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Stationary solutions of a reaction-diffusion equation with a nonlocal convection

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Abstract: We are concerned with an ecological model described by a nonlinear diffusion equation with a nonlocal convection. The conditions under which stationary solutions exist are investigated. We also discuss the stability problem of stationary solutions.

1. Introduction

Reaction-diffusion equations are widely used in the modelling in biology, chemistry and other fields. Kawasaki [3] has proposed an ecological model described by a nonlinear diffusion equation with a nonlocal convection. The model of this type has been further studied by Nagai & Mimura [5], Mimura & Ohara [4], Ikeda [2] in the whole line of $\boldsymbol{R}^1$, whereas Ei [1] has considered the model in the finite interval. In the latter case the equation of interest takes the form

$$u_t = u_{xx} - [(K * u)u]_x + F(u), \qquad x \in I = (-1/2, 1/2) \tag{1.1}$$

subject to the boundary condition

$$u_x - (K * u)u = 0 \qquad \text{at } x = \pm 1/2 \tag{1.2}$$

and the initial condition

$$u(x, 0) = u_0(x) \geq 0, \qquad x \in I. \tag{1.3}$$

Here $u = u(t, x)$ denotes the population density at time t and the position x. The convection term $[(K * u)u]_x$ corresponds to aggregating mechanism of the population, where $(K * u)(x) = \int_I K(x - y)u(y)\,dy$ and $K(x)$ is an appropriate odd function satisfying $K(x) < 0$ for $x > 0$.

A representative kernel $K(x)$ is

$$K(x) = \begin{cases} \gamma e^{\beta x} & (x < 0), \\ -\gamma e^{-\beta x} & (x > 0), \end{cases} \tag{1.4}$$

where γ, β are nonnegative constants. One knows that when

$$\int_x^{1/2} e^{\beta(x-y)}u(y)\,dy - \int_{-1/2}^{x} e^{-\beta(x-y)}u(y)\,dy > 0 \qquad (\text{resp. } < 0)$$

the individuals move in the right (resp. left) direction, which means that the individuals move in the direction of higher distribution. In this paper we restrict the kernel $K(x)$ to the above type.

The parameter β represents, so to speak, *perception ability* of the species. When $\beta = 0$, the individual's movement depends evenly on the information of the whole habitat, which is unrealistic from the biological point of view. When β is large, the individual's movement depends heavily on the *nearby* information.

The function $F(u)$ represents the rate of growth of the population. In this paper we assume that the growth process is much *slower* than the dispersion process, so that we write the function $F(u)$ as $\varepsilon f(u)$ with $0 < \varepsilon \ll 1$ (For an ecological interpretation, see Shigesada [6]).

We are concerned with the asymptotic behaviour of solutions of (1.1), (1.2), (1.3). The case of the simplest kernel ($\beta = 0$ in (1.4)) was analyzed in [1]. It was shown there that the situation crucially depends on the choice of the function $f(u)$ and the kernel $K(x)$.

We would like to find the global picture of stationary solutions of (1.1), (1.2). In the rest of this section we specify $f(u)$ to be a cubic function $u(1-u)(u-a)$, where $0 < a < 1$, and fix $\gamma > 0$.

When $\beta = 0$, the global picture of stationary solutions of (1.1), (1.2) with respect to the parameter a is shown in Figure 1. There exsists $0 < a^* < 1$ depending on γ such that for $a^* < a < 1$ there is only one stable stationary solution $v_0 \equiv 0$ and for $0 < a < a^*$ there are two stable statinary solutions v_0, v_2 and one unstable stationary solution v_1 (see [1]). When β tends to infinity in (1.4), we formally obtain $K(x) \equiv 0$. In this case (1.1) $\sim$ (1.3) is a simple semilinear parabolic equation with the homogeneous Neumann boundary condition and we can also have complete global picture of stationary solutions with respect to a as in Figure 2. Extending the results of [1] to the case $0 < \beta < \infty$ is the main concern of this paper.

The structure of stationary solutions of (1.1), (1.2) with respect to the parameter β is an interesting problem. Our theoretical results and some numerical simulations suggest that the global picture of stationary solutions is as follows: There exists $0 < a^*(\beta) < 1$ depending on β such that for $a > a^*(\beta)$ there is only one stationary solution $v_0 \equiv 0$ and for $0 < a < a^*(\beta)$ there are three stationary solutions v_0, v_1, v_2 (see Figure 3). Figures $1 \sim 3$ suggest that the global picture of the set of stationary solutions change continuously with respect to β and that no structural changes occur irrespective of the parameter β.

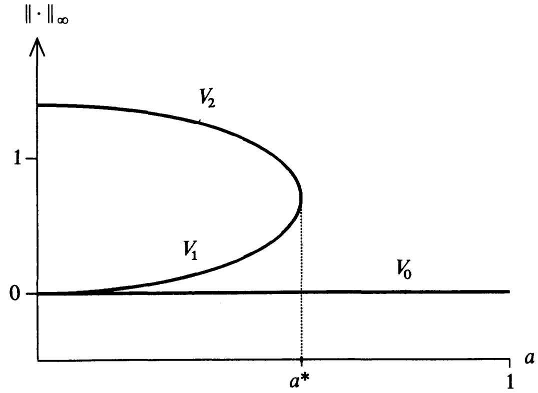

Figure 1. The global picture of stationary solutions of (1.1), (1.2) with respect to a in the case $\beta = 0$, $\gamma > 0$, $f(u) = u(1-u)(u-a)$.

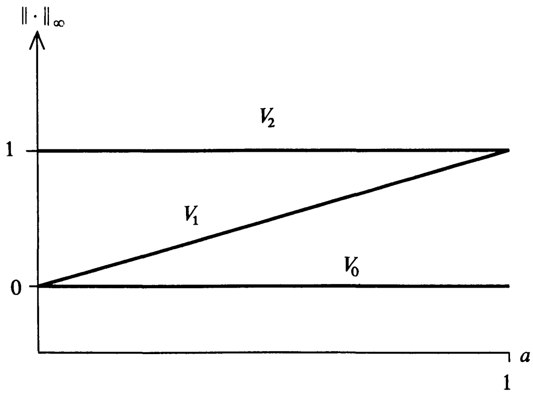

Figure 2. The global picture of stationary solutions of (1.1), (1.2) with respect to a in the case $\beta = \infty$, $\gamma > 0$, $f(u) = u(1-u)(u-a)$.

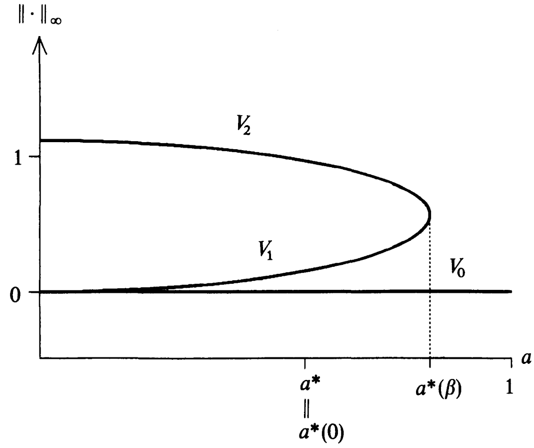

Figure 3. The global picture of stationary solutions of (1.1), (1.2) with respect to a in the case $0 < \beta < \infty$, $\gamma > 0$, $f(u) = u(1-u)(u-a)$.

2. Main results

Throughout this paper, we assume that $f \in C^2(R)$ and $f(0) \geq 0$. Let v be a stationary solution of (1.1), (1.2). Then v satisfies

$$\{v_x - (K * v)v\}_x + \varepsilon f(v) = 0 \quad \text{in } (-1/2, 1/2), \tag{2.1}$$

$$v_x - (K * v)v = 0 \quad \text{at } x = \pm 1/2. \tag{2.2}$$

Let w be a solution of the following equation.

$$w_x - (K * w)w = 0 \quad \text{in } \bar{I}, \tag{2.3}$$

$$w(0) = c. \tag{2.4}$$

Note that v satisfies (2.3) when $\varepsilon = 0$.

THEOREM 1. *Let c be an arbitrary nonnegative number. Then there exsists a unique solution $w(\cdot; c, \beta, \gamma)$ of* (2.3), (2.4), *which is nonnegative in $\bar{I}$. Moreover $w(\cdot; c, \beta, \gamma) \in C^\infty([0, \infty) \times (0, \infty) \times (0, \infty); B) \cap C^0([0, \infty) \times [0, \infty) \times [0, \infty); B)$, where B denotes the Banach space $C(\bar{I})$ with sup-norm $\|\cdot\|_\infty$.*

Let $H(c; \beta, \gamma) = \int_I f(w(x; c, \beta, \gamma))\,dx$, where $w(x; c, \beta, \gamma)$ is a solution of (2.3), (2.4).

THEOREM 2. *Suppose that there exist $c_0 \geq 0$, $\beta_0 > 0$, $\gamma_0 > 0$ such that*

$$H(c_0, \beta_0, \gamma_0) = 0, \quad \frac{\partial}{\partial c} H(c, \beta_0, \gamma_0)\Big|_{c = c_0} \neq 0.$$

Then there exists a unique function $v(\varepsilon, \beta, \gamma; c_0, \beta_0, \gamma_0) \in C^1((-\varepsilon_0, \varepsilon_0) \times (\beta_0 - \varepsilon_0, \beta_0 + \varepsilon_0) \times (\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0); B)$ for sufficiently small $\varepsilon_0 > 0$ such that $v(\varepsilon, \beta, \gamma; c_0, \beta_0, \gamma_0)(x)$ is a solution of (2.1), (2.2) *satisfying $v(0, \beta_0, \gamma_0; c_0, \beta_0, \gamma_0)(x) = w(x; c_0, \beta_0, \gamma_0)$.*

THEOREM 3. *Assume that $\limsup_{u \to \infty} \dfrac{f(u)}{u^3} < 0$. Let q be a solution of the following equation,*

$$\begin{cases} q_t = \{q_x - (K * v)q - (K * q)v\}_x + \varepsilon f'(v)q, & x \in I, \\ q_x - (K * v)q - (K * q)v = 0 & \text{at } x = \pm 1/2, \\ q(x, 0) = \phi(x), & x \in I, \end{cases}$$

where v is a solution of (2.1), (2.2). *Let u be a solution of* (1.1) $\sim$ (1.3) *with an initial value $u_0(x) = \phi(x) + v(x)$. Set $p(x, t) = u(x, t) - v(x)$. Suppose that there exist positive constants C, k, δ_0 such that*

$$\|q(\cdot, t)\| \le Ce^{-kt}\|\phi\| \qquad \text{for } \phi \in L^2(I) \text{ with } \|\phi\| \le \delta_0,$$

where $\|\cdot\|$ *means* $\|\cdot\|_{L^2(I)}$. *Then there exist positive constants* C', m, δ *and a subset* $V_1 \subset L^2(I)$ *containing* 0 *such that*

$$\|p(\cdot, t)\| \le C'e^{-mt}\|\phi\| \qquad \textit{for } \phi \in V_1 \textit{ with } \|\phi\| \le \delta.$$

3. Proofs of Theorems 1 and 2

In this section proofs of Theorems 1 and 2 will be given in several steps. Let w be a solution of (2.3), namely, w satisfies

$$w_x - \gamma\left\{\int_x^{1/2} e^{\beta(x-y)}w(y)\,dy - \int_{-1/2}^{x} e^{-\beta(x-y)}w(y)\,dy\right\}w = 0 \qquad \text{in } \bar{I}. \tag{3.1}$$

If we introduce two functions w_+, w_- defined by

$$w_+ = \int_x^{1/2} e^{\beta(x-y)}w(y)\,dy, \; w_- = \int_{-1/2}^{x} e^{-\beta(x-y)}w(y)\,dy, \tag{3.2}$$

then (3.1) is equivalent to the following system of differential equations with the boundary conditions.

$$\begin{cases} w' = \gamma(w_+ - w_-)w \\ w'_+ = \beta w_+ - w \qquad \text{in } \bar{I}, \\ w'_- = -\beta w_- + w \end{cases} \tag{3.3}$$

$$w_+(1/2) = 0, \; w_-(-1/2) = 0. \tag{3.4}$$

We note that the initial value problem of the system (3.3) has a unique solution.

LEMMA 1 *Let* (w, w_+, w_-) *be a solution of* (3.3) *with* $w \neq 0$. *Then the following equations hold.*

$$w_+ + w_- - \frac{\beta}{\gamma}\log|w| \equiv \text{const.} \tag{3.5}$$

$$w_+ w_- - \frac{w}{\gamma} \equiv \text{const.} \tag{3.6}$$

PROOF. By the equation (3.3) we have

$$\frac{d}{dx}\left(w_+ + w_- - \frac{\beta}{\gamma}\log|w|\right) = \beta w_+ - w - \beta w_- + w - \frac{\beta w'}{\gamma w} = 0.$$

The equation (3.6) can be proved similarly. □

LEMMA 2 *(a) Let* (w, w_+, w_-) *be a solution of* (3.3). *If* $w(0) > 0$ *and* $w_-(0) > 0$, *then* $w(x) > 0$ *in* $\bar{I}$, $w_-(x) > 0$ *in* $[0, 1/2]$ *and* $w_+(x)$ *is bounded below in* $\bar{I}$. *Moreover* w, w_+, w_- *are bounded above in* $\bar{I}$.
(b) Let (w, w_+, w_-) *be a solution of* (3.3), (3.4). *Then* $w(-x) = w(x)$, $w_+(x) = w_-(-x)$ *in* $\bar{I}$. *In particular* $w_+(0) = w_-(0)$ *and* $w'(0) = 0$. *Moreover* $w_+(x) > 0$ *in* I *if* $w(0) > 0$.

PROOF. (a) From (3.3) we have

$$w(x) = w(0)\exp\left\{\int_0^x (w_+(t) - w_-(t))\,dt\right\}.$$

Therefore $w(x) > 0$ *in* $\bar{I}$. From the inequality $w'_+ \leq \beta w_+$, we know that w_+ is bounded above. The rest of the statements can be proved similarly. (b) From (3.6) we have

$$w_+(-1/2)w_-(-1/2) - (1/\gamma)w(-1/2) = w_+(1/2)w_-(1/2) - (1/\gamma)w(1/2),$$

which implies that $w(-1/2) = w(1/2)$. Thus from (3.5) we have $w_+(-1/2) = w_-(1/2)$. If we put $(\tilde{w}(x), \tilde{w}_+(x), \tilde{w}_-(x)) = (w(-w), w_-(-x), w_+(-x))$, then $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)$ satisfies (3.3). The uniqueness of solutions of (3.3) implies that $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)(x) \equiv (w, w_+, w_-)(x)$. If $w(0) > 0$ we have $w(x) > 0$ in $\bar{I}$. Then from (3.2) we know that $w_+(x) > 0$ in I. □

By Lemma 2, in order to find a nonnegative solution of (3.3), (3.4) it is sufficient to seek a solution of (3.3) with an initial condition $w(0) \geq 0$, $w_+(0) = w_-(0) \geq 0$ in the interval $\bar{J} = [0, 1/2]$ which satisfies the following boundary condition:

$$w_+(1/2) = 0. \tag{3.7}$$

LEMMA 3 *Let* (w, w_+, w_-) *be a solution of* (3.3) *such that* $w(0) > 0$. *If there exists an* $x_0 \in \bar{J}$ *such that* $w'(x_0) = 0$, *then* $w''(x_0) \neq 0$ *unless* $(w, w_+, w_-)(x) \equiv \mathrm{const}$.

PROOF. Suppose to the contrary that $w'(x_0) = w''(x_0) = 0$. Then we have $w_+(x_0) = w_-(x_0)$ and $\beta w_+(w_0) - w(x_0) = -\beta w_-(x_0) + w(x_0) = 0$. If we put $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)(x) \equiv (w(x_0), (1/\beta)w(x_0), (1/\beta)w(x_0))$, then $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)$ satisfies (3.3). By the uniqueness of solutions of (3.3), we have $(w, w_+, w_-)(x) \equiv (\tilde{w}, \tilde{w}_+, \tilde{w}_-)(x)$. This contradicts the assumption. □

LEMMA 4. *Let* (w, w_+, w_-) *be a solution of* (3.3) *such that* $w(0) > 0$, $w_+(0) = w_-(0) > 0$, $w'_+(0) > 0$. *Then we have* $w_+(x) > 0$ in $\bar{J}$.

PROOF. Suppose that there exists an $x_0 \in (0, 1/2]$ such that $w_+(x_0) = 0$. Then it is easy to see that $w_+(x) < 0$ in $(x_0, 1/2]$. From Lemma 1

$w_+(0)w_-(0) - (1/\gamma)w(0) = w_+(1/2)w_-(1/2) - (1/\gamma)w(1/2)$, which implies that $w(0) > w(1/2)$. By the assumption we have $w''(0) > 0$, which means that $w(x) > w(0)$ for small $x > 0$. Hence there is an $x_1 \in J$ such that $w(x_1) = w(0)$. From Lemma 1 we have $w_+(x_1) + w_-(x_1) = w_+(0) + w_-(0)$ and $w_+(x_1)w_-(x_1) = w_+(0)w_-(0)$. Since $w_+(0) = w_-(0)$ we have $w_+(x_1) = w_-(x_1)$ and $w'(x_1) = 0$. By Lemma 3 $w''(x_1) \neq 0$, which contradicts the fact that $w(0) > w(1/2)$. □

LEMMA 5. *Let c be an arbitrary nonnegative number. Then there exists a $\xi \geq 0$ such that the solution of* (3.3) *with an initial value* $(w, w_+, w_-)(0) = (c, \xi, \xi)$ *satisfies* (3.7).

PROOF. The case $c = 0$ is trivial. For $c > 0$, $(\tilde{w}, \tilde{w}_+, \tilde{w}_-) \equiv (c, c/\beta, c/\beta)$ satisfies (3.3) and $\tilde{w}_+(1/2) > 0$. On the other hand the solution $(\bar{w}, \bar{w}_+, \bar{w}_-)$ of (3.3) with an initial condition $(\bar{w}, \bar{w}_+, \bar{w}_-)(0) = (c, 0, 0)$ satisfies $\bar{w}_+(1/2) < 0$. By the standard shooting argument there is a ξ $(0 < \xi < \beta/c)$ such that the solution (w, w_+, w_-) of (3.3) with an initial value (c, ξ, ξ) satisfies $w_+(1/2) = 0$. □

LEMMA 6. *Let (w, w_+, w_-) be a solution of* (3.3), (3.7). *If* $w(0) > 0$, $w_+(0) = w_-(0) > 0$, *then we have* $w'(x) < 0$ *in* $(0, 1/2]$. *In particular* $w_+(x) < w_-(x)$ *in* $(0, 1/2]$.

PROOF. By Lemma 4 $w'_+(0) < 0$, which implies that $w''(0) < 0$. Hence $w'(x) < 0$ for small $x > 0$. Suppose that there exists the smallest $x_0 \in J$ such that $w'(x_0) = 0$. By Lemma 3 we have $w''(x_0) > 0$. On the other hand we know that $w(1/2) < w(x_0)$ by Lemma 1. Thus there is an $x_1(> x_0)$ such that $w(x_1) = w(x_0)$. Following the proof of Lemma 4 we get $w'(x_1) = 0$ and $w''(x_1) \neq 0$. This leads to a contradiction. □

LEMMA 7. *Let (w, w_+, w_-) be a solution of* (3.3), (3.7) *such that* $w(0) > 0$, $w_+(0) = w_-(0) > 0$. *Then* $w'_+(x) < 0$ *in* $\bar{J}$.

PROOF. Let x_0 be the smallest x, if any, such that $w'_+(x) = 0$. Then $w''_+(x_0) > 0$. Since $w_+(1/2) = 0$, there must be an $x_1 > x_0$ such that $w'_+(x_1) = 0$, $w''_+(x_1) \leq 0$. But this contradicts the fact that $w''_+(x_1) = -w'(x_1) > 0$. □

LEMMA 8. *For a given $c > 0$, there exists a unique $\xi > 0$ such that the solution of* (3.3) *with an initial condition* $(w, w_+, w_-)(0) = (c, \xi, \xi)$ *satisfies* (3.7).

PROOF. Suppose to the contrary that there exist two solutions (w, w_+, w_-), $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)$ of (3.3) with initial conditions $w(0) = \tilde{w}(0) = c > 0$, $w_+(0) = w_-(0) = \xi > \tilde{\xi} = \tilde{w}_+(0) = \tilde{w}_-(0)$ which satisfy the boundary conditions $w_+(1/2) = \tilde{w}(1/2) = 0$. From the assumption $\xi > \tilde{\xi}$, we have $w''(0) > \tilde{w}''(0)$, which implies that $w(x) > \tilde{w}(x)$ for small $x > 0$. By Lemma 1 there exist constants $c_1, \tilde{c}_1, c_2, \tilde{c}_2$ such that

$$(3.8)\qquad \begin{cases} w_+ + w_- = c_1 + (\beta/\gamma)\log w, \\ \tilde{w}_+ + \tilde{w}_- = \tilde{c}_1 + (\beta/\gamma)\log \tilde{w}, \\ w_+ w_- = (1/\gamma)w + c_2, \\ \tilde{w}_+ \tilde{w}_- = (1/\gamma)\tilde{w} + \tilde{c}_2. \end{cases}$$

From the inequality $c_2 > \tilde{c}_2$ and (3.8), we obtain $w(1/2) < \tilde{w}(1/2)$. Let $x_1 \in J$ be the largest x such that $w(x) = \tilde{w}(x)$. We will show that $\tilde{w}_+(x_1) \geq w_+(x_1)$. Assume that $w_+(x_1) > \tilde{w}_+(x_1)$. Then for $x \in (x_1, 1/2]$

$$(w_+ - \tilde{w}_+)' = \beta(w_+ - \tilde{w}_+) - (w - \tilde{w}) \geq \beta(w_+ - \tilde{w}_+),$$

which implies that $w_+(x) - \tilde{w}_+(x) > 0$ in $(x_1, 1/2]$. This contradicts $w_+(1/2) = \tilde{w}_+(1/2) = 0$. Hence $\tilde{w}_+(x_1) \geq w_+(x_1)$. On the other hand at $x = x_1$

$$(3.9)\qquad \begin{cases} w_- = c_1 - \tilde{c}_1 + \tilde{w}_+ + \tilde{w}_- - w_+, \\ w_+ w_- - \tilde{w}_+ \tilde{w}_- = c_2 - \tilde{c}_2. \end{cases}$$

From (3.9) we obtain

$$(c_1 - \tilde{c}_1)w_+ + (w_+ - \tilde{w}_-)(\tilde{w}_+ - w_+) = c_2 - \tilde{c}_2 \qquad \text{at } x = x_1.$$

Since $w_+(x_1) \leq \tilde{w}_+(x_1) < \tilde{w}_-(x_1)$, we have

$$(c_1 - \tilde{c}_1)w_+(x_1) \geq c_2 - \tilde{c}_2.$$

On the other hand

$$\begin{aligned} (c_1 - \tilde{c}_1)w_+(x_1) &= 2(\xi - \tilde{\xi})w_+(x_1) \leq (\xi - \tilde{\xi})(w_+(x_1) + \tilde{w}_+(x_1)) \\ &< (\xi - \tilde{\xi})(\xi + \tilde{\xi}) = c_2 - \tilde{c}_2. \end{aligned}$$

This is a contradiction. $\square$

Combining Lemmas 5 and 8, it is proved that for a given $c \geq 0$ there exists a unique $\xi \geq 0$ such that the solution of (3.3) with an initial condition (c, ξ, ξ) satisfies (3.7). We will write $\xi = \xi(c)$ hereafter. The value ξ also depends on the parameters β, γ. We will write $\xi = \xi(c, \beta, \gamma)$ if necessary.

LEMMA 9. *Let* (w, w_+, w_-), $(\tilde{w}, \tilde{w}_+, \tilde{w}_-)$ *be two solutions of* (3.3), (3.7) *such that* $w_+(0) = w_-(0) = \xi(c)$, $\tilde{w}_+(0) = \tilde{w}_-(0) = \xi(\tilde{c})$ *and assume that* $w(0) = c > \tilde{w}(0) = \tilde{c} > 0$. *Then we have* $\xi(c) > \xi(\tilde{c})$.

PROOF. Suppose to the contrary that $\xi(\tilde{c}) \geq \xi(c)$.

Case (1). $\xi(\tilde{c}) = \xi(c)$

As in the proof of Lemma 8 we obtain the equation (3.8). From the assumption we also have $\tilde{c}_1 > c_1$, $\tilde{c}_2 > c_2$, which implies that $\tilde{w}(1/2) < w(1/2)$ and $w'_+(1/2) < \tilde{w}'_+(1/2) < 0$. Hence $w_+(x) > \tilde{w}_+(x)$ for x near $1/2$. Let x^* be the largest $x \in (0, 1/2)$ such that $w_+(x) = \tilde{w}_+(x)$. If $w(x) > \tilde{w}(x)$ in $x \in [x^*, 1/2]$ then we have $(\tilde{w}_+ - w_+)' \geq \beta(\tilde{w}_+ - w_+)$, which implies that $\tilde{w}_+ > w_+$ for $x \in (x^*, 1/2]$. This contradicts $w_+(1/2) = \tilde{w}_+(1/2) = 0$. Therefore there is an $x_2 \in (x^*, 1/2)$ such that $w(x_2) = \tilde{w}(x_2)$ and $w(x) > \tilde{w}(x)$ in $(x_2, 1/2]$. Let y^* be the smallest $x \in (0, 1/2)$ such that $w_+(x) = \tilde{w}_+(x)$. Reasoning similarly as above, there is an $x_1 \in (0, y^*)$ such that $w(x_1) = \tilde{w}(x_1)$ and $w(x) > \tilde{w}(x)$ in $[0, x_1)$. At $x = x_1, x_2$, we have

$$(3.10) \qquad \begin{cases} \tilde{w}_+ + \tilde{w}_- - w_+ - w_- = \tilde{c}_1 - c_1 > 0, \\ \tilde{w}_+ \tilde{w}_- - w_+ w_- = \tilde{c}_2 - c_2 > 0. \end{cases}$$

From (3.10) we obtain

$$\tilde{c}_2 - c_2 = (\tilde{c}_1 - c_1)\tilde{w}_+(x_2) + \{w_+(x_2) - \tilde{w}_+(x_2)\}\{\tilde{w}_+(x_2) - w_-(x_2)\}.$$

Since $\tilde{w}_+(x_2) < w_+(x_2) < w_-(x_2)$ we have

$$(3.11) \qquad \tilde{c}_2 - c_2 < (\tilde{c}_1 - c_1)\tilde{w}_+(x_2).$$

On the other hand

$$\begin{aligned} (\tilde{c}_1 - c_1)\tilde{w}_+(x_2) &< (\tilde{c}_1 - c_1)\tilde{w}_+(x_1) \\ &= \{\tilde{w}_+(x_1) + \tilde{w}_-(x_1) - w_+(x_1) - w_-(x_1)\}\tilde{w}_+(x_1), \\ \tilde{c}_2 - c_2 &= \tilde{w}_+(x_1)\tilde{w}_-(x_1) - w_+(x_1)w_-(x_1). \end{aligned}$$

Hence $\tilde{c}_2 - c_2 - (\tilde{c}_1 - c_1)\tilde{w}_+(x_1) = \{\tilde{w}_+(x_1) - w_+(x_1)\}\{w_-(x_1) - \tilde{w}_+(x_1)\}$. Since $w_-(x_1) > \tilde{w}_-(x_1) > \tilde{w}_+(x_1) > w_+(x_1)$ we have

$$\tilde{c}_2 - c_2 > (\tilde{c}_1 - c_1)\tilde{w}_+(x_1) > (\tilde{c}_1 - c_1)w_+(x_2),$$

which contradicts (3.11).

Case (2). $\xi(\tilde{c}) > \xi(c)$

By noting that the inequalities $\tilde{c}_1 > c_1$, $\tilde{c}_2 > c_2$ are also valid, this ease can be treated similarly as in the Case (1). □

LEMMA 10. *The function* $\xi = \xi(c, \beta, \gamma)$ *is continuous on* $[0, \infty) \times [0, \infty) \times [0, \infty)$.

PROOF. We consider the case $c > 0$, $\beta > 0$, $\gamma > 0$. Other cases can be treated similarly. Fix $c_0, \beta_0, \gamma_0 > 0$. Let $\{c_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences such that $c_n \to c_0$, $\beta_n \to \beta_0$, $\gamma_n \to \gamma_0$ $(n \to \infty)$. Suppose that $\limsup_{n\to\infty} \xi(c_n, \beta_n, \gamma_n) > \xi(c_0, \beta_0, \gamma_0)$. Let ξ_0 be a number such that $\min\{c_0/\beta_0, \limsup_{n\to\infty} \xi(c_n, \beta_n, \gamma_n)\}$

$> \xi_0 > \xi(c_0, \beta_0, \gamma_0)$. Consider the solution (w, w_+, w_-) of (3.3) with an initial condition (c_0, ξ_0, ξ_0) and with parameters β_0, γ_0. If $w_+(1/2) \leq 0$, there exists a ξ_1 $(\xi_0 \leq \xi_1 < c_0/\beta_0)$ such that the solution of (3.3) with an initial condition (c_0, ξ_1, ξ_1) satisfies (3.7) (by shooting argument as in Lemma 5). But this contradicts Lemma 8. If $w_+(1/2) > 0$, the solution (w^n, w^n_+, w^n_-) of (3.3) with an initial condition (c_n, ξ_0, ξ_0) and with parameters β_n, γ_n also satisfies $w^n_+(1/2) > 0$ for sufficiently large n. Since $\limsup_{n\to\infty} \xi(c_n, \beta_n, \gamma_n) > \xi_0$, there is a large n such that $\xi(c_n, \beta_n, \gamma_n) > \xi_0$. Then there exists a ξ_2 $(0 < \xi_2 < \xi_0)$ such that the solution of (3.3) with an initial condition (c_n, ξ_2, ξ_2) and with parameters β_n, γ_n (for some fixed large n) satisfies (3.7). This again contradicts Lemma 8. Thus $\limsup_{n\to\infty} \xi(c_n, \beta_n, \gamma_n) \leq \xi(c_0, \beta_0, \gamma_0)$. Similarly we can show that $\liminf_{n\to\infty} \xi(c_n, \beta_n, \gamma_n) \geq \xi(c_0, \beta_0, \gamma_0)$. □

The next Proposition combined with the differentiability of solutions of (3.3) with respect to the initial values and parameters completes the proof of Theorem 1.

PROPOSITION 1. $\xi = \xi(c, \beta, \gamma) \in C^\infty([0, \infty) \times (0, \infty) \times (0, \infty))$

PROOF. Fix $c_0 > 0$, $\beta_0 > 0$, $\gamma_0 > 0$ and define the sets U, V for small $\varepsilon > 0$ as follows:

$$U = \{\xi \mid \inf_{(c,\beta,\gamma)\in V} \xi(c, \beta, \gamma) < \xi < \sup_{(c,\beta,\gamma)\in V} \xi(c, \beta, \gamma)\},$$

$$V = \{(c, \beta, \gamma) \mid |c - c_0| < \varepsilon,\ |\beta - \beta_0| < \varepsilon,\ |\gamma - \gamma_0| < \varepsilon\}.$$

Lemma 9 guarantees that U is non-empty. The condition imposed on ε is explained in the course of the proof. First of all ε is taken sufficiently small so that $(\xi, c, \beta, \gamma) \in \bar{U} \times V$ implies $\xi < c/\beta$. Let us define the map $T\colon \bar{U} \times V \to \bar{U}$ as follows:

$$T(\xi, c, \beta, \gamma) = \xi - (1/K) w_+(1/2, \xi, c, \beta, \gamma), \tag{3.12}$$

where $w_+(x, \xi, c, \beta, \gamma)$ is a unique solution of (3.3) with an initial condition $(w, w_+, w_-)(0) = (c, \xi, \xi)$ and K is a sufficiently large number as explained below. Note that (w, w_+, w_-) satisfies (3.7) if and only if $T(\xi, c, \beta, \gamma) = \xi$. We will show that the map T is a uniform contraction on $\bar{U}$. Namely we prove that there exists a constant k $(0 < k < 1)$ such that

$$|T(\xi, c, \beta, \gamma) - T(\xi', c, \beta, \gamma)| \leq k|\xi - \xi'| \tag{3.13}$$

for (ξ, c, β, γ), $(\xi', c, \beta, \gamma) \in \bar{U} \times V$. The inequality (3.13) immediately finishes the proof of Proposition 1. We will need the following lemma to prove that the map T is well-defined.

LEMMA 11. $w_+(1/2, \sup_{(c,\beta,\gamma)\in V} \xi(c, \beta, \gamma), c, \beta, \gamma) > 0$ *for* $(\xi, c, \beta, \gamma) \in \bar{U} \times V$.

PROOF. Suppose that $w_+(1/2, \sup \xi(c, \xi, \gamma), c, \beta, \gamma) \leq 0$. Since $\xi(c, \beta, \gamma) < \sup \xi(c, \beta, \gamma) < c/\beta$, there exists a ξ_1 ($\sup \xi(c, \beta, \gamma) < \xi_1 < c/\beta$) such that the solution of (3.3) with an initial value (c, ξ_1, ξ_1) satisfies (3.7). This contradicts Lemma 8. □

The continuous dependence of solutions of (3.3) with respect to the initial values implies that there exists a $K > 0$ such that

$$(3.14) \qquad \begin{aligned} w_+(1/2, \xi, c, \beta, \gamma) - w_+(1/2, \sup \xi(c, \beta, \gamma), c, \beta, \gamma) \\ \geq -K(\sup \xi(c, \beta, \gamma) - \xi). \end{aligned}$$

Since $w_+(1/2, \xi, c, \beta, \gamma) = w_+(1/2, \xi, c, \beta, \gamma) - w_+(1/2, \sup \xi(c, \beta, \gamma), c, \beta, \gamma) + w_+(1/2, \sup \xi(c, \beta, \gamma), c, \beta, \gamma)$, (3.14) and Lemma 11 imply that

$$(3.15) \qquad \xi - (1/K) w_+(1/2, \xi, c, \beta, \gamma) \leq \sup \xi(c, \beta, \gamma).$$

Similarly there exists a $K > 0$ such that

$$(3.16) \qquad \inf \xi(c, \beta, \gamma) \leq \xi - (1/K) w_+(1/2, \xi, c, \beta, \gamma).$$

The inequalities (3.15), (3.16) show that the map T is well-defined. The inequality (3.13) is a consequence of the following lemma.

LEMMA 12. *Assume that (ξ, c, β, γ), $(\xi', c, \beta, \gamma) \in \bar{U} \times V$ and $\xi > \xi'$. Then there exist $L > M > 0$ such that*

$$M(\xi - \xi') \leq w_+(1/2, \xi, c, \beta, \gamma) - w_+(1/2, \xi', c, \beta, \gamma) \leq L(\xi - \xi').$$

PROOF. The right hand side inequality is an immediate consequence of the continuous dependence of solutions of (3.3) on the initial values. To prove the other inequality, we will show that there exists an $M > 0$ such that

$$(3.17) \qquad w_+(1/2, \xi + h, c, \beta, \gamma) - w_+(1/2, \xi, c, \beta, \gamma) \geq Mh$$

for $(\xi + h, c, \beta, \gamma)$, $(\xi, c, \beta, \gamma) \in \bar{U} \times V$ and $h > 0$. To show that (3.17) is valid when ε is sufficiently small, it is sufficient to prove (3.17) in the case $\xi = \xi(c_0, \beta_0, \gamma_0)$, $c = c_0$, $\beta = \beta_0$, $\gamma = \gamma_0$. For brevity we will denote $w_+(x)$ for $w_+(x, \xi(c_0, \beta_0, \gamma_0) + h, c_0, \beta_0, \gamma_0)$ and $\tilde{w}_+(x)$ for $w_+(x, \xi(c_0, \beta_0, \gamma_0), c_0, \beta_0, \gamma_0)$.

Case (1) $w(1/2) \geq \tilde{w}(1/2)$

By Lemma 1, we have

$$\begin{aligned} &w_+(1/2) w_-(1/2) - \tilde{w}_+(1/2) \tilde{w}_-(1/2) \\ &= (1/\gamma)\{w(1/2) - \tilde{w}(1/2)\} + 2\xi h + h^2 \geq 2\xi h, \end{aligned}$$

where $\xi = \xi(c_0, \beta_0, \gamma_0)$. Since $\tilde{w}_+(1/2) = 0$ and there exists a K_1 such that $K_1 > w_- > 0$, $w_+(1/2) \geq (2\xi h)/K_1 \geq Mh$ for some constant $M > 0$.

Cose (2) $w(1/2) < \tilde{w}(1/2)$

To treat this case we will need the following three lemmas.

LEMMA 13. Let x_0 be any point such that $w(x) = \tilde{w}(x)$. Then $w_+(x_0) > \tilde{w}_+(x_0)$.

PROOF. From Lemma 1, we have

$$\begin{cases} w_+ + w_- - \tilde{w}_+ - \tilde{w}_- = 2h \\ w_+ w_- - \tilde{w}_+ \tilde{w}_- = 2\xi h + h^2 \end{cases} \tag{3.18}$$

at $x = x_0$. From (3.18) we obtain

$$2hw_+ + (\tilde{w}_- - w_+)(w_+ - \tilde{w}_+) = 2\xi h + h^2 \qquad \text{at } x = x_0.$$

Suppose that $w_+(x_0) \le \tilde{w}_+(x_0)$. Since $\tilde{w}_+(x_0) < \tilde{w}_-(x_0)$, we have $2hw_+(x_0) \ge 2\xi h + h^2$. On the other hand $2hw_+(x_0) \le h\{w_+(x_0) + \tilde{w}_+(x_0)\} < hw_+(0) + h\tilde{w}_+(0) = 2\xi h + h^2$. This is a contradiction. □

LEMMA 14. *Let x_1 be the smallest x, if any, such that $w_+(x) - \tilde{w}_+(x) = h/2$. Then there exists an $l > 0$ such that $x_1 \ge l$ for $(\xi + h, c, \beta, \gamma)$, $(\xi, c, \beta, \gamma) \in \bar{U} \times V$.*

PROOF. From (3.3) we have

$$w_+(x_1) - \tilde{w}_+(x_1) - \{w_+(0) - \tilde{w}_+(0)\} = \beta \int_0^{x_1} (w_+ - \tilde{w}_+)\,dx - \int_0^{x_1} (w - \tilde{w})\,dx.$$

Since $w_+ > \tilde{w}_+$ in $[0, x_1)$, we have $\int_0^{x_1} (w - \tilde{w})\,dx \ge h/2$. Since there exists an $L > 0$ such that $|w - \tilde{w}| \le Lh$, we have $Lhx_1 \ge h/2$. Hence $x_1 \ge 1/2L$. □

LEMMA 15. *Let $x^* \in J$ be the largest x such that $w(x) = \tilde{w}(x)$. Then there exists an $M > 0$ such that $w_+(x^*) - \tilde{w}_+(x^*) \ge Mh$.*

PROOF. By Lemma 13, it is sufficient to treat the case $0 < w_+(x^*) - \tilde{w}_+(x^*) < h/2$. As in the proof of Lemma 13, we have

$$2hw_+ + (\tilde{w}_- - w_+)(w_+ - \tilde{w}_+) = 2\xi h + h^2 \qquad \text{at } x = x^*.$$

By Lemma 14, $2hw_+(x^*) \le 2hw_+(l)$. Put $\tilde{w}_+(l) = \eta$. Then for sufficiently small $\varepsilon > 0$, $w_+(l) \le \{\xi(c_0, \beta_0, \gamma_0) + \eta\}/2$. Hence $2hw_+(x^*) \le \{\xi(c_0, \beta_0, \gamma_0) + \eta\}h$. Since there exists an $L > 0$ such that $|\tilde{w}_- - w_+| \le L$, we have

$$w_+ - \tilde{w}_+ \ge (1/L)\{\xi(c_0, \beta_0, \gamma_0) - \eta\}\,h.$$

Hence $w_+(x^*) - \tilde{w}_+(x^*) \ge \min(h/2, (1/L)\{\xi(c_0, \beta_0, \gamma_0) - \eta\}h)$, which is to be proved. □

Using Lemma 15, the proof of (3.17) in the Case (2) is now immediate. Since $w < \tilde{w}$ in $(x^*, 1/2]$, we have $w_+(1/2) - \tilde{w}_+(1/2) \geq w_+(x^*) - \tilde{w}_+(x^*)$. Thus $w_+(1/2) - \tilde{w}_+(1/2) \geq Mh$. We have completed the proof of Proposition 1 as well as Theorem 1. □

REMARK. We assumed in the proof of Proposition 1 that $c_0 > 0$, $\beta_0 > 0$, $\gamma_0 > 0$. The case $c_0 = 0$, $\beta_0 > 0$, $\gamma_0 > 0$ can be treated similarly. We omit the details.

The proof of Theorem 2 is analogous to the proof of Theorem 2 in [1]. Therefore we will briefly outline the proof and only some nontrivial points are explained in detail. Let B denote the Banach space $C(\bar{I})$ with sup-norm $\|\cdot\|_\infty$. Let us define the operator $G: B \times B \times \boldsymbol{R}_+^2 \to B$, where $\boldsymbol{R}_+ = [0, \infty)$, as follows:

$$G(v, p, \beta, \gamma) = v - \int_0^x (K * v) v ds - p. \tag{3.19}$$

Then the solution $w(\cdot; c)$ of (2.3), (2.4) satisfies $G(w(\cdot; c), c, \beta, \gamma) = 0$. (For brevity we write $w(\cdot; c)$ for $w(\cdot; c, \beta, \gamma)$.) From this equatin we have

$$\frac{\partial G}{\partial v}(w(\cdot; c), c, \beta, \gamma)h = h - \int_0^x \{(K * w)h + (K * h)w\} ds \qquad \text{for } h \in B. \tag{3.20}$$

Put $L = \dfrac{\partial G}{\partial v}(w(\cdot; c), c, \beta, \gamma)$. We will show the invertibility of L. Since $L - I$ is a compact operator, it suffices to show that $Lh = 0$ implies $h = 0$. Suppose that $Lh = 0$. Then $h \in C^1(I)$ and we obtain

$$h_x - \{(K * w)h + (K * h)w\} = 0. \tag{3.21}$$

Let us define the operator L_B in B with the domain $D(L_B)$ as follows:

$$L_B = \{v_x - (K * w)v - (K * v)w\}_x,$$
$$D(L_B) = \{v \in C^2(\bar{I}) | v_x - (K * w)v - (K * v)w = 0 \quad \text{at } x = \pm 1/2\}.$$

It can be shown that $\operatorname{Ker} L_B = \operatorname{span}\left\{\dfrac{\partial w}{\partial c}\right\}$. To prove this we need the following Proposition.

PROPOSITION 2. *There exists at most one* φ *which satisfies the following equations.*

$$\begin{cases} \varphi_x - (K * w)\varphi - (K * \varphi)w = 0 & \text{in } \bar{I}, \\ \varphi(0) = 1, \end{cases} \tag{3.22}$$

where w *is a solution of* (2.3), (2.4).

PROOF. We will transform (3.22) into equivalent system of differential equations as in the proof of Theorem 1.

$$\begin{cases} \varphi' = \gamma(w_+ - w_-)\varphi + \gamma(\bar{w}_+ - \bar{w}_-)w \\ \bar{w}'_+ = \beta\bar{w}_+ - \varphi \\ \bar{w}'_- = -\beta\bar{w}_- + \varphi \end{cases} \quad \text{in } I, \tag{3.23}$$

$$\varphi(0) = 1,\ \bar{w}_+(1/2) = \bar{w}_-(-1/2) = 0. \tag{3.24}$$

Here w_+, w_- are defined in (3.2) and

$$\bar{w}_+ = \int_x^{1/2} e^{\beta(x-y)}\varphi(y)\,dy,\ \bar{w}_- = \int_{-1/2}^{x} e^{-\beta(x-y)}\varphi(y)\,dy.$$

Recall that (w, w_+, w_-) is a solution of (3.3), (3.4). First we state two lemmas.

LEMMA 16. *Let* $(\varphi, \bar{w}_+, \bar{w}_-, w, w_+, w_-)$ *be a solution of* (3.3), (3.23). *Then the next equations hold.*

$$\begin{cases} \bar{w}_+ w_- + w_+\bar{w}_- - \dfrac{\varphi}{\gamma} \equiv \text{const}, \\ \bar{w}_+ + \bar{w}_- - \dfrac{\beta\varphi}{\gamma w} \equiv \text{const}. \end{cases} \tag{3.25}$$

The proof is straightforward from (3.3), (3.23).

LEMMA 17. *Let* φ *be a solution of* (3.22). *Then* $\varphi(x) = \varphi(-x)$ *in* $\bar{I}$. *In particular* $\varphi'(0) = 0$ *and* $\bar{w}_+(0) = \bar{w}_-(0)$.

The proof of this lemma can be similarly done as Lemma 2 using (3.5), (3.6), (3.25).

We will continue the proof of Proposition 2. Let φ, φ^* be two solutions of (3.22). Put $v = \varphi - \varphi^*$, $\tilde{w}_+ = w_+ - \bar{w}_+$, $\tilde{w}_- = w_- - \bar{w}_-$. Then v, $\tilde{w}_+$, $\tilde{w}_-$ satisfy

$$\begin{cases} v' = \gamma(w_+ - w_-)v + \gamma(\tilde{w}_+ - \tilde{w}_-)w, \\ \tilde{w}'_+ = \beta\tilde{w}_+ - v, \\ \tilde{w}'_- = -\beta\tilde{w}_- + v, \\ v(0) = 0,\ \tilde{w}_+(1/2) = \tilde{w}_-(-1/2) = 0. \end{cases} \tag{3.26}$$

To prove that $v \equiv 0$, it is sufficient to show that $\tilde{w}_+(0)$ $(= \tilde{w}_-(0)) = 0$. Suppose to the contrary that $\tilde{w}_+(0) \neq 0$.

Case (1). $\tilde{w}_+(0) > 0$

In this case we have $v(0) = v'(0) = 0$, $v''(0) > 0$. Hence $v(x) > 0$ for small

$x > 0$. By Lemma 16 we obtain

$$\tilde{w}_+(1/2)w_-(1/2) + w_+(1/2)\tilde{w}_-(1/2) - (1/\gamma)v(1/2) = 2\tilde{w}_+(0)w_+(0) > 0.$$

Since $w_+(1/2) = \tilde{w}_+(1/2) = 0$, we know that $v(1/2) < 0$. Let $x_0 \in (0, 1/2)$ be the largest x such that $v(x) = 0$. By Lemma 16 we obtain

$$\begin{cases} \tilde{w}_+(x_0) + \tilde{w}_-(x_0) = 2\tilde{w}_+(0), \\ \tilde{w}_+(x_0)w_-(x_0) + w_+(x_0)\tilde{w}_-(x_0) = 2\tilde{w}_+(0)w_+(0). \end{cases} \tag{3.27}$$

From (3.27) we have

$$2\tilde{w}_+(0)w_+(x_0) + \tilde{w}_+(x_0)\{w_-(x_0) - w_+(x_0)\} = 2\tilde{w}_+(0)w_+(0). \tag{3.28}$$

Suppose that $\tilde{w}_+(x_0) > 0$. Since $v < 0$ in $(x_0, 1/2]$, we have $\tilde{w}'_+ \geq \beta\tilde{w}_+$ in $(x_0, 1/2]$. Then we know that $\tilde{w}_+ > 0$ in $(x_0, 1/2]$, which contradicts $\tilde{w}_+(1/2) = 0$. Thus $\tilde{w}_+(x_0) \leq 0$. Since $w_-(x_0) > w_+(x_0)$ and $w_+(x_0) < w_+(0)$, the equation (3.28) can not hold.

Case (2) $\tilde{w}_+(0) < 0$

This case can be treated similarly. □

It is easy to see that $\dfrac{\partial w}{\partial c} \in \operatorname{Ker} L_B$. Proposition 2 shows that the kernel of L_B is one-dimensional. From (3.21), there exists an $l \in \boldsymbol{R}$ such that $h = l\dfrac{\partial w}{\partial c}$. On the other hand it is easy to see that

$$\frac{\partial w}{\partial c} - \int_0^x \left\{\left(k * \frac{\partial w}{\partial c}\right)w + (K * w)\frac{\partial w}{\partial c}\right\}ds - 1 = 0.$$

Hence $L\left(\dfrac{\partial w}{\partial c}\right) = 1$. Since $Lh = lL\left(\dfrac{\partial w}{\partial c}\right) = l$, we have proved that $Lh = 0$ implies $h = 0$. This shows that $0 \in \rho(L)$, as desired. By the implicit function theorem there exists a map $R \in C^2(U; B)$, where U is a neighborhood of (c, β, γ) in $B \times \boldsymbol{R}_+^2$, such that

$$G(R(p, \beta, \gamma), p, \beta, \gamma) = 0 \quad \text{for } (p, \beta, \gamma) \in U, \quad R(c, \beta, \gamma) = w(\cdot\,; c, \beta, \gamma).$$

Using the map R, we can transform (2.1), (2.2) into the following boundary value problem.

$$\begin{cases} p_{xx} + \varepsilon f(R(p, \beta, \gamma)) = 0 & \text{in } I, \\ p_x = 0 & \text{at } x = \pm 1/2. \end{cases} \tag{3.29}$$

When c_0, β_0, γ_0 satisfy the assumption of Theorem 2, standard Lyapunov-

Schmidt method leads to the existence of a solution of (3.29), say $p(x; \varepsilon, \beta, \gamma)$, satisfying $p(x; 0, \beta_0, \gamma_0) \equiv c_0$. Defining $v(\varepsilon, \beta, \gamma; c_0, \beta_0, \gamma_0)(x) = R(p(\cdot; \varepsilon, \beta, \gamma), \beta, \gamma)(x)$, we get a solution of the original equation (2.1), (2.2).

4. Proof of Theorem 3

In this section, we consider the stability of stationary solutions whose existence has been already proved in Theorem 2. First, we will give the next general proposition.

PROPOSITION 3. *Let B be a Banach space with the norm $\|\cdot\|$ and $\{S(t)\}_{t\geq 0}$, $\{U(t)\}_{t\geq 0}$ be (C_0)-semigroups in B. Moreover, assume that there are sets V_1 and V_2 in B with the properties below.*
(i) $0 \in V_1 \subset V_2$ *and* $S(t)V_1 \subset V_2$ *for* $t \geq 0$.
(ii) *There exist positive constants C, k, δ_0 such that*

$$\|U(t)\phi\| \leq Ce^{-kt}\|\phi\| \qquad \text{for } \phi \in B \text{ with } \|\phi\| \leq \delta_0.$$

(iii) *There exist continuous functions $k_1(t)$, $k_2(t)$ for $t > 0$ and constants $\alpha > 1$, $\delta_0 > 0$ such that* $\limsup_{t\to\infty} k_1(t) < 1$, $k_2(t) \geq 0$ *for* $t > 0$ *and*

$$\|S(t)\phi - U(t)\phi\| \leq k_1(t)\|\phi\| + k_2(t)\|\phi\|^{\alpha}$$

for $\phi \in V_2$ with $\|\phi\| \leq \delta_0$.

Then, there exist positive constants m, δ, C' such that

$$\|S(t)\phi\| \leq C'e^{-mt}\|\phi\|$$

for $\phi \in V_1$ with $\|\phi\| \leq \delta$.

PROOF. From the assumptions (ii) and (iii). we obtain

$$\|S(t)\phi\| \leq (Ce^{-kt} + k_1(t) + k_2(t)\|\phi\|^{\alpha-1})\|\phi\| \tag{4.1}$$

for $\phi \in V_2$ with $\|\phi\| \leq \delta_0$. Since $\limsup_{t\to\infty}(Ce^{-kt} + k_1(t)) < 1$, there exist $0 < r_0 < 1$ and $t_0 > 0$ such that $(Ce^{-kt} + k_1(t)) < r_0$ for $t \geq t_0$, which implies that there exist $\delta > 0$ and $0 < \lambda < 1$ so that

$$\|S(t_0)\phi\| \leq \lambda\|\phi\| \tag{4.2}$$

for $\phi \in V_2$ with $\|\phi\| \leq \delta$. Let $\phi \in V_1$ with $\|\phi\| \leq \delta$. Then $S(t_0)\phi \in V_2$ and (4.2) leads to $\|S(t_0)\phi\| \leq \delta$, which implies that $\|S(2t_0)\phi\| \leq \lambda^2\|\phi\|$. Thus, iterating this procedure, we have

$$\|S(nt_0)\phi\| \leq \lambda^n\|\phi\| \tag{4.3}$$

for $\phi \in V_1$ with $\|\phi\| \le \delta$. The inequality (4.3) gives the proof of this proposition. $\square$

We would like to apply this proposition to our problems. Let v be a solution of (2.1), (2.2) and $S(t)\phi$, $U(t)\phi$ be solutions p, q of the following equations, respectively:

$$\text{(4.4)} \qquad \begin{cases} p_t = J^s(p)_x + (F(v+p) - F(v)), & x \in I, \\ J^s(p) = 0, & x \in \partial I, \\ p(x, 0) = \phi(x), & x \in I, \end{cases}$$

and

$$\text{(4.5)} \qquad \begin{cases} q_t = J^u(q)_x + F'(v)q, & x \in I, \\ J^u(q) = 0, & x \in \partial I, \\ q(x, 0) = \phi(x), & x \in I, \end{cases}$$

where $J^s(p) = p_x - (K * v)p - (K * p)v - (K * p)p$ and $J^u(q) = q_x - (K * v)q - (K * q)v$ and $F(u) = \varepsilon f(u)$. By the general existence theorem, it is easily shown that $\{S(t)\}_{t \ge 0}$ and $\{U(t)\}_{t \ge 0}$ are (C_0)-semigroups. Then, the assumption (ii) in Proposition 3 means that the stationary solution v of (1.1), (1.2) is stable in the linearized sense, while we have not yet been able to analyze the spectrum of the eigenvalue problem of the linearized equation of (1.1), (1.2) with respect to the stationary solutions. In the rest of this section, we show that $S(t)$ and $U(t)$ defined by (4.4), (4.5) satisfy the assumptions (i), (iii) in Proposition 3 under suitable conditions.

Throughout this section, we assume $\limsup_{u \to \infty} \dfrac{F(u)}{u^3} < 0$, which gives the boundedness of the solution of (1.1) $\sim$ (1.3) (Ei [1]). Let $B = L^2(I)$ with the usual L^2-norm, say $\|\cdot\|$, and $V_1 = \{\phi \in B \cap L^\infty(I) | \, \|\phi\|_\infty \le L_1\}$ for arbitrarily fixed $L_1 > 0$. Then, there exists $L_2 (\ge L_1) > 0$ such that $\|S(t)\phi\|_\infty \le L_2$ for any $t \ge 0$ and $\phi \in V_1$. So that we define $V_2 = \{\phi \in B \cap L^\infty(I) | \, \|\phi\|_\infty \le L_2\}$ and (i) holds.

Proposition 4. *$S(t)$ and $U(t)$ satisfy* (iii).

Proof. First, we show the following lemma.

Lemma 18. *Let $\phi \in V_1$ in* (4.4). *Then, there exist $C > 0$ and $M > 0$ such that*

$$\text{(4.6)} \qquad \|p(\cdot, t)\| \le Ce^{Mt}\|\phi\|,$$

$$\text{(4.7)} \qquad \int_0^t \|p_x(\cdot, s)\|^2 ds \le Ce^{Mt}\|\phi\|^2,$$

$$\int_0^t \|p(\cdot, s)\|_\infty^2\, ds \le Ce^{Mt}\|\phi\|^2. \tag{4.8}$$

PROOF. From (4.4), we have

$$\frac{1}{2}\frac{d}{dt}\|p(\cdot, t)\|^2 = \int_I p_t \cdot p\, dx$$

$$= \int_I J^s(p)_x \cdot p\, dx + \int_I (F(p+v) - F(v)) \cdot p\, dx$$

$$= -\int_I J^s(p) \cdot p_x\, dx + \int_I (F_{L_3}(p+v) - F_{L_3}(v)) \cdot p\, dx + L_3 \int_I p^2\, dx,$$

where $F_{L_3}(u) = F(u) - L_3 u$. Since $p \in V_2$, we can take L_3 so that $F'(u) \le L_3$ for any $u \ge 0$. Hence, F_{L_3} is monotone decreasing and so $(F_{L_3}(p+v) - F_{L_3}(v)) \cdot p \le 0$, which leads to

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 \le -\int_I J^s(p) \cdot p_x dx + L_3 \int_I p^2\, dx.$$

Here, $-\int_I J^s(p) \cdot p_x dx = -\int_I p_x^2\, dx + H_1$, where $H_1 = \int_I (K * (p+v)) \cdot p \cdot p_x\, dx + \int_I (K \cdot p) \cdot v \cdot p_x\, dx$ and so $H_1 \le C_1 \|p\| \cdot \|p_x\| + \|v\|_\infty \cdot \|K * p\| \cdot \|p_x\|$ for some $C_1 > 0$. $\|K * p\|$ is estimated by

$$\|K * p\| \le \left\{\int_I \left(\int_I K(x-y) \cdot p(y)\, dx\right)^2 dx\right\}^{1/2}$$

$$\le \|K\|_\infty \left\{\int_I \left(\int_I |p(y)|^2\, dy\right) dx\right\}^{1/2}$$

$$\le \|K\|_\infty \cdot \|p\|.$$

So that we have

$$H_1 \le C_1 \|p\| \cdot \|p_x\| + \|v\|_\infty \cdot \|K\|_\infty \cdot \|p\| \cdot \|p_x\|$$

$$\le C_2 \|p\| \cdot \|p_x\|$$

$$\le \frac{1}{2}\|p_x\|^2 + C_3 \|p\|^2$$

for some $C_2 > 0$ and $C_3 > 0$. Thus, it follows that

$$\frac{1}{2}\frac{d}{dt}\|p(\cdot, t)\|^2 + \frac{1}{2}\|p_x\|^2 \le C_3 \|p\|^2,$$

which gives (4.6) and (4.7). The inequality (4.8) holds from (4.6), (4.7) and the relation $\|p\|_\infty \le C_4(\|p_x\| + \|p\|)$ for some $C_4 > 0$. $\square$

Assume $\phi \in V_1$. We note that $\|p\|_\infty \le L_2$ holds. Let $\mathscr{L}r = J^u(r)_x + F'(v)r$ for $r \in D(\mathscr{L})$, $N(r) = F(v+r) - F(v) - F'(v)r$ and $G(r) = (K * r)\cdot r$, where $D(\mathscr{L}) = \{r \in H^2(I) | J^u(r) = 0 \text{ on } \partial I\}$. Then, $U(t)\phi$ is represented by $U(t)\phi = e^{t\mathscr{L}}\phi$, where $e^{t\mathscr{L}}$ is the semigroup generated by $\mathscr{L}$. Let $w(x,t) = p(x,t) - q(x,t)$. w satisfies

$$(4.9) \qquad \begin{cases} w_t = \mathscr{L}w - G(p)_x + N(p), & t > 0,\ x \in I, \\ J^u(w) = G(p), & x \in \partial I, \\ w(x,0) = 0, & x \in I. \end{cases}$$

Then, we have

$$(4.10) \qquad \frac{1}{2}\frac{d}{dt}\|w\|^2 = \int_I (\mathscr{L}w)\cdot w\,dx + \int_I \{-G(p)_x + N(p)\}\cdot w\,dx.$$

The first term in the right hand side of (4.10) is $\int_I (\mathscr{L}w)\cdot w\,dx = \int_I J^u(w)_x \cdot w\,dx + \int_I F'(v)\cdot w^2\,dx$, and we have

$$\int_I J^u(w)_x\cdot w\,dx = J^u(w)\cdot w|_{\partial I} - \int_I J^u(w)\cdot w_x\,dx$$
$$= G(p)\cdot w|_{\partial I} - \|w_x\|^2 + \int_I (K * v)\cdot w\cdot w_x\,dx + \int_I (K * w)\cdot v\cdot w_x\,dx.$$

So

$$\int_I J^u(w)_x\cdot w\,dx \le G(p)\cdot w|_{\partial I} - \|w_x\|^2 + \frac{1}{2}\|w_x\|^2 + C_1\|w\|^2$$

for some $C_1 > 0$, which shows from (4.10) that

$$(4.11) \quad \frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{1}{2}\|w_x\|^2 \le C_1\|w\|^2 + G(p)\cdot w|_{\partial I} + (\|G(p)_x\| + \|N(p)\|)\|w\|.$$

Here, the inequality

$$(4.12) \qquad \|w\|_\infty \le C_2(\|w\| + \|w\|^{1/2}\cdot\|w_x\|^{1/2})$$

leads to

$$G(p)\cdot w|_{\partial I} \le \|G(p)\|_\infty \cdot \|w\|_\infty$$
$$\le C_2\|G(p)\|_\infty\cdot\|w\| + C_2\|G(p)\|_\infty\cdot\|w\|^{1/2}\cdot\|w_x\|^{1/2}$$

$$\leq C_2 \| G(p) \|_\infty \cdot \| w \| + \frac{1}{(4/3)} (C_2 \| G(p) \|_\infty \cdot \| w \|^{1/2})^{4/3} + \frac{1}{4} (\| w_x \|^{1/2})^4$$

$$\leq \frac{1}{4} \| w_x \|^2 + C_3 (\| G(p) \|_\infty \cdot \| w \| + \| G(p) \|_\infty^{4/3} \cdot \| w \|^{2/3})$$

for some C_2 and $C_3 > 0$. The third inequality follows from the relation $\alpha\beta \leq \frac{\alpha^{4/3}}{(4/3)} + \frac{\beta^4}{4}$ for $\alpha \geq 0$, $\beta \geq 0$. Hence we see from (4.11) that

$$\frac{1}{2} \frac{d}{dt} \| w \|^2 + \frac{1}{4} \| w_x \|^2 \leq C_1 \| w \|^2 + B(t) \| w \|^{2/3}, \tag{4.13}$$

where $B(t) = C_4 \{ (\| G(p) \|_\infty + \| G(p)_x \| + \| N(p) \|) \| w \|^{1/3} + \| G(p) \|_\infty^{4/3} \}$ for some $C_4 > 0$. The inequality (4.13) shows

$$\| w(t) \|^2 \leq 2 \int_0^t e^{2C_1(t-s)} \cdot B(s) \cdot \| w(s) \|^{2/3} ds$$

and so

$$\| w(t) \|^{4/3} \leq 2 \int_0^t e^{2C_1(t-s)} \cdot B(s) ds \leq 2e^{2C_1 t} \int_0^t B(s) ds. \tag{4.14}$$

LEMMA 19. $\int_0^t B(s) ds \leq Ce^{Mt} \| \phi \|^2$.

PROOF. Since $\| w \|^{1/3} \leq C$ for some $C > 0$, we have

$$\int_0^t B(s) ds \leq C \int_0^t \{ \| G(p) \|_\infty + \| G(p)_x \| + \| N(p) \| \} + C \int_0^t \| G(p) \|_\infty^{4/3} ds$$

for some $C > 0$. First, we obtain from (4.8),

$$\int_0^t \| G(p) \|_\infty ds \leq \| K \|_\infty \cdot \int_0^t \| p(s) \|_\infty^2 ds \leq Ce^{Mt} \| \phi \|^2 \tag{4.15}$$

for some $C > 0$ and $M > 0$.

Next, we shall estimate $\int_0^t \| G(p)_x \| ds$. It follows that

$$\| G(p)_x \| \leq \| (K * p)_x \cdot p \| + \| (K * p) \cdot p_x \|$$

and $\| (K * p)_x \cdot p \|^2 = \int_I |(K * p)_x \cdot p|^2 dx$. Since $|(K * p)_x| \leq C \| p \|_\infty$ holds for some $C > 0$, $\int_I |(K * p)_x \cdot p|^2 dx \leq C \| p \|_\infty^2 \cdot \| p \|^2$ for some $C > 0$. So we have from (4.8)

$$\int_0^t \|(K*p)_x\cdot p\|\,ds \le C\int_0^t \|p\|_\infty\cdot\|p\|\,ds \le C'\int_0^t \|p\|_\infty^2\,ds \le C''e^{Mt}\|\phi\|^2$$

for some C, C' and $C''>0$. Similarly, we obtain from (4.6), (4.7)

$$\int_0^t \|(K*p)\cdot p_x\|\,ds \le C\int_0^t \|p\|\cdot\|p_x\|\,ds \le C'e^{Mt}\|\phi\|\cdot\int_0^t \|p_x\|\,ds$$
$$\le C'e^{Mt}\|\phi\|\cdot t^{1/2}\cdot\left(\int_0^t \|p_x\|^2\,ds\right)^{1/2} \le C''e^{Mt}\|\phi\|^2$$

for some C, C', C'' and M, $M'>0$. Thus, it follows that

$$\int_0^t \|G(p)_x\|\,ds \le Ce^{Mt}\|\phi\|^2 \tag{4.16}$$

for some C and $M>0$.

The estimation of $\int_0^t \|N(p)\|\,ds$ is as follows. Since $|N(p)|\le C|p|^2$ for some $C>0$, we have from (4.8),

$$\int_0^t \|N(p)\|\,ds \le C\int_0^t \|p^2\|\,ds \le C\int_0^t \|p\|_\infty^2\,ds \le C'e^{Mt}\|\phi\|^2 \tag{4.17}$$

for some C' and $M>0$.

Finally, we give the estimate of $\int_0^t \|G(p)\|_\infty^{4/3}\,ds$. From (4.8),

$$\int_0^t \|G(p)\|_\infty^{4/3}\,ds \le \|K\|_\infty^{4/3}\int_0^t (\|p\|_\infty^2)^{4/3}\,ds \tag{4.18}$$
$$\le C\cdot\sup_{0\le s\le t}\|p(s)\|_\infty^{2/3}\cdot\int_0^t \|p(s)\|_\infty^2\,ds \le C'e^{Mt}\|\phi\|^2$$

holds for some C, C' and $M>0$. Thus, (4.15) $\sim$ (4.18) gives the proof of this lemma. □

The inequality (4.14) and Lemma 19 leads to the estimate of

$$\|w(t)\|^{4/3} \le 2e^{2C_1t}\cdot Ce^{Mt}\|\phi\|^2,$$

so that we have

$$\|w(t)\| \le Ce^{Mt}\|\phi\|^{3/2} \tag{4.19}$$

for some C and $M>0$ and (iii) is satisfied. □

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