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On a Lemma of Peetre

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By H^s, $-\infty < s < \infty$, we shall understand the space of temperate distributions f defined on Euclidean n-space R^n such that the Fourier transform $\hat{f}$ is a function satisfying

$$\|f\|_s^2 = \int |\hat{f}(\xi)|^2(1 + |\xi|^2)^s d\xi < \infty.$$

Let $\mathscr{L}^s$ be the space of distributions f such that $f\phi \in H^s$ for any $\phi \in \mathscr{D}$, and $\mathscr{K}^s$ be the space of distributions composed of elements of H^s with compact support [4]. A sequence of functions $\psi_j \in \mathscr{D}$, $j = 1, 2, \cdots$, is called *uniform partition* of a function $\psi \in \mathscr{B}$ when the following conditions are satisfied:

(i) $\sum_j \psi_j(x) = \psi(x)$ for any $x \in R^n$.

(ii) $\{\psi_j\}$ is bounded in $\mathscr{B}$.

(iii) For any compact set $A \subset R^n$, at most n_A of the supports of ψ_j can meet A, where n_A is a positive integer depending on the diameter of A.

(iv) The diameters of the supports of ψ_j are uniformly bounded. If, in addition, $\psi = 1$ and $\psi_j \geqq 0$, $j = 1, 2, \cdots$, we shall say that $\{\psi_j\}$ is a *uniform partition of the unity.*

In connection with the estimates of differential inequalities J. Peetre has established the following lemma with slightly weaker definition of uniform partition ([2], Lemma 1, p. 65).

Lemma. *Let $s \geqq 0$. If $\{\psi_j\}$ is a uniform partition of $\psi \in \mathscr{B}$, there exists a constant $C_{s, \{\psi_j\}}$ such that*

$$\sum_j \|\psi_j f\|_s^2 \leqq C_{s, \{\psi_j\}} \|f\|_s^2$$

for any $f \in H^s$.

Conversely, if f is a distribution of $\mathscr{L}^s$ such that there exists a uniform partition $\{\phi_j\}$ of the unity with $\sum_j \|\phi_j f\|_s^2 < \infty$, then $f \in H^s$.

He carried out the proof by making use of the norm ${}_0\|f\|_s^{*2} = \int \|f_a - f\|^2 / |a|^{n+2s}\, da$, $0 < s < 1$, and of induction with respect to s. But he says nothing about the case $s < 0$. His method of the proof seems not to be available in this case. The main purpose of this paper is to show the following lemma which may be regarded as a generalization of his lemma.

Lemma A. *Let s be any real number.*

(α) *If $\{\psi_j\}$ is a uniform partition of $\psi \in \mathscr{B}$, there exists a constant $C_{s, \{\psi_j\}}$ such that we have*

$$(1) \qquad \sum_j \|\psi_j f\|_s^2 \leqq C_{s, \{\psi_j\}} \|f\|_s^2 \quad \textit{for any} \quad f \in H^s.$$

(β) *If $\{\phi_j\}$ is a uniform partition of the unity, there exist two positive constants $C_{s, \{\phi_j\}}$, $C'_{s, \{\phi_j\}}$ such that we have*

$$(2) \qquad C_{s, \{\phi_j\}} \|f\|_s^2 \leqq \sum_j \|\phi_j f\|_s^2 \leqq C'_{s, \{\phi_j\}} \|f\|_s^2$$

for any $f \in H^s$. Therefore $f \to \|f\|_s$ and $f \to (\sum_j \|\phi_j f\|_s^2)^{\frac{1}{2}}$ are equivalent norms in H^s.

(γ) *Conversely, if f is a distribution of $\mathscr{L}^s$ such that there exists a uniform partition $\{\phi_j\}$ of the unity with $\sum_j \|\phi_j f\|_s^2 < \infty$, we have $f \in H^s$.*

The lemma may be effectively applied to the estimates of differential inequalities in the uniformly hypoelliptic case contemplated in Peetre's work ([2], pp. 65–69).

The definitions and the notations of L. Schwartz [3] with respect to the spaces of functions or distributions will be used without further reference.

1. Let ρ be a fixed indefinitely differentiable function defined on R^n with support in the unit ball B_1 such that $\rho \geqq 0$ and $\int \rho(x)dx = 1$. We put $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$.

In H^s, $s<0$, there have been considered by L. Hörmander [1] the following two norms $\|\cdot\|_{s,\varepsilon_0}$ and $|||\cdot|||_{s,\varepsilon_0}$, each equivalent to the original norm $\|\cdot\|_s$ of H^s:

$$(3) \qquad \|f\|_{s,\varepsilon_0}^2 = \int (|\xi|^2 + \varepsilon_0^{-2})^s |\hat{f}(\xi)|^2 \, d\xi.$$

$$(4) \qquad |||f|||_{s,\varepsilon_0}^2 = -s \int_0^{\varepsilon_0} \|f * \rho_\varepsilon\|_{L^2}^2 \varepsilon^{-2s-1} d\varepsilon.$$

In these expressions ε_0 denotes any positive number. He proved that there are positive constants C_1 and C_2 depending only on s such that $C_1 \|f\|_{s,\varepsilon_0} \leqq |||f|||_{s,\varepsilon_0} \leqq C_2 \|f\|_{s,\varepsilon_0}$ for any $f \in H^s$.

For our later use we need the Friedrichs' lemma established by Hörmander ([1], Lemma 5.2) but in somewhat precise form:

LEMMA 1. *Let $a \in \mathscr{D}$, $s < 0$. Then there exists a constant C_s depending on s such that*

$$(5) \qquad \int_0^{\varepsilon_0} \|a(f * \rho_\varepsilon) - (af) * \rho_\varepsilon\|_{L^2}^2 \varepsilon^{-2s-1} \, d\varepsilon$$
$$\leqq C_s \|f\|_{s-1,\varepsilon_0}^2 \left(\int |\hat{a}(\xi)| (1 + |\xi|)^{-s+2} d\xi \right)^2, \quad f \in H^{s-1}, \ 0 < \varepsilon_0 \leqq 1.$$

When s is bounded, C_s is also bounded.

This is immediately verified by estimating the constants C_4 and C_5 considered in his proof of the lemma.

From this lemma we have

COROLLARY. *If $\{\psi_j\}$ is a uniform partition of $\psi \in \mathscr{B}$, then there exists a constant C depending on $\{\psi_j\}$ and s such that*

$$\int_0^{\varepsilon_0} \|\psi_j(f*\rho_\varepsilon) - (\psi_j f)*\rho_\varepsilon\|_{L^2}^2 \varepsilon^{-2s-1} d\varepsilon$$
$$\leqq C|||f|||_{s-1,\varepsilon_0}^2, \qquad f \in H^{s-1},\ 0 < \varepsilon_0 \leqq 1.$$

PROOF. Let a_j be any vector which lies in the support of ψ_j, $j = 1, 2, \cdots$. By the definition of uniform partition of ψ the set $\{\tau_{a_j} \psi_j\}$ forms a bounded subset of $\mathscr{D}$. Hence the set $\{\widehat{\tau_{a_j}\psi_j}\}$ is bounded in $\mathscr{S}$, so that there exists a constant C' depending only on the set $\{\widehat{\tau_{a_j}\psi_j}\}$ such that $(1 + |\xi|)^{-s+n+3}\ |\widehat{\tau_{a_j}\psi_j}| < C'$, $j = 1, 2, \cdots$. Consequently we have

$$\int |\hat{\psi}_j(\xi)|(1 + |\xi|)^{-s+2}\, d\xi$$
$$= \int |\widehat{\tau_{a_j}\psi_j}(\xi)|(1 + |\xi|)^{-s+2} d\xi \leqq C' \int (1 + |\xi|)^{-n-1}\, d\xi < \infty.$$

The preceding lemma together with these inequalities will complete the proof of the corollary.

2. This section is devoted to the proof of Lemma A. Let $\{\phi_j\}$ (resp. $\{\psi_j\}$) be a uniform partition of the unity (resp. of any element $\psi \in \mathscr{B}$).

We shall begin with the proof for the case $s<0$. Since the set {supp. $\phi_j + B_1$} (resp. {supp. $\psi_j + B_1$}), B_1 being the closed unit ball with center 0 in R^n, is bounded in diameter, there exists, by definition, a positive integer l such that at most l of ψ_k (resp. ϕ_k) cannot vanish identically on supp. $\phi_j + B_1$ (resp. supp. $\psi_j + B_1$) for any given j, $j=1, 2, \cdots$. Let f be any element of H^s. Let $0 < \varepsilon_0 < 1$. Then

$$|||\psi_j f|||_{s,\varepsilon_0}^2 = -s \int_0^{\varepsilon_0} \|(\psi_j f)*\rho_\varepsilon\|_{L^2}^2\ \varepsilon^{-2s-1} d\varepsilon$$
$$\leqq -2s \int_0^{\varepsilon_0} \|(\psi_j f)*\rho_\varepsilon - \psi_j(f*\rho_\varepsilon)\|_{L^2}^2 \varepsilon^{-2s-1}\, d\varepsilon - \tag{6}$$
$$- 2s \int_0^{\varepsilon_0} \|\psi_j(f*\rho_\varepsilon)\|_{L^2}^2\ \varepsilon^{-2s-1}\, d\varepsilon.$$

We write $\sum' \phi_k$ to denote the sum of ϕ_k whose support intersects supp. $\psi_j + B_1$. Noting that the number of such ϕ_k is at most l, and that $(\psi_j f)*\rho_\varepsilon = (\psi_j (\sum' \phi_k) f)*\rho_\varepsilon$ and $\psi_j(f*\rho_\varepsilon) = \psi_j((\sum' \phi_k f)*\rho_\varepsilon)$, we get by the Corollary to Lemma 1

$$
\begin{aligned}
& - 2s\int_0^{\varepsilon_0} \|(\psi_j f)*\rho_\varepsilon - \psi_j(f*\rho_\varepsilon)\|_{L^2}^2\, \varepsilon^{-2s-1}\, d\varepsilon \\
(7) \qquad & \leqq -2sC\, |||(\textstyle\sum' \phi_k)f|||_{s-1,\varepsilon_0}^2 \\
& \leqq 2(-s+1)\, C\varepsilon_0^2 l \textstyle\sum' |||\phi_k f|||_{s,\varepsilon_0}^2,
\end{aligned}
$$

where C is a constant depending on $\{\psi_j\}$ and s but not on ε_0.

Combining (6) and (7) and summing up with respect to j, we have

$$
\begin{aligned}
(8) \qquad & \sum_j |||\psi_j f|||_{s,\varepsilon_0}^2 \leqq 2(-s+1)\, C\varepsilon_0^2\, l^2 \sum_j |||\phi_j f|||_{s,\varepsilon_0}^2 - \\
& - \sum_j 2s\int_0^{\varepsilon_0} \|\psi_j(f*\rho_\varepsilon)\|_{L^2}^2 \varepsilon^{-2s-1}\, d\varepsilon.
\end{aligned}
$$

Setting $M = \sup \sum_j |\psi_j(x)|^2$, we have

$$
-\sum_j 2s\int_0^{\varepsilon_0} \|\psi_j(f*\rho_\varepsilon)\|_{L^2}^2\, \varepsilon^{-2s-1}\, d\varepsilon \leqq 2M |||f|||_{s,\varepsilon_0}^2,
$$

which together with (8) yields

$$
(9) \qquad \sum_j |||\psi_j f|||_{s,\varepsilon_0}^2 \leqq 2(-s+1)\, C\, \varepsilon_0^2\, l^2 \sum_j |||\phi_j f|||_{s,\varepsilon_0}^2 + 2M|||f|||_{s,\varepsilon_0}^2.
$$

Now suppose that $\sum_j |||\phi_j f|||_{s,\varepsilon_0}^2 < \infty$. Substituting ψ_j by ϕ_j in (9) (with C', M' in place of C, M) and taking ε_0 so small that $2(-s+1)\, C'\, \varepsilon_0^2\, l^2$ *and* $2(-s+1)\, C\, \varepsilon_0^2\, l^2 < \frac{1}{2}$, we get

$$
(10) \qquad \sum_j |||\phi_j f|||_{s,\varepsilon_0}^2 \leqq 4M' |||f|||_{s,\varepsilon_0}^2,
$$

which together with (9) yields

$$
(11) \qquad \sum_j |||\psi_j f|||_{s,\varepsilon_0}^2 \leqq 2(M+M')|||f|||_{s,\varepsilon_0}^2.
$$

We shall show that (10) and (11) hold for any $f \in H^s$. To this end we consider a sequence of multiplicators α_i such that $\alpha_i f \to f$ in H^s. The inequalities (10) and (11) hold for $\alpha_i f$ since $\sum_j |||\psi_j\, \alpha_i f|||_{s,\varepsilon_0}^2$ is finite. Hence passing to the limit as $i \to \infty$, we see that (10) and (11) hold for any $f \in H^s$. On account of the equivalence of two norms $|||\cdot|||_{s,\varepsilon_0}$ and $\|\cdot\|_s$, we see that the inequality (1) and the second part of the inequalities (2) hold for any $f \in H^s$, $s < 0$.

As for the first part of the inequalities (2) we start with the inequalities:

$$
\begin{aligned}
(12) \qquad & 2\sum_j |||\phi_j f|||_{s,\varepsilon_0}^2 \geqq 2s\sum_j \int_0^{\varepsilon_0} \|(\phi_j f)*\rho_\varepsilon - \phi_j(f*\rho_\varepsilon)\|_{L^2}^2 \varepsilon^{-2s-1} d\varepsilon - \\
& - s\sum_j \int_0^{\varepsilon_0} \|\phi_j(f*\rho_\varepsilon)\|_{L^2}^2\, \varepsilon^{-2s-1}\, d\varepsilon = -J_1 + J_2.
\end{aligned}
$$

As before we can take ε_0 so small that $J_1 < \frac{1}{2}\sum_j |||\phi_j f|||^2_{s,\varepsilon_0}$. Setting $m = \inf \sum_j |\phi_j(x)|^2$, we see from (12) that

$$\sum_j |||\phi_j f|||^2_{s,\varepsilon_0} \geqq \frac{2}{5} m |||f|||^2_{s,\varepsilon_0}, \tag{13}$$

which proves the first part of the inequalities (2) since the two norms $\|\cdot\|_s$ and $|||\cdot|||_{s,\varepsilon_0}$ are equivalent.

The general case will be proved by using induction on s. We assume that the inequalities (1) and (2) hold for $s<s_0$. It then follows that for any $f\in H^{s+1}$, $s<s_0$, we have

$$\begin{aligned}\sum_j \|\psi_j f\|^2_{s+1} &= \sum_j \|\psi_j f\|^2_s + \frac{1}{4\pi^2}\sum_j\sum_{i=1}^n \|\frac{\partial}{\partial x_i}(\psi_j f)\|^2_s \\ &\leqq \sum_j \|\psi_j f\|^2_s + \frac{1}{2\pi^2}\sum_j\sum_{i=1}^n \|\psi_j \frac{\partial f}{\partial x_i}\|^2_s + \\ &\quad + \frac{1}{2\pi^2}\sum_j\sum_{i=1}^n \|\frac{\partial \psi_j}{\partial x_i} f\|^2_s \\ &\leqq C(\|f\|^2_s + \frac{1}{4\pi^2}\sum_{i=1}^n \|\frac{\partial f}{\partial x_i}\|^2_s) \\ &= C\|f\|^2_{s+1},\end{aligned}$$

where C is a constant depending on $\{\psi_j\}$ and s.

Noting that $\sum_j \|\frac{\partial \phi_j}{\partial x_i} f\|^2_s \leqq C'\|f\|^2_s \leqq C''\sum_j \|\phi_j f\|^2_s \leqq C''\sum_j \|\phi_j f\|^2_{s+1}$, where C', C'' are constants depending on $\{\phi_j\}$ and s, we have

$$\begin{aligned}\|f\|^2_{s+1} &= \|f\|^2_s + \frac{1}{4\pi^2}\sum_{i=1}^n \|\frac{\partial f}{\partial x_i}\|^2_s \\ &\leqq C_1(\sum_j \|\phi_j f\|^2_s + \frac{1}{8\pi^2}\sum_j\sum_{i=1}^n \|\phi_j \frac{\partial f}{\partial x_i}\|^2_s) \\ &\leqq C_1(\sum_j \|\phi_j f\|^2_s + \frac{1}{4\pi^2}\sum_j\sum_{i=1}^n \|\frac{\partial}{\partial x_i}(\phi_j f)\|^2_s + \\ &\quad + \frac{1}{4\pi^2}\sum_j\sum_{i=1}^n \|\frac{\partial \phi_j}{\partial x_i} f\|^2_s) \\ &\leqq C_2\sum_j \|\phi_j f\|^2_{s+1}\end{aligned}$$

where C_1, C_2 are constants depending on $\{\phi_j\}$ and s.

Thus we have shown that the inequalities (1) and (2) hold for any s.

Now we turn to the proof of the last part of our lemma. Let f be any element of $\mathscr{L}^s$ such that $\sum_j \|\phi_j f\|_s^2 < \infty$.

Let $\alpha \in \mathscr{D}$ be a function such that α is 1 near the origin. If we put $\alpha_k(x) = \alpha\left(\frac{x}{k}\right)$, $\{\alpha_k\}$ is bounded in $\mathscr{B}$ and forms a sequence of multiplicators. To complete the proof, since H^s is complete it suffices to show that $\{\alpha_k f\}$ is a Cauchy sequence in H^s. We have by (2)

$$\|\alpha_k f - \alpha_{k'} f\|_s^2 \leqq C_{s,\{\phi_j\}} \sum_j \|\phi_j(\alpha_k - \alpha_{k'})f\|_s^2.$$

If k, k' are taken so large that $\phi_j(\alpha_k - \alpha_{k'}) = 0$ for $j = 1, 2, \cdots, N$, then, noting that since $\{\alpha_k\}$ is bounded in $\mathscr{B}$ there exists a constant M such that $\|\phi_j(\alpha_k - \alpha'_k)f\|_s^2 \leqq M\|\phi_j f\|_s^2$ for any j, k and k', we have

$$\|\alpha_k f - \alpha_{k'} f\|_s^2 \leqq C_{s,\{\phi_j\}} M \sum_{j \geqq N} \|\phi_j f\|_s^2,$$

whence it is clear that $\{\alpha_k f\}$ is a Cauchy sequence, which completes the proof.

3. In this section we shall concern ourselves with an application of Lemma A to the estimate of differential inequalities.

To write our differential operators, let $D_j = \frac{1}{2\pi i} \frac{\partial}{\partial x_j}$ for $1 \leqq j \leqq n$. Then if $p = (p_1, p_2, \cdots, p_n)$ is any n-tuple of non-negative integers and ξ is an n-dimensional vector $(\xi_1, \xi_2, \cdots, \xi_n)$, we shall write $p! = p_1!\, p_2! \cdots p_n!$, $|p| = p_1 + p_2 + \cdots + p_n$, $\xi^p = \xi_1^{p_1} \xi_2^{p_2} \cdots \xi_n^{p_n}$ and $D^p = D_1^{p_1} D_2^{p_2} \cdots D_n^{p_n}$.

Let $P(x, D) = \sum_{|p| \leqq m} a_p(x) D^p$ be a differential operator of order m with coefficients $a_p \in \mathscr{B}$. When x is fixed, $P(x, D)$, which we shall write $P_x(D)$, is a differential operator with constant coefficients. Let $M(D)$ be a hypoelliptic differential operator of order m with constant coefficients, i.e. in any domain any distribution solution T of $M(D)\, T = 0$ is indefinitely differentiable. We denote by $M(\xi)$ the polynomial in ξ obtained by substituting ξ for D in $M(D)$. $M^{(p)}(D)$ stands for a differential operator corresponding to the polynomial $\left(\frac{\partial}{\partial \xi_1}\right)^{p_1} \left(\frac{\partial}{\partial \xi_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{p_n} M(\xi)$. $P^{(p)}(x, D)$ and $P_x^{(p)}(D)$ will have obvious meanings.

The symbol C with various subscripts is used to denote a constant, not necessarily the same at each occurrence, which depends only on the variables displayed.

In the sequel we shall assume that $P(x, D)$ is uniformly of type (M), that is, $M(D)$ satisfies the condition:

$$\frac{1}{C} \leqq \frac{1 + |P(x, \xi)|^2}{1 + |M(\xi)|^2} \leqq C, \tag{14}$$

where C is a constant. Then $P(x, D)$ is expressed as $\sum_{j=1}^{N} \beta_j(x) M_j(D)$, $\beta_j \in \mathscr{B}$, where $M_j(D)$ are chosen among $\{P_x(D)\}_{x\in R^n}$.

Our aim of the present section is to show the following proposition, a special case of which is found in Peetre ([2], p. 69).

PROPOSITION. *Let $P(x, D)$ be uniformly of type (M). If $f\in H^t$ and $Pf\in H^s$, then $Mf\in H^s$ and we have*

$$(*)\qquad \|Mf\|_s\leqq C_s\|Pf\|_s+C_{s,t}\|f\|_t.$$

Before proving the proposition, we shall state some lemmas for our later use.

LEMMA 2. *Let $f\in H^t$. If any of M_jf, Mf, P_xf lies in H^s, so do the others and we have the estimates:*

$$(15)\qquad \|M_jf\|_s\leqq\sqrt{2C}\,\|Mf\|_s+C_{s,t}\|f\|_t.$$

$$(16)\qquad \|Mf\|_s\leqq\sqrt{2C}\,\|P_xf\|_s+C_{s,t}\|f\|_t.$$

$$(17)\qquad \|M^{(p)}f\|_s,\ \|M_j^{(p)}f\|_s\leqq\varepsilon\|Mf\|_s+C_{s,t,\varepsilon}\|f\|_t,\ \varepsilon>0,\ |p|>0.$$

$$(18)\qquad \|M_jf\|_t\leqq\varepsilon\|Mf\|_s+C_{s,t,\varepsilon}\|f\|_t,\ t<s,\ \varepsilon>0.$$

PROOF. Since M is hypoelliptic, it follows that $M(\xi)\to\infty$ as $|\xi|\to\infty$. Hence from (14) we get

$$(1+|\xi|^2)^s|M_j(\xi)|^2\leqq 2C(1+|\xi|^2)^s|M(\xi)|^2+C^2_{s,t}(1+|\xi|^2)^t.$$

Consequently, if $Mf\in H^s$, then $\|M_jf\|_s<\infty$ and we have the inequality (15). The other cases may be proved similarly, so the proof is omitted.

LEMMA 3. *Let $f\in H^s\cap H^t$. For any $\phi\in\mathscr{B}$ we have*

$$\|\phi f\|_s\leqq(\sup|\phi(x)|+\varepsilon)\|f\|_s+C_{s,t,\varepsilon}\|f\|_t,\ \varepsilon>0.$$

PROOF. The estimate has been essentially established by Peetre ([2], p. 19) for the case $s\geqq 0$, to which the general case may be reduced by considering a function $f_1\in H^{s+2k}$ with $\left(1-\frac{\Delta}{4\pi^2}\right)^k f_1=f$, where k is a positive integer such that $2k+s\geqq 0$. The proof is not supplied here since it is only a matter of calculations often used in Peetre [2].

LEMMA 4. *Let $B^r_{x_0}$ be a ball with center x_0 and radius r, then for any $f\in H^s{}_{B^r_{x_0}}\cap H^t$ we have*

$$(19)\qquad \|(\beta_j(x)-\beta_j(x_0))f\|_s\leqq rC'\|f\|_s+C_{s,t,r}\|f\|_t$$

where $C'=2\sup_j\sup_x|grad.\ \beta_j|+1$.

PROOF. Let ψ_r be a fixed function of $\mathscr{D}$ such that $0\leqq\psi_r(x)\leqq 1$, $\psi_r(x)=1$

for $|x| \leqq 1$ and $\psi(x) = 0$ for $|x| \geqq 2$. Setting $\psi_{r,x_0}(x) = \psi\left(\frac{x-x_0}{r}\right)$, we have $(\beta_j(x) - \beta_j(x_0))f(x) = (\beta_j(x) - \beta_j(x_0))\psi_{r,x_0}(x)f(x)$. Now we can use Lemma 3 to establish (19). The details are omitted.

THE PROOF OF THE PROPOSITION. (*a*) First we shall show that the proposition is valid if the inequality (*) holds for any function of $\mathcal{D}_{L^2}$ and for any s, t. Suppose that $f \in H^t$ and $Pf \in H^s$. Mf lies in an $H^{s'}$. Put $\sigma = \min(s-1, s')$. Let $\{\rho_\varepsilon\}$ be a sequence of regularizations considered in Section 1. Since $f*\rho_\varepsilon \in \mathcal{D}_{L^2}$, we have by hypothesis

$$(20) \qquad \|M(f*\rho_\varepsilon)\|_{\sigma+1} \leqq C_{\sigma+1}\|P(f*\rho_\varepsilon)\|_{\sigma+1} + C_{\sigma+1,t}\|f*\rho_\varepsilon\|_t.$$

Noting that $M_jf \in H^\sigma$ by Lemma 2, we get from (20)

$$(21) \qquad \|Mf*\rho_\varepsilon\|_{\sigma+1} \leqq C_{\sigma+1}\|(Pf)*\rho_\varepsilon\|_{\sigma+1} + C_{\sigma+1}\|\sum_j \beta_j(M_jf)*\rho_\varepsilon - \sum_j \beta_j((M_jf)*\rho_\varepsilon)\|_{\sigma+1}$$
$$+ C_{\sigma+1,t}\|f*\rho_\varepsilon\|_t.$$

On the other hand, $\|f*\rho_\varepsilon\|_t \to \|f\|_t$ and $\|Pf*\rho_\varepsilon\|_{\sigma+1} \to \|Pf\|_{\sigma+1}$ as $\varepsilon \to 0$ since $f \in H^t$ and $Pf \in H^{\sigma+1}$. By Friedrichs' lemma ([2], p. 22), the second term of the right side of (21) tends to zero as $\varepsilon \to 0$. Therefore from (21) we see that $\{\|Mf*\rho_\varepsilon\|_{\sigma+1}\}$ is bounded, so that $Mf*\rho_\varepsilon \to Mf$ in $H^{\sigma+1}$ as $\varepsilon \to 0$. Hence we have from (20)

$$(22) \qquad \|Mf\|_{\sigma+1} \leqq C_{\sigma+1}\|Pf\|_{\sigma+1} + C_{\sigma+1,t}\|f\|_t.$$

By repeating this process if necessary, we can see that $Mf \in H^s$ and the inequality (*) holds, as desired.

(*b*) To complete the proof, it remains to show the inequality (*) for any $f \in \mathcal{D}_{L^2}$. Since $\|f\|_t$ is an increasing function of t, we can assume $t < s$ without loss of generality.

Let $\{\phi_j\}$ be a uniform partition of the unity such that the diameter of each supp. ϕ_j is less than r, where r is a fixed number chosen so small that $8CC'Nr < 1$. Let x_j be any point of supp. ϕ_j. We have

$$(23) \qquad \|\phi_jMf\|_s \leqq \|M(\phi_jf)\|_s + \sum_{|q|>0} \frac{1}{q!}\|(D^q\phi_j)(M^{(q)}f)\|_s.$$

Using Lemma 2 we have

$$(24) \qquad \|M(\phi_jf)\|_s \leqq \sqrt{2C}\,\|P_{x_j}(\phi_jf)\|_s + C_{s,t}\|\phi_jf\|_t$$
$$\leqq \sqrt{2C}\,\|(P - P_{x_j})(\phi_jf)\|_s + \sqrt{2C}\,\|P(\phi_jf)\|_s + C_{s,t}\|\phi_jf\|_t.$$

On the other hand, we have by Lemma 2 and Lemma 4

$$\|(P - P_{x_j})(\phi_j f)\|_s \leqq \sum_{k=1}^{N} \|(\beta_k(x) - \beta_k(x_j))M_k(\phi_j f)\|_s \tag{25}$$

$$\leqq r\, C' \sum_k \|M_k(\phi_j f)\|_s + C_{s,t} \sum_k \|M_k(\phi_j f)\|_t$$

$$\leqq \sqrt{2C}\, C'Nr \|M(\phi_j f)\|_s + C'_{s,t}\varepsilon \|M(\phi_j f)\|_s + C_{s,t} \|\phi_j f\|_t$$

and also

$$\|P(\phi_j f)\|_s \leqq \|\phi_j Pf\|_s + \sum_{|q|>0} \frac{1}{q!} \|(D^q \phi_j) \sum_k \beta_k(x) M_k^{(q)} f\|_s \tag{26}$$

$$\leqq \|\phi_j Pf\|_s + C_s \sum_{|q|>0} \sum_k \|(D^q \phi_j)(M_k^{(q)} f)\|_s.$$

(24) together with (25) and (26) yields

$$(1 - 2CC'Nr - \sqrt{2C}\, C'_{s,t}\varepsilon)\|M(\phi_j f)\|_s$$

$$\leqq \sqrt{2C}\, \|\phi_j Pf\|_s + C_s \sum_{|q|>0} \sum_k \|(D^{(q)} \phi_j)(M_k^{(q)} f)\|_s + C_{s,t} \|\phi_j f\|_t,$$

in which we take ε so small that $\sqrt{2C}\, C'_{s,t}\varepsilon < \frac{1}{4}$. Then

$$\|M(\phi_j f)\|_s \leqq 2\sqrt{2C}\, \|\phi_j Pf\|_s + C_s \sum_{|q|>0} \sum_k \|(D^q \phi_j)(M_k^{(q)} f)\|_s + \tag{27}$$

$$+ C_{s,t} \|\phi_j f\|_t.$$

(23) and (27) give

$$\begin{aligned} \|\phi_j Mf\|_s \leqq 2\sqrt{2C}\, \|\phi_j Pf\|_s &+ C_s \{ \sum_{|q|>0} \sum_k \|(D^q \phi_j)(M_k^q f)\|_s + \\ &+ \sum_{|q|>0} \|(D^q \phi_j)(M^{(q)} f)\|_s \} + C_{s,t} \|\phi_j f\|_t, \end{aligned} \tag{28}$$

whence

$$\begin{aligned} \|\phi_j Mf\|_s^2 \leqq 8Cl \|\phi_j Pf\|_s^2 &+ C_s \{ \sum_{|q|>0} \sum_k \|(D^q \phi_j)(M_k^{(q)} f)\|_s^2 + \\ &+ \sum_{|q|>0} \|(D^q \phi_j)(M^{(q)} f)\|_s^2 \} + C_{s,t} \|\phi_j f\|_t^2, \end{aligned} \tag{29}$$

where l is the number of terms on the right side of (28). Summing up (29) with respect to j and using Lemma A we have

$$\|Mf\|_s^2 \leqq C_s \|Pf\|_s^2 + \tag{30}$$

$$+ C'_s \{ \sum_{|q|>0} \sum_k \|M_k^{(q)} f\|_s^2 + \sum_{|q|>0} \|M^{(q)} f\|_s^2 \} + C_{s,t} \|f\|_t^2.$$

Since $\|M_k^{(q)} f\|_s \leqq \varepsilon \|Mf\|_s + C_{s,t,\varepsilon} \|f\|_t$ and $\|M^{(q)} f\|_s \leqq \varepsilon \|Mf\|_s + C_{s,t,\varepsilon} \|f\|_t$ by Lemma 2, we can choose ε so small that we may obtain from (30)

$$\|Mf\|_s^2 \leqq C_s\|Pf\|_s^2 + C_{s,t}\|f\|_t^2,$$

whence

$$\|Mf\|_s \leqq C_s\|Pf\|_s + C_{s,t}\|f\|_t.$$

Thus the proof of the proposition is complete.

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