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Maximum of the Amplitude of the Periodic Solution of van der Pol's Equation

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1. Introduction

Previously, by Urabe and his collaborators [3, 5, 6][1], the periodic solutions of van der Pol's equation

$$\frac{d^2x}{dt^2}-\lambda(1-x^2)\frac{dx}{dt}+x=0 \quad (\lambda>0) \tag{1.1}$$

have been computed for various values of λ up to 20. One of the important facts found by their computation is the behavior of the amplitude of the periodic solution as the damping coefficient varies from 0 to infinity. The amplitudes a obtained for various values of λ are as follows:

Table 1

λ	a	λ	a
0	2.000	6	2.0199
1	2.009	8	2.0169
2	2.0199	10	2.0145
3	2.0235	20	2.0077
4	2.0231		
5	2.0216	∞	2.0000

As was pointed out by Urabe [4], the above value for $\lambda=10$ differs by only 0.0007 from that given by the asymptotic expression of Дородницын [1]

$$a=2+\frac{\alpha}{3}\lambda^{-4/3}-\frac{16}{27}\frac{\log\lambda}{\lambda^2}+\frac{1}{9}(3b_0-1+2\log 2 \tag{1.2}$$

$$-8\log 3)\frac{1}{\lambda^2}+O(\lambda^{-8/3})$$

$$(\alpha=2.338107,\ b_0=0.1723)$$

and the value for $\lambda=20$ coincides exactly with that given by the above asymptotic expression. From this, it may be supposed that, for λ greater than 10, the amplitude behaves quite approximately in accordance with the asymptotic

1) The numbers in square brackets refer to the references listed at the end of the report.

formula (1. 2). Then the behavior of the amplitude in the whole range of λ can be seen from the values of the above table continued by the asymptotic formula (1.2). It is shown graphically in Fig. 1.

From this figure, it is seen that there exists a unique maximum value of the amplitude for λ near 3.

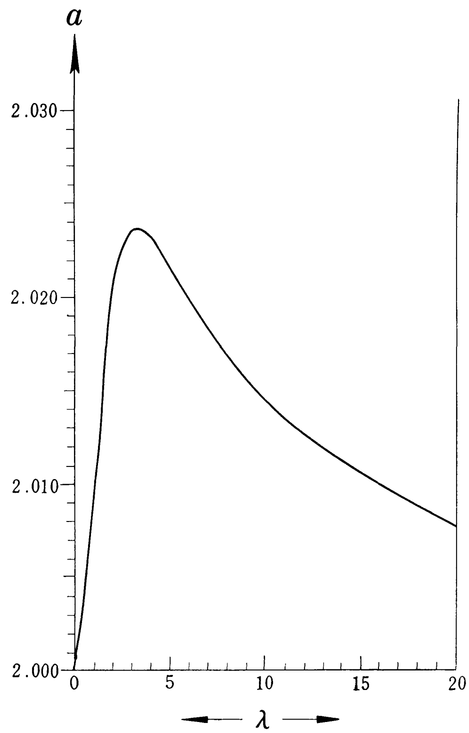

Fig. 1

In this report, this maximum value and the value λ for which the amplitude attains the maximum are calculated by computing the derivative of the amplitude with respect to λ.

2. Formula for $da/d\lambda$

Let us write the equation (1.1) in the simultaneous form as follows:

$$(2.1)\qquad \begin{cases} \dfrac{dx}{dt}=y & (\overset{\text{def}}{=} X(x, y, \lambda)), \\[2ex] \dfrac{dy}{dt}=-x+\lambda(1-x^2)y & (\overset{\text{def}}{=} Y(x, y, \lambda)). \end{cases}$$

Let

$$(2.2)\qquad x=\varphi(t, a), \qquad y=\psi(t, a) \qquad (a\geqq 0)$$

be the equation of the closed orbit C of (2. 1) such that

(2.3) $$\varphi(0, a)=a, \qquad \psi(0, a)=0.$$

Then, as is seen from the shape of C, a is the amplitude of the periodic solution of (1. 1) corresponding to the orbit (2. 2). Clearly a depends upon the value of λ.

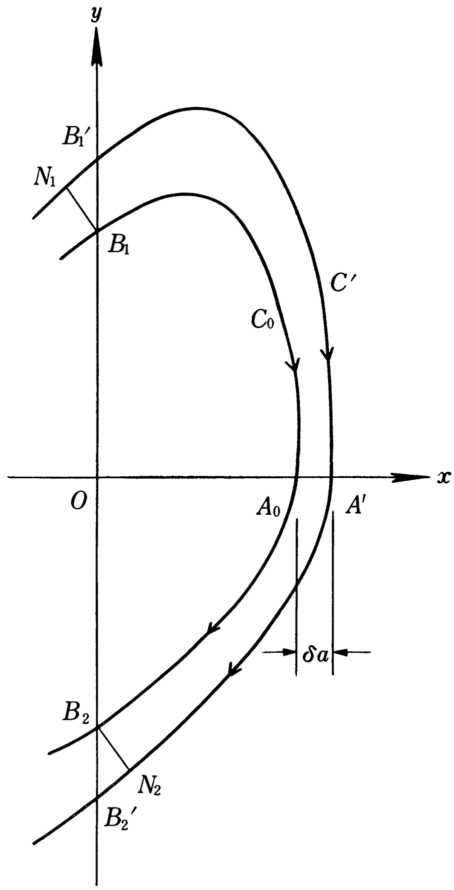

Fig. 2

In order to find the formula for $da/d\lambda$, let us consider the closed orbits C_0 and C' corresponding to λ_0 and $\lambda_0+\delta\lambda$ respectively.

Let a_0 and $a'=a_0+\delta a$ be the amplitudes of the periodic solutions of (1. 1) corresponding to C_0 and C' respectively. Let A_0, B_1, and B_2 be respectively the points where C_0 cuts the x-axis, the positive y-axis and the negative y-axis. Further let A', B_1' and B_2' be respectively the points where C' cuts the x-axis, the positive y-axis and the negative y-axis.

Let B_1N_1 and B_2N_2 be the normal distances at B_1 and B_2 from C_0 to C'. Then, since C_0 is perpendicular to the x-axis at A_0, by the theory of variation of orbits obtained by Urabe [2], we see that

$$\overrightarrow{B_iN_i}=\frac{\sqrt{X_0^2+Y_0^2}}{\sqrt{X_i^2+Y_i^2}}e^{h(T_i)}\delta a \tag{2.4}$$
$$+\frac{\delta\lambda}{\sqrt{X_i^2+Y_i^2}}e^{h(T_i)}\int_0^{T_i}e^{-h(t)}(XK-YH)_{\substack{x=\varphi(t,\, a_0)\\ y=\psi(t,\, a_0)\\ \lambda=\lambda_0}}dt$$
$$+o(|\delta a|+|\delta\lambda|) \qquad (i=1,\ 2),$$

where

X_0 and Y_0 are the values of X and Y at A_0 for $\lambda=\lambda_0$;
X_i and Y_i are the values of X and Y at B_i for $\lambda=\lambda_0$;

$$h(t)=\int_0^t\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)_{\substack{x=\varphi(t,\, a_0)\\ y=\psi(t,\, a_0)\\ \lambda=\lambda_0}}dt; \tag{2.5}$$

T_i are the times required to reach B_i from A_0 along C_0;

$$\begin{cases} H=H(x,\ y)=\dfrac{\partial X(x,\ y,\ \lambda)}{\partial\lambda}\Big|_{\lambda=\lambda_0}, \\ K=K(x,\ y)=\dfrac{\partial Y(x,\ y,\ \lambda)}{\partial\lambda}\Big|_{\lambda=\lambda_0}. \end{cases} \tag{2.6}$$

Now the inclinations of C_0 at the points B_i $(i=1,\ 2)$ are Y_i/X_i $(i=1,\ 2)$. Therefore, if we denote by B_1' and B_2' the points where C' cuts the positive and negative y-axis, we have

$$\overrightarrow{B_iB_i'}=\frac{\sqrt{X_i^2+Y_i^2}}{X_i}\overrightarrow{B_iN_i}+O(\overline{B_i\mathrm{N}_i}^2) \qquad (i=1,\ 2),$$

from which, due to (2.4), follows

$$\overrightarrow{B_iB_i'}=\frac{\sqrt{X_0^2+Y_0^2}}{X_i}e^{h(T_i)}\delta a+\frac{1}{X_i}e^{h(T_i)}I(T_i)\delta\lambda \tag{2.7}$$
$$+o(|\delta a|+|\delta\lambda|) \qquad (i=1,\ 2),$$

where

$$I(t)=\int_0^t e^{-h(t)}(XK-YH)_{\substack{x=\varphi(t,\, a_0)\\ y=\psi(t,\, a_0)\\ \lambda=\lambda_0}}dt. \tag{2.8}$$

On the other hand, since C_0 and C' are both closed orbits of (2. 1),

$$\overrightarrow{B_1B_1'}+\overrightarrow{B_2B_2'}=0 \tag{2.9}$$

as is remarked in the paper [5] (due to the symmetric character of the orbits of (2. 1) with respect to the origin).

The condition (2. 9) is written by (2. 7) as follows:

$$\sqrt{X_0^2+Y_0^2}\left(\frac{e^{h(T_1)}}{X_1}+\frac{e^{h(T_2)}}{X_2}\right)\delta a \tag{2.10}$$
$$+\left(\frac{e^{h(T_1)}I(T_1)}{X_1}+\frac{e^{h(T_2)}I(T_2)}{X_2}\right)\delta\lambda+o(|\delta a|+|\delta\lambda|)=0.$$

Now, in the present problem, from (2.1), (2.5), (2.6) and (2.8),

$$\begin{cases} X_0=0, \qquad Y_0=-a_0, \\ X_1=b_0, \qquad X_2=-b_0 \qquad (b_0\text{: the length of } \overline{OB_1}), \\ h(t)=\lambda_0\int_0^t[1-\varphi^2(s,\,a_0)]ds, \\ I(t)=\int_0^t e^{-h(s)}[1-\varphi^2(s,\,a_0)]\psi^2(s,\,a_0)ds. \end{cases}$$

Therefore (2.10) is written as follows:

$$\frac{a_0}{b_0}(e^{h(T_1)}-e^{h(T_2)})\delta a \tag{2.11}$$
$$+\frac{1}{b_0}[e^{h(T_1)}I(T_1)-e^{h(T_2)}I(T_2)]\delta\lambda+o(|\delta a|+|\delta\lambda|)=0.$$

But, as is remarked in the papers [3, 5, 6],

$$e^{h(T_1)}-e^{h(T_2)}>0.$$

Therefore, from (2. 11), we see

$$\delta a=-\frac{e^{h(T_1)}I(T_1)-e^{h(T_2)}I(T_2)}{a_0[e^{h(T_1)}-e^{h(T_2)}]}\delta\lambda+o(|\delta\lambda|),$$

from which, in the limit where $\delta\lambda\to 0$, follows

$$\frac{da}{d\lambda}=-\frac{e^{h(T_1)}I(T_1)-e^{h(T_2)}I(T_2)}{a_0[e^{h(T_1)}-e^{h(T_2)}]}. \tag{2.12}$$

This is the desired formula for $da/d\lambda$.

3. Actual computation

For computation of $h(t)$, there is used the integrated *Bessel's interpolation formula*

$$\int_0^k f(x)dx=\frac{k}{1440}(11f_{-2}-93f_{-1}+802f_0+802f_1-93f_2+11f_3),$$

where $f_r=f(rk)$.

For computation of $I(T_i)$ $(i=1, 2)$, there are used the *Simpson's rule* and the integrated *Bessel's interpolation formula*

$$\int_0^{x=uk} f(x)\,dx=k\Big[u\cdot\mu f_{\frac{1}{2}}+\frac{1}{2}u(u-1)\delta f_{\frac{1}{2}}+\frac{1}{6}u^2\Big(u-\frac{3}{2}\Big)\mu\delta^2 f_{\frac{1}{2}}$$
$$+\frac{1}{24}u^2(u-1)^2\delta^3 f_{\frac{1}{2}}$$
$$+\frac{1}{720}u^2(6u^3-15u^2-10u+30)\mu\delta^4 f_{\frac{1}{2}}$$
$$+\frac{1}{720}u^2(u-1)^2(u^2-u-3)\,\delta^5 f_{\frac{1}{2}}\Big].$$

First, for $\lambda=3$,

$$a=2.0235, \qquad T_1=-0.7173, \qquad T_2=3.7133$$

as is seen from the results of [5] and, from these values,

$$\frac{da}{d\lambda}=85\times 10^{-5}(>0)$$

is found by (2.12).

For $\lambda=3.5$,

$$a=2.0233, \qquad T_1=-0.6548, \qquad T_2=4.1054$$

are found by the methods described in [5] and, from these values,

$$\frac{da}{d\lambda}=-244\times 10^{-5}(<0)$$

is found by (2.12).

For $\lambda=3.3$, there are found two values of a:

$$a=2.0234, \qquad a=2.0235$$

and, for these values,

$$\frac{da}{d\lambda}=10^{-5}(>0), \qquad \frac{da}{d\lambda}=-2\times 10^{-5}(<0)$$

are respectively obtained.

Further, for $\lambda=3.2$, there are found

$$a=2.0234 \quad\text{and}\quad \frac{da}{d\lambda}=28\times 10^{-5}.$$

From these results, there are suggested two values of a:

$$a=2.0234, \qquad a=2.0235$$

for the values of λ near 3.3.

In fact, by the actual computation, there are obtained the following results:

Table 2

a	λ	$da/d\lambda \times 10^5$
2.0234	3.2000	+ 28
	3.3000	+ 1
	3.3005	+ 3
	3.3006	+ 2
	3.3007	− 1
	3.3008	− 1
	3.3010	− 0
2.0235	3.0000	+ 85
	3.2400	+ 9
	3.2650	+ 2
	3.2651	+ 0
	3.2652	− 1
	3.2654	− 4
	3.2662	− 3
	3.2675	− 3
	3.2700	− 8
	3.3000	− 2

4. Conclusions

The table 2 suggests that the maximum amplitude $\bar{a}$ and the damping coefficient $\bar{\lambda}$ yielding the maximum amplitude are respectively either

$$\bar{a}=2.0234 \quad \text{and} \quad \bar{\lambda}=3.3007$$

or

$$\bar{a}=2.0235 \quad \text{and} \quad \bar{\lambda}=3.2651.$$

But, of these two sets of values, the latter, i.e.

$$\bar{a}=2.0235, \quad \bar{\lambda}=3.2651$$

would be preferable to the former, because

$$a=2.0235 \quad \text{and} \quad \frac{da}{d\lambda}=85\times 10^{-5}>0$$

for $\lambda=3$.

It is clear that the ambiguity of the above results could be avoided if the computation were carried out more minutely.

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