\}) \\ &\qquad \times i^2\exp[\lambda_{n,p}(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1}\lambda_j(1 - e^{ist_j})]| \\ &\quad \le \left\{C_1(m, p, 2)\max_{j,i} p_{n,i}^{(j)} + C_2(2)\varphi(m + 1)\right\}\textstyle\sum_{j=1}^{p-1}|t_j| \\ &\qquad + \left\{C_1(m, p, 1)\max_{j,i} p_{n,i}^{(j)} + C_2(1)\varphi(m + 1)\right\}\textstyle\sum_{j=1}^{p-1}|\lambda_j t_j| \\ &\quad \le \varepsilon\textstyle\sum_{j=1}^{p-1}|t_j|(1 + \lambda_j) .\end{aligned} \tag{3.12}$$

Furthermore, by (3.10) and (3.11),

$$\begin{aligned}
&\int_0^t \int_0^s \frac{\partial^2}{\partial t\, \partial s} \varphi_n(t, s) \exp\left[\lambda_{n,p}(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1} \lambda_j(1 - e^{ist_j})\right] dt\, ds \\
&\quad = \varphi_n(t, s) \exp\left[\lambda_{n,p}(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1} \lambda_j(1 - e^{ist_j})\right] \\
&\qquad - E\{\exp[itW_n^{(p)}]\} \exp\left[\lambda_{n,p}(1 - e^{it})\right] \\
&\qquad - E\{\exp[i \textstyle\sum_{j=1}^{p-1} st_j W_n^{(j)}]\} \exp\left[\textstyle\sum_{j=1}^{p-1} \lambda_j(1 - e^{ist_j})\right] + 1 \\
&\quad \to \lim_{n\to\infty} \varphi_n(t, s) \exp\left[\lambda_{n,p}(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1} \lambda_j(1 - e^{ist_j})\right] - 1\,.
\end{aligned}$$

By (3.12), the left hand side of the above relation converges to 0 as $n \to \infty$. Hence, by (A.1)′,

$$\lim_{n\to\infty} \varphi_n(t, s) \exp\left[\lambda_p(1 - e^{it}) + \textstyle\sum_{j=1}^{p-1} \lambda_j(1 - e^{ist_j})\right] - 1 = 0\,.$$

Replacing $t = t_p$ and $s = 1$, we have (3.7), and the proof of the theorem is completed.

§4. The asymptotic distribution of symmetric statistics

In this section, we extend the Poisson limit theorems in [17] to the case of dependent random elements. Let $X_{n,1}, \ldots, X_{n,k_n} (1 \le k_n \le \infty)$, $n = 1, 2, \ldots$, be sequences of random elements on $\mathfrak{X}$ as in Section 2 which satisfy the assumptions (A.1), (A.2) and (A.3). For a symmetric function $h_k(x_1, \ldots, x_k)$, we define symmetric statistics by

$$\sigma_k^n(h_k) \equiv \begin{cases} \displaystyle\sum_{1 \le s_1 < \cdots < s_k \le k_n} h_k(X_{n,s_1}, \ldots, X_{n,s_k}) & \text{for } k \le k_n \\ 0 & \text{for } k > k_n\,. \end{cases}$$

Let $\bar{\mathscr{K}}(\mathfrak{X})$ be the family of sequences of continuous symmetric functions $\{h_k\}_{k\ge 0}$ defined by Notation 1 of [17] and let $\bar{\mathscr{E}}(\mathfrak{X})$ be the family of sequences of symmetric step functions of special form defined in §3 of [17]. We investigate the asymptotic distribution of the symmetric statistics

$$Y_n(h) \equiv \textstyle\sum_{k=0}^{k_n} \sigma_k^n(h_k)\,,$$

for $h = (h_0, h_1, \ldots) \in \bar{\mathscr{K}}(\mathfrak{X})$. We show that the limiting distribution is expressed in terms of multiple Poisson-Wiener-Ito integrals with respect to a Poisson random measure P_λ with intensity λ;

$$W(h) = \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int h_k(x_1, \ldots, x_k)\, dP_\lambda(x_1) \ldots dP_\lambda(x_k)\,.$$

We denote

$$\boldsymbol{x}^k = (x_1, \ldots, x_k) \in \mathfrak{X}^k \quad \text{and} \quad d\lambda_n^k(\boldsymbol{x}^k) = d\lambda_n(x_1) \ldots d\lambda_n(x_k)\,.$$

By Lemma 2.1 and the same way as in [17] (cf. [18]), we have the following estimation of the covariance,

$$\begin{aligned}
&E|\sigma_k^n(h_k)\sigma_l^n(g_l)| \\
&= \sum_{1 \le s_1 < \cdots < s_k \le k_n} \sum_{1 \le r_1 < \cdots < r_l \le k_n} E|h_k(X_{n,s_1}, \ldots, X_{n,s_k}) g_l(X_{n,r_1}, \ldots, X_{n,r_l})| \\
&= \sum_{j=0}^{k \wedge l} \sum\nolimits^{\#} E|h_k(X_{n,s_1}, \ldots, X_{n,s_j}, X_{n,s_{j+1}}, \ldots, X_{n,s_k}) \\
&\quad \times g_l(X_{n,s_1}, \ldots, X_{n,s_j}, X_{n,s_{k+1}}, \ldots, X_{n,s_{k+l-j}})| \\
&\le \sum_{j=0}^{k \wedge l} \frac{1}{(k-j)!} \frac{1}{(l-j)!} \frac{1}{j!} \sum_{1 \le s_1, \ldots, s_{k+l-j} \le k_n} \\
&\quad \times E|h_k(X_{n,s_1}, \ldots, X_{n,s_j}, X_{n,s_{j+1}}, \ldots, X_{n,s_k}) \\
&\quad \times g_l(X_{n,s_1}, \ldots, X_{n,s_j}, X_{n,s_{k+1}}, \ldots, X_{n,s_{k+l-j}})| \\
&\le \sum_{j=1}^{k \wedge l} \frac{1}{(k-j)!} \frac{1}{(l-j)!} \frac{1}{j!} \alpha^{k+l-j-1} \\
&\quad \times \int \cdots \int |h_k(\boldsymbol{x}^j, \boldsymbol{y}^{k-j}) g_l(\boldsymbol{x}^j, \boldsymbol{z}^{l-j})|\, d\lambda_n^j(\boldsymbol{x}^j)\, d\lambda_n^{k-j}(\boldsymbol{y}^{k-j})\, d\lambda_n^{l-j}(\boldsymbol{z}^{l-j})\,,
\end{aligned}$$

for $k_n \ge k$, $k_n \ge l$ and

$$E|\sigma_k^n(h_k)\sigma_l^n(g_l)| = 0\,,$$

for $k_n < k$ or $k_n < l$, where $k \wedge l$ is the minimum of k and l, the sum $\sum^{\#}$ is extended over all different s_i, $1 \le i \le k + l - j$ such that $1 \le s_1 < \cdots < s_j \le k_n$, $1 \le s_{j+1} < \cdots < s_k \le k_n$, $1 \le s_{k+1} < \cdots < s_{k+l-j} \le k_n$. Then we have

$$\begin{aligned}
\text{(4.1)} \quad E|Y_n(h)|^2 &\le \sum_{k,l=0}^{k_n} \sum_{j=0}^{k \wedge l} \frac{1}{(k-j)!} \frac{1}{(l-j)!} \frac{1}{j!} \alpha^{k+l-j} \\
&\quad \times \int \cdots \int |h_k(\boldsymbol{x}^j, \boldsymbol{y}^{k-j}) g_l(\boldsymbol{x}^j, \boldsymbol{z}^{l-j})|\, d\lambda_n^j(\boldsymbol{x}^j)\, d\lambda_n^{k-j}(\boldsymbol{y}^{k-j})\, d\lambda_n^{l-j}(\boldsymbol{z}^{l-j})\,.
\end{aligned}$$

As in [17] (cf. [18]), for a given Radon measure ν, we define a norm $\|h\|_\nu$ of a sequence of symmetric functions $h = \{h_k\}_{k=0}^{\infty}$ by

(4.2) $$\|h\|_\nu^2 \equiv \sum_{k,l=0}^{\infty} \sum_{j=0}^{k\wedge l} \frac{1}{(k-j)!}\frac{1}{(l-j)!}\frac{1}{j!} \times \int \cdots \int |h_k(x^j, y^{k-j}) h_l(x^j, z^{l-j})| \, d\nu^j(x^j)\, d\nu^{k-j}(y^{k-j})\, d\nu^{l-j}(z^{l-j}).$$

By (4.2), we have

(4.3) $$E|Y_n(h)|^2 \leq \|h\|_{\alpha\lambda_n}^2 .$$

Furthermore, we define a norm $\|\cdot\|$ by

(4.4) $$\|h\| \equiv \overline{\lim_{n\to\infty}} \|h\|_{\alpha\lambda_n} \left(\geq \underline{\lim_{n\to\infty}} \|h\|_{\alpha\lambda_n} \geq \|h\|_\lambda \right).$$

NOTATION 1. Denote by $\bar{\mathscr{H}}(\mathfrak{X})$ the set of all sequences $h = \{h_k\}_{k\geq 0}$ which can be approximated by elements of $\bar{\mathscr{E}}(\mathfrak{X})$ with respect to the norm $\|\cdot\|$, that is, for any $h \in \bar{\mathscr{H}}(\mathfrak{X})$ and any $\varepsilon > 0$, there exists an $h^\varepsilon \in \bar{\mathscr{E}}(\mathfrak{X})$ such that

$$\|h - h^\varepsilon\| < \varepsilon .$$

LEMMA 4.1. *For* $h \in \bar{\mathscr{E}}(\mathfrak{X})$,

$$Y_n(h) \xrightarrow{d} W(h) = \sum_{k=1}^{p} \frac{1}{k!} \sum_{1\leq i_1,\ldots,i_k\leq p} h_{i_1,\ldots,i_k} P_\lambda(B_{i_1}) \ldots P_\lambda(B_{i_k})$$

as $n \to \infty$.

PROOF. By 15.7.2 in [15] (cf. Lemma 2.1 in [17]), we know that for each $n \geq 1$, $X_{n,i} = (\chi_{B_1}(X_{n,i}), \ldots, \chi_{B_p}(X_{n,i}))$, $1 \leq i \leq k_n$, is a dependent p-dimensional Bernoulli array which satisfies (A.1)′ and (A.2)′, i.e.

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \Pr(\chi_{B_j}(X_{n,i}) = 1) = \lim_{n\to\infty} \sum_{i=1}^{k_n} \nu_{n,i}(B_j) = \lambda(B_j), \qquad j = 1, \ldots, p,$$

$$\lim_{n\to\infty} \max_{j,i} \Pr(\chi_{B_j}(X_{n,i}) = 1) = \lim_{n\to\infty} \max_{j,i} \nu_{n,i}(B_j) = 0 .$$

Therefore, by Theorem 3.2,

$$\left(\sum_{i=1}^{k_n} \chi_{B_1}(X_{n,i}), \ldots, \sum_{i=1}^{k_n} \chi_{B_p}(X_{n,i})\right) \xrightarrow{d} (P_\lambda(B_1), \ldots, P_\lambda(B_p))$$

as $n \to \infty$, where $P_\lambda(B_j)$ are independent Poisson random variables with means $\lambda(B_j)$, $j = 1, \ldots, p$. Then by Corollary 5.1 of [4], we have the assertion of the lemma.

THEOREM 4.2. *For* $h \in \bar{\mathscr{H}}(\mathfrak{X})$, $Y_n(h) \xrightarrow{d} W(h)$ *as* $n \to \infty$. *Particularly, for* $h \in \bar{\mathscr{K}}(\mathfrak{X})$, $Y_n(h) \xrightarrow{d} W(h)$ *as* $n \to \infty$.

PROOF. By Lemma 4.1, (4.3) and (4.4), the following estimation

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h)]\} - E\{\exp[itW(h)]\}| \\
&\quad \le \overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h)]\} - E\{\exp[itY_n(h^\varepsilon)]\}| \\
&\qquad + \overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h^\varepsilon)]\} - E\{\exp[itW(h^\varepsilon)]\}| \\
&\qquad + |E\{\exp[itW(h^\varepsilon)]\} - E\{\exp[itW(h)]\}| \\
&\quad \le \overline{\lim_{n\to\infty}} |t|\,\|h - h^\varepsilon\|_{\alpha\lambda_n} + |t|\,\|h - h^\varepsilon\|_\lambda \\
&\quad \le 2|t|\,\|h - h^\varepsilon\| \le 2|t|\varepsilon
\end{aligned}$$

is shown, for $h^\varepsilon \in \bar{\mathscr{E}}(\mathfrak{X})$ with $\|h - h^\varepsilon\| < \varepsilon$. Thus the first assertion is seen. To prove the second assertion, it is sufficient to show that $\bar{\mathscr{K}}(\mathfrak{X}) \subset \mathscr{H}(\mathfrak{X})$. Define $h^\varepsilon \equiv \{h_k^\varepsilon\}_{k\ge 0}$ by (3.7) in §3 of [17]. Then we have

$$\begin{aligned}
&\|h - h^\varepsilon\|^2_{\alpha\lambda_n} \\
&= \sum_{k,l=0}^{\infty} \sum_{j=0}^{k\wedge l} \frac{\alpha^{k+l-j}}{(k-j)!\,(l-j)!\,j!} \\
&\quad \times \int\cdots\int |(h_k - h_k^\varepsilon)(x^j, y^{k-j})(h_l - h_l^\varepsilon)(x^j, z^{l-j})|\, d\lambda_n^j(x^j)\, d\lambda_n^{k-j}(y^{k-j})\, d\lambda_n^{l-j}(z^{l-j}) \\
&\le \sum_{k,l=0}^{L} \sum_{j=0}^{k\wedge l} \frac{(\lambda(K)+1)^{-2L}\alpha^{k+l-j}}{(k-j)!\,(l-j)!\,j!} (4\varepsilon^2\lambda_n(K)^j + \varepsilon H^{k+l-2L}\lambda_n(K)^{k+l-j-1}) \\
&\quad + 2\sum_{k=L+1}^{\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{k\wedge l} \frac{1}{(k-j)!\,(l-j)!\,j!} H^{k+l+2}(\alpha\lambda_n(K))^{k+l-j} \\
&\le (4\varepsilon^2 + \varepsilon)e^{3\alpha} + 2\varepsilon\,.
\end{aligned}$$

Therefore, any $h \in \bar{\mathscr{K}}(\mathfrak{X})$ can be approximated by elements of $\bar{\mathscr{E}}(\mathfrak{X})$ with respect to the norm $\|\cdot\|$ defined by (4.4). Hence, $\bar{\mathscr{K}}(\mathfrak{X}) \subset \mathscr{H}(\mathfrak{X})$ is clear.

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