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Extremal Length of Level Surfaces and Orthogonal Trajectories

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Introduction

In [6] we computed the extremal length of a family of collections of parallel segments in the (x, y)-plane. The vertical lines can be regarded as the orthogonal trajectories of the harmonic function y. From this point of view we shall generalize a part of the results in [6].

The space will be an n-dimensional space $\mathscr{E}$ in the sense of Brelot-Choquet [1]. We shall use terminologies of the case $n \geqq 3$ although the results are valid in the case $n=2$ too. In this case we need some modifications. When we say that a set is measurable in this paper, we mean that it is Lebesgue measurable.

In §1 we introduce orthogonal trajectories and regular tubes for a given harmonic function. Their existence can be proved just as for Green lines and regular tubes consisting of Green arcs, and so the proof is omitted. Then we define harmonic flows and subflows as in the two-dimensional case which was treated in [3] and [5]. The notion of extremal length (with weight) is introduced in §2 and the extremal length of harmonic subflows is computed in §3. In §4 an extremal length in a more general sense is considered and Theorems 1 and 2 in [6] are generalized. The extremal length of the family of all orthogonal trajectories is calculated in §5 with the aid of a theorem on the decomposition of the domain of definition into disjoint harmonic subflows. Finally the extremal length of level surfaces is computed in §6.

§ 1. Harmonic flows

Let G be an open set in $\mathscr{E}$ and $H(P)$ be a harmonic function in G which is not constant in any component of G. A non-empty set of the form $\{P; H(P)=\text{const.}\}$ will be called a *level surface* or an *equipotential surface* of H. It consists of a countable number of $(n-1)$-dimensional analytic surfaces, of isolated points at infinity and of an $(n-2)$-dimensional relatively closed subset of G where $\operatorname{grad} H=0$; see Lemme 12 of [1]. A point with $\operatorname{grad} H=0$ will be called *critical.* Excluding the set of all points at infinity and critical points from a level surface, we call the rest the *regular part* of the level surface.

Through every non-critical point of G which is not a point at infinity, passes an analytic curve on which $\mathrm{grad}\, H \neq 0$ and whose tangent at every point is parallel to the vector $\mathrm{grad}\, H$ (cf. [1], p. 232). A maximal curve in G with this property is called an *orthogonal trajectory*. Since H increases or decreases strictly along every orthogonal trajectory, no orthogonal trajectory is a closed curve. No two orthogonal trajectories intersect each other. Each orthogonal trajectory clusters to a subset of the union of the boundary of G and the set of critical points, unless it terminates at a point at infinity.

Let σ be an $(n-1)$-dimensional domain with piecewise smooth boundary on the regular part of a level surface such that its closure σ^a is contained in the regular part. Let $d_1 < d_2$ and let c be an orthogonal trajectory which passes through a point P_c of σ^a. Suppose that there is a subarc $c(d_1, d_2)$ of c on which H assumes all the values of $[d_1, d_2]$. When this is true for every c intersecting σ^a, $\bigcup_c c(d_1, d_2)$ as a point set is called a *regular compact tube* (cf. p. 233 of [1]). Its interior is a domain and called a *regular tube*. We shall call the part of the boundary on which $H(P)=d_1$ (d_2 resp.) the lower (upper resp.) base of the tube. One sees easily that, for any two points P and Q of any orthogonal trajectory, there is a regular tube containing P and Q.

Now let τ be any $(n-1)$-dimensional domain on the regular part of a level surface. We shall call the bundle of orthogonal trajectories which pass through τ a *harmonic flow* (for $H(P)$ through τ). It is easy to observe that a harmonic flow F is a domain as a point set. We shall call it the *domain of* F and denote it by $[F]$. We denote by c_P the orthogonal trajectory passing through $P \in [F]$. Let X be any subset of $[F]$. We shall call the set $\{Q \in \tau;\ c_Q \cap X \neq \emptyset\}$ the *projection of* X *on* τ, and denote it by $p(X)$.

A subbundle Γ of a harmonic flow which meets τ at a set measurable in the $(n-1)$-dimensional sense will be called a *harmonic subflow*. As a point set it will be denoted by $[\Gamma]$. If the $(n-1)$-dimensional measure of $[\Gamma] \cap \tau$ is positive (null resp.), it is called a *positive* (*null* resp.) subflow. We shall say that a flow or a subflow is *finite* if the variation of H along each trajectory is finite. Let E be a measurable subset of the regular part of a level surface. We shall call the surface integral $\int_E \partial H/\partial \nu d\sigma$ of the normal derivative $\partial H/\partial \nu = |\mathrm{grad}\, H|$ the *flux* on E and denote it by $\varphi(E)$. For any tube the flux on one base is equal to the flux on the other.

Let F be a harmonic flow passing through τ. We shall show, for any subflow Γ of F and any non-negative (Lebesgue) measurable function f in $\mathscr{E}$, that $\int_{c_Q} f/|\mathrm{grad}\, H| ds$ is a measurable function defined for a.e. Q on τ, and that

$$(1) \qquad \int_{[\Gamma]} f dv = \int_{[\Gamma]\cap\tau} \left(\int_{c_Q} \frac{f}{|\text{grad } H|} ds \right) d\varphi(Q)$$

is valid, where dv is the volume element in $\mathscr{E}$. It is sufficient to prove

$$(2) \qquad \int_{[F]} f dv = \int_{\tau} \left(\int_{c_Q} \frac{f}{|\text{grad } H|^2} dt \right) d\varphi(Q) = \int_{\tau} \left(\int_{c_Q} \frac{f}{|\text{grad } H|} ds \right) d\varphi(Q),$$

where H is taken as a variable on each orthogonal trajectory and denoted by t. For, applying (2) to $f\chi_{[\Gamma]}$ we obtain (1), where $\chi_{[\Gamma]}$ is the characteristic function of $[\Gamma]$.

We denote by $d\sigma$ the $(n-1)$-dimensional surface element on a level surface, and set $d\varphi = |\text{grad } H| d\sigma$. We have $dv = ds d\sigma = |\text{grad } H|^{-2} dt d\varphi$ where ds is taken along an orthogonal trajectory. Hence $\int_{[F]} f dv = \iint f |\text{grad} H|^{-2} dt d\varphi$. Since the flux is invariant on any section of tube, $d\varphi$ at P is equal to $d\varphi$ at $p(P)$ on τ. For this reason we can identify $[F]$ with a domain in the product space $\tau \times \{-\infty < t < \infty\}$ with the product measure of φ and the linear measure on the t-axis, and apply Fubini's theorem. We infer that $\int_{c_Q} f |\text{grad } H|^{-2} dt$ is a measurable function defined for a.e. Q on τ and that $\iint f |\text{grad } H|^{-2} dt d\varphi = \int_{\tau} \left(\int_{c_Q} f |\text{grad } H|^{-2} dt \right) d\varphi(Q)$. Thus (2) is derived.

§ 2. Extremal length with weight

By a measure in $\mathscr{E}$ we shall mean a countably additive non-negative set-function, defined on a σ-field of sets containing the Borel class in $\mathscr{E}$ and admitting ∞. Let $\mathscr{M}$ be a class of measures none of which is identically zero. A non-negative (Lebesgue) measurable function ρ in $\mathscr{E}$ will be called *admissible* (in association with $\mathscr{M}$) if $\int \rho d\mu$ is well-defined[1)] and $\geqq 1$ for each $\mu \in \mathscr{M}$. For a non-negative measurable function π defined in $\mathscr{E}$, the *module* $M_p(\mathscr{M}; \pi)$ *of* $\mathscr{M}$ *with weight* π is defined by $\inf \int \pi \rho^p dv$ $(0 < p < \infty)$, where ρ is admissible. The *extremal length* $\lambda_p(\mathscr{M}; \pi)$ *of* $\mathscr{M}$ *with weight* π is defined by $1/M_p(\mathscr{M}; \pi)$. An admissible ρ is called *extremal* if $\int \pi \rho^p dv = M_p(\mathscr{M}; \pi)$.

1) This means that, if $\mathfrak{S}_\mu$ is the σ-field on which μ is defined, there is a set $E_\mu \in \mathfrak{S}_\mu$ such that $\mu(\mathscr{E} - E_\mu) = 0$ and the restriction of ρ to E_μ is an $\mathfrak{S}_\mu$-measurable function.

The present definition of extremal length is a generalization of the extremal length with weight π in [4] and a special case of the one considered by Fuglede [2]. In case $\pi\equiv 1$, we write simply $M_p(\mathscr{M})$ and $\lambda_p(\mathscr{M})$.

On account of Vitali-Carathéodory's theorem ([7], p. 75) we find a Borel measurable function ρ' which is equal to ρ a.e. and $\geqq\rho$ everywhere in $\mathscr{E}$. Hence we obtain the same value of $M_p(\mathscr{M};\pi)$ if we restrict admissible ρ to be Borel measurable.[2)]

We shall use the following properties of $M_p(\mathscr{M};\pi)$; see [2].

(3) $M_p(\mathscr{M};\pi)\leqq M_p(\mathscr{M}';\pi)$ if $\mathscr{M}\subset\mathscr{M}'$.

(4) $M_p(\bigcup_n \mathscr{M}_n;\pi)\leqq\sum_n M_p(\mathscr{M}_n;\pi)$ for any countable family $\{\mathscr{M}_n\}$.

(5) $M_p(\mathscr{M};\pi)=M_p(\mathscr{M}-\mathscr{M}';\pi)$ if $M_p(\mathscr{M}';\pi)=0$.

(6) $M_p(\bigcup_n \mathscr{M}_n;\pi)=\sum_n M_p(\mathscr{M}_n;\pi)$ if there are mutually disjoint measurable sets $\{A_n\}$ such that, for every measure μ of each $\mathscr{M}_n$, there is a set $A_\mu\in\mathfrak{S}_\mu$ contained in A_n and satisfying $\mu(\mathscr{E}-A_\mu)=0$, where $\mathfrak{S}_\mu$ is the σ-field on which μ is defined.

With a given family $\{\gamma\}$ of locally rectifiable curves in $\mathscr{E}$ we associate $\mathscr{M}$ as follows: Take $\gamma\in\{\gamma\}$ and let $\{P_\gamma(s);\ s\in I_\gamma=\text{an interval}\}$ be a representation of γ in terms of arc-length. Let E be a set in $\mathscr{E}$ such that the set $\{s\in I_\gamma;\ P_\gamma(s)\in E\}$ is linearly measurable. We shall denote by $\mathfrak{S}_\gamma$ the class of such sets E. We define the value $\mu_\gamma(E)$ by the linear measure of $\{s\in I_\gamma;\ P_\gamma(s)\in E\}$. In such a way we obtain a measure μ_γ defined on the class $\mathfrak{S}_\gamma$. When $\{\mu_\gamma;\ \gamma\in\{\gamma\}\}$ is taken for $\mathscr{M}$, $M_p(\mathscr{M};\pi)$ will be denoted by $M_p(\{\gamma\};\pi)$ and called the module of $\{\gamma\}$ with weight π. The extremal length $\lambda_p(\{\gamma\};\pi)$ of $\{\gamma\}$ with weight π is defined by $1/M_p(\{\gamma\};\pi)$. The extremal length of a family of surfaces with weight will be defined in §6.

§ 3. Extremal length of harmonic subflows

In §1 we observed that $\int_{c_Q}(|\operatorname{grad} H|/\pi)^{1/(p-1)}ds$ is a measurable function defined for a.e. Q on τ. *Hereafter we shall write simply g for* $|\operatorname{grad} H|/\pi$.

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2) We can show furthermore that we may restrict ρ to be lower semicontinuous. Actually, given an admissible ρ, we take a Borel measurable function π' which is equal to π a.e. and apply Theorem 6.10 at p. 71 of [7] and Vitali-Carathéodory's theorem to find a decreasing sequence $\{\rho_n\}$ of lower semicontinuous functions which are $\geqq\rho$ and satisfy

$$\lim_{n\to\infty}\int\pi\rho_n dv=\lim_{n\to\infty}\int\pi'\rho_n dv=\int\pi'\rho\, dv=\int\pi\rho\, dv.$$

First we prove

LEMMA. *The extremal length $\lambda_p(\Gamma;\pi)$ of any null subflow Γ is infinite.*

PROOF. We note that $[\Gamma]$ is of measure zero because of the relation

$$\int_{[\Gamma]} dv = \int_{[\Gamma]\cap\tau}\left(\int_{c_Q}\frac{1}{|\mathrm{grad}\,H|}ds\right)d\varphi(Q)$$

as obtained in (1). Consider the function $\rho=\infty$ on $[\Gamma]$ and $=0$ elsewhere. It is admissible in association with Γ and $\int\pi\rho^p dv=0$. Hence $M_p(\Gamma;\pi)=0$.

On account of (5) we obtain

COROLLARY. *If Γ is a harmonic subflow and Γ' is a null subflow, then $\lambda_p(\Gamma;\pi)=\lambda_p(\Gamma-\Gamma';\pi)$.*

The following theorem is the main result.

THEOREM 1. *Let Γ be a harmonic subflow passing through τ. If $p>1$, then*

$$M_p(\Gamma;\pi)=\int_{[\Gamma]\cap\tau}\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi(Q). \tag{7}$$

If $0<\int_{c_Q}g^{1/(p-1)}ds<\infty$ for a.e. $Q\in[\Gamma]\cap\tau$,[3] *an extremal function is given by*

$$\rho_0(P)=\begin{cases}\dfrac{g^{\frac{1}{p-1}}}{\int_{c_P}g^{\frac{1}{p-1}}ds} & \text{for } P\in[\Gamma-\Gamma'],\\ \infty & \text{for } P\in[\Gamma'],\\ 0 & \text{for } P\in\mathscr{E}-[\Gamma],\end{cases}$$

where

$$\Gamma'=\left\{c_Q\in\Gamma;\ \int_{c_Q}g^{\frac{1}{p-1}}ds=0 \text{ or } \infty\right\}.$$

PROOF. Let ρ be an admissible function and assume $0<\int_c g^{1/(p-1)}ds<\infty$

3) Namely, the $(n-1)$-dimensional measure of the set of points Q with $\int_{c_Q}g^{1/(p-1)}ds=0$ or ∞ is zero.

for every $c\in\Gamma$. We apply Hölder's inequality to $1\leqq\int_{c_Q}\rho ds$ and derive

$$1\leqq\left(\int_{c_Q}\frac{\pi\rho^p}{|\mathrm{grad}\,H|}ds\right)^{\frac{1}{p}}\left(\int_{c_Q}\left(\frac{|\mathrm{grad}\,H|}{\pi}\right)^{\frac{1}{p-1}}ds\right)^{1-\frac{1}{p}}$$

or

$$\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}\leqq\int_{c_Q}\frac{\pi\rho^p}{|\mathrm{grad}\,H|}ds.$$

Using (1) we obtain

$$\int_{[\Gamma]\cap\tau}\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi(Q)\leqq\int_{[\Gamma]\cap\tau}\int_{c_Q}\frac{\pi\rho^p}{|\mathrm{grad}\,H|}dsd\varphi=\int_{[\Gamma]}\pi\rho^p dv.$$

Hence

$$\int_{[\Gamma]\cap\tau}\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi(Q)\leqq M_p(\Gamma;\pi).$$

On the other hand, we note that ρ_0 is measurable and check that $\int_{c_Q}\rho_0 ds\geqq 1$ for every $Q\in[\Gamma]\cap\tau$. We infer by (1)

$$M_p(\Gamma;\pi)\leqq\int\pi\rho_0^p dv=\int_{[\Gamma]}\frac{\pi g^{\frac{p}{p-1}}}{\left(\int g^{\frac{1}{p-1}}ds\right)^p}dv=\int_{[\Gamma]\cap\tau}\int_{c_Q}\frac{g^{\frac{1}{p-1}}}{\left(\int g^{\frac{1}{p-1}}ds\right)^p}dsd\varphi(Q)$$
$$=\int_{[\Gamma]\cap\tau}\left(\int g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi.$$

Thus (7) is concluded and ρ_0 is extremal. We obtain the same conclusion if $0<\int_{c_Q}g^{1/(p-1)}ds<\infty$ for a.e. $Q\in[\Gamma]\cap\tau$, because $[\Gamma']$ is of measure zero as noted in the proof the Lemma.

If $\int_{c_Q}g^{1/(p-1)}ds=0$ for a set E of points $Q\in[\Gamma]\cap\tau$ of positive $(n-1)$-dimensional measure, $\pi=\infty$ a.e. (=except for a set of n-dimensional measure zero) on the set $A=[\{c_Q;\,Q\in E\}]$. Since ρ is positive on a subset of A of positive measure, both sides of (7) are equal to ∞.

Next we consider the case where $\int_{c_Q}g^{1/(p-1)}ds=\infty$ for each $Q\in[\Gamma]\cap\tau$. We exclude from G the set of points at infinity and critical points and denote the remaining open set by G'. First we assume that τ is relatively compact in G', and approximate G' by an increasing sequence $\{G_n\}$ of open sets such that $\tau\subset G_1$ and $G_n\cup\partial G_n\subset G_{n+1}$. We denote by $c_Q^{(n)}$ the orthogonal trajectory in G_n which starts from $Q\in\tau$. Evidently $c_Q^{(n)}\nearrow c_Q$ as $n\to\infty$. We set $\pi_n=\max(\pi, 1/n)$ in $\mathscr{E}$. First assume $\int_{c_Q^{(1)}}(|\mathrm{grad}\, H|/\pi_1)^{1/(p-1)}ds\geqq 1$ for each $Q\in[\Gamma]\cap\tau$. Since $\int_{c_Q^{(n)}}(|\mathrm{grad}\, H|/\pi_n)^{1/(p-1)}ds$ is a bounded function of Q on $[\Gamma]\cap\tau$,

$$M_p(\Gamma;\pi)\leqq M_p(\{c_Q^{(n)};\, Q\in[\Gamma]\cap\tau\};\pi_n)$$

$$=\int_{[\Gamma]\cap\tau}\left(\int_{c_Q^{(n)}}\left(\frac{|\mathrm{grad}\, H|}{\pi_n}\right)^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi$$

$$\searrow\int_{[\Gamma]\cap\tau}\left(\int_{c_Q}\left(\frac{|\mathrm{grad}\, H|}{\pi}\right)^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi=0 \quad\text{as}\quad n\to\infty.$$

If $\int_{c_Q^{(1)}}(|\mathrm{grad}\, H|/\pi_1)^{1/(p-1)}ds\geqq 1$ is not true for $Q\in[\Gamma]\cap\tau$, there is n such that $\int_{c_Q^{(n)}}(|\mathrm{grad}\, H|/\pi_n)^{1/(p-1)}ds\geqq 1$. Consequently, by the aid of (4) we can conclude $M_p(\Gamma;\pi)=0$. We obtain the same conclusion on account of (4) even if τ is not relatively compact in G'.

In the general case, $\int_{c_Q}g^{1/(p-1)}ds$ is a measurable function defined for a.e. Q on τ as observed in §1. Thus Γ' is a harmonic subflow. We have already established (7) in case $\int_{c_Q}g^{1/(p-1)}ds=0$ for a set of points $Q\in[\Gamma]\cap\tau$ of positive $(n-1)$-dimensional measure. Hence we assume that $\Gamma_0=\{c_0\in\Gamma;\int_{c_Q}g^{1/(p-1)}ds=0\}$ is a null subflow. By the Lemma, $M_p(\Gamma_0;\pi)=0$. Hence we have $M_p(\Gamma';\pi)=0$ and

$$M_p(\Gamma;\pi)=M_p(\Gamma-\Gamma';\pi)=\int_{[\Gamma-\Gamma']\cap\tau}\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi$$

$$=\int_{[\Gamma]\cap\tau}\left(\int_{c_Q}g^{\frac{1}{p-1}}ds\right)^{1-p}d\varphi.$$

COROLLARY. *The extremal length $\lambda_2(\Gamma)$ of any positive finite subflow is finite.*

§ 4. Generalizations

Let us consider the following definition of extremal length. Let κ be a non-negative (Lebesgue) measurable function in $\mathscr{E}$. A non-negative measurable function ρ in $\mathscr{E}$ will be called κ-admissible in association with a family $\{\gamma\}$ of locally rectifiable curves if $\int_\gamma \kappa\rho ds$ is well-defined in the Lebesgue sense and $\geqq 1$ for each γ. The module $M_p(\{\gamma\}; \pi, \kappa)$ is defined by $\inf \int \pi\rho^p dv$. If there is no κ-admissible ρ, we set $M_p(\{\gamma\}; \pi, \kappa)=\infty$.

If we take a harmonic subflow Γ for $\{\gamma\}$, $M_p(\Gamma; \pi, \kappa)=M_p(\Gamma; \pi/\kappa^p)$ for κ which is positive and finite on $[\Gamma]$. Actually we can prove more, namely,

THEOREM 2. *Let Γ be a harmonic subflow, and π and κ be non-negative measurable functions in $\mathscr{E}$. Set $E_\kappa^0=\{P\in\mathscr{E}; \kappa(P)=0\}$, $E_\kappa^\infty=\{P\in\mathscr{E}; \kappa(P)=\infty\}$, $E_\pi^\infty=\{P\in\mathscr{E}; \pi(P)=\infty\}$ and $E=E_\kappa^0\cup(E_\kappa^\infty\cap E_\pi^\infty)$. If the linear measure of $c-E$ is positive for each $c\in\Gamma$, then*

$$M_p(\Gamma; \pi, \kappa)=M_p\left(\Gamma; \frac{\pi}{\kappa^p}\right),$$

where $\pi/\kappa^p=\infty$ if $\kappa=0$ or if $\pi=\infty$.

PROOF. a). $M_p(\Gamma; \pi, \kappa)\leqq M_p(\Gamma; \pi/\kappa^p)$. We suppose $M_p(\Gamma; \pi/\kappa^p)<\infty$ and take ρ admissible in association with Γ such that $\int(\pi/\kappa^p)\rho^p dv<\infty$. Since $\pi/\kappa^p=\infty$ on E, $\rho=0$ a.e. there. Therefore the linear measure of $E\cap\{\rho>0\}\cap c$ is zero for a.e. $c\in\Gamma$, i.e. except for c belonging to a null subflow of Γ. If c is non-exceptional, we change the values of ρ on $E\cap c$ so that it is zero everywhere on $E\cap c$. By this change the values of $\int_c \rho ds$ and $\int(\pi/\kappa^p)\rho^p dv$ are not altered. If c is exceptional, namely, if the linear measure of $E\cap\{\rho>0\}\cap c$ is positive, we change the values of ρ on c so that it is zero on $E\cap c$ and is equal to ∞ on $c-E$. By this change, the value of $\int(\pi/\kappa^p)\rho^p dv$ is invariant and the new $\int\rho ds=\infty$. Thus we may assume from the beginning that $\rho=0$ on E. Now we consider ρ' which is equal to 0 on E, to ρ/κ on $\mathscr{E}-E-E_\kappa^\infty$ and to ρ_1 on $E_\kappa^\infty-E$, where ρ_1 is a measurable function in $\mathscr{E}$ which is positive on $E_\kappa^\infty-E$ and satisfies $\int_{E_\kappa^\infty-E}\pi\rho_1^p dv<\varepsilon$ for any given $\varepsilon>0$. Then

$$\int_c \kappa\rho' ds = \int_{c-E-E_\kappa^\infty} \rho ds + \int_{(c-E)\cap E_\kappa^\infty} \kappa\rho_1 ds$$
$$= \int_{c-E-E_\kappa^\infty} \rho ds + \int_{(c-E)\cap E_\kappa^\infty} \infty\cdot ds \geqq \int_c \rho ds \geqq 1$$

and

$$M_p(\Gamma;\pi,\kappa) \leqq \int \pi\rho'^p dv = \int_{\mathscr{E}-E-E_\kappa^\infty} (\pi/\kappa^p)\rho^p dv + \int_{E_\kappa^\infty-E} \pi\rho_1^p dv$$
$$\leqq \int (\pi/\kappa^p)\rho^p dv + \varepsilon.$$

There follows a).

b). $M_p(\Gamma;\pi/\kappa^p) \leqq M_p(\Gamma;\pi,\kappa)$. We suppose $M_p(\Gamma;\pi,\kappa)<\infty$ and take ρ' such that $\int_c \kappa\rho' ds \geqq 1$ for each $c\in\Gamma$ and $\int \pi\rho'^p dv<\infty$. We may assume $\rho'=0$ on E_κ^0. In the same way as in a) we can show that we may assume $\rho'=0$ on $E_\kappa^\infty \cap E_\pi^\infty$. Therefore we suppose $\rho'=0$ on E from the beginning. We set $\kappa\rho'=\rho$ and have $\int_c \rho ds = \int_c \kappa\rho' ds \geqq 1$ and

$$M_p\left(\Gamma;\frac{\pi}{\kappa^p}\right) \leqq \int_{\mathscr{E}-E} \frac{\pi}{\kappa^p}\rho^p dv = \int_{\mathscr{E}-E-E_\kappa^\infty} \pi\rho'^p dv + \int_{E_\kappa^\infty-E} 0 dv \leqq \int \pi\rho'^p dv.$$

This establishes b).

Remark. In the general case, the two extremal lengths may not coincide. For instance, if $\kappa\equiv 0$ and $\pi\equiv 1$ and if Γ consists of only one orthogonal trajectory, then $M_p(\Gamma;1,0)=\infty$ and $M_p(\Gamma;\infty)=0$.

As a special case we consider a finite-valued measurable function $\kappa\geqq 0$ in $\mathscr{E}$. We shall denote by Γ the family

(8) $\{c\in F; c\cap\{\kappa>0\}$ is linearly measurable on c and its linear measure is positive$\}$.

Then, if $p>1$,

$$M_p(\Gamma;\pi,\kappa) = M_p\left(\Gamma;\frac{\pi}{\kappa^p}\right) = \int_{[\Gamma]\cap\tau}\left(\int_{c_Q}\left(\frac{\kappa^p|\text{grad } H|}{\pi}\right)^{\frac{1}{p-1}} ds\right)^{1-p} d\varphi(Q)$$

by Theorems 1 and 2.

Let κ be again a finite-valued measurable function in $\mathscr{E}$. Suppose that it is expressed as a finite or infinite sum $\kappa_1+\kappa_2+\cdots$ of non-negative measurable functions in $\mathscr{E}$ such that at most one of them is positive at each point of $[F]$. We define Γ_n for κ_n like Γ in (8), and set $\Gamma=\bigcup_n \Gamma_n$. We see by (6) that $\inf\int \pi\rho^p dv$, taken with respect to ρ satisfying $\int_{c\cap\{\kappa_n>0\}}\kappa\rho ds\geqq 1$ for all $c\in\Gamma_n$ $(n=1, 2, \cdots)$, is equal to

$$\sum_n M_p(\Gamma_n; \pi, \kappa_n)=\int_{[\Gamma]\cap\tau}\left(\sum_n\left(\int_{c_Q}\left(\frac{\kappa_n^p|\mathrm{grad}\ H|}{\pi}\right)^{\frac{1}{p-1}}ds\right)^{1-p}\right)d\varphi(Q).$$

This result generalizes Theorem 2 in [6].

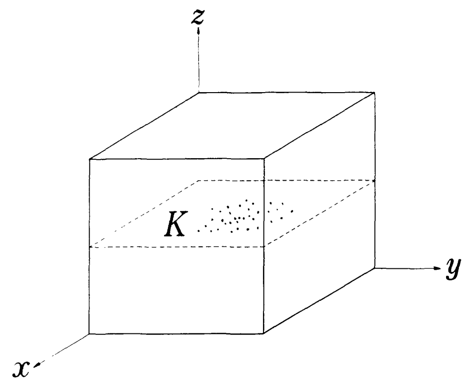
Furthermore, we can extend Theorem 1 of [6] in the following way. With every $c\in F$ let a family $\mathscr{K}_c$ of non-negative finite-valued linearly measurable functions k on c be associated such that at most one of them is positive at each point of c and $0<\int_c h_k ds<\infty$ for each $k\in\mathscr{K}_c$ where $h_k=(k^p|\mathrm{grad}\ H|/\pi)^{1/(p-1)}$. For some $c\in F$, $\mathscr{K}_c$ may be empty. Consider the function in $\mathscr{E}$ which is equal to $h_k(P)/\int_{c_P} h_k ds$ at P with $h_k(P)>0$ and which vanishes at P where no $h_k(P)$ is positive. If it is (Lebesgue) measurable, then the following conclusion is drawn: $\inf\int\pi\rho^p dv$, taken with respect to ρ satisfying $\int_c k\rho ds\geqq 1$ for all $c\in F$ and $k\in\mathscr{K}_c$, is equal to

$$\int\left(\sum_{k\in\mathscr{K}_{c_Q}}\left(\int_{c_Q}h_k ds\right)^{1-p}\right)d\varphi(Q),$$

where Q ranges over the set $\{Q\in\tau;\ \mathscr{K}_{c_Q}\neq\emptyset\}$.

§ 5. Decomposition of G into subflows

We exclude from G all critical points and points at infinity and denote by G' the remaining open set. We ask whether or not it is always possible to cover G' by mutually disjoint harmonic flows, possibly except for a subset of G' of measure zero. This is negatively answered by the following example: Exclude from the cube $0<x<1$, $0<y<1$, $0<z<1$, a totally disconnected compact subset K



of positive two-dimensional measure on the intersection of the cube with the plane $z=1/2$. Let G be the remaining domain, and take z in G as $H(P)$.

However, we prove

THEOREM 3. *We can decompose G' into a countable number of mutually disjoint harmonic subflows.*

PROOF. We cover G' by a countable number of regular tubes $T_1, T_2, \cdots$. We associate a harmonic flow F_n with T_n for each n. The subflows $F_1, F_2-F_1, F_3-F_2-F_1, \cdots$ fulfil the condition.

We obtain easily

THEOREM 4. *The module M_p of the family of all orthogonal trajectories is given by*

$$\sum_k \int_{[\Gamma_k]\cap\tau_k} \left(\int_{c_Q} g^{\frac{1}{p-1}} ds \right)^{1-p} d\varphi(Q),$$

where $\{\Gamma_k\}$ is a decomposition of G' into mutually disjoint subflows and τ_k is a domain on a level surface with which Γ_k is associated.

§ 6. Extremal length of level surfaces

Let S denote a countable collection of smooth surfaces in $\mathscr{E}$ and $\Sigma=\{S\}$ be a family of such collections. Let $\pi \geqq 0$ be a Borel measurable function in $\mathscr{E}$. We shall call a measurable function $\rho \geqq 0$ in $\mathscr{E}$ admissible if the surface integral $\int_S \rho d\sigma$ exists and $\geqq 1$ for each $S \in \Sigma$. The module $M_p(\Sigma; \pi)$ of Σ with weight π is defined by $\inf_\rho \int \pi\rho^p dv$ and the extremal length $\lambda_p(\Sigma; \pi)$ of Σ with weight π is set to be $1/M_p(\Sigma; \pi)$. We can regard this as a special case of Fuglede's definition.

In the same way as in §3 we can establish

THEOREM 5. *Let $p>1$, and T be a one-dimensional measurable subset of the range of values of $H(P)$. Denote by S_t the regular part of the level surface on which $H=t$. Then we have*

$$M_p(\{S_t; t\in T\}; \pi) = \int_T \left(\int_{S_t} g^{\frac{1}{p-1}} d\sigma \right)^{1-p} dt,$$

where $g=|\text{grad } H|/\pi$ as before. If $0<\int_{S_t} g^{1/(p-1)} d\sigma < \infty$ for a.e. $t\in T$, an *extremal function is given by*

$$\rho_0(P)=\begin{cases}\dfrac{g^{\frac{1}{p-1}}}{\int_{S_t} g^{\frac{1}{p-1}}d\sigma} & \textit{if } P\in S_t \textit{ and } t\in T', \\ \infty & \textit{if } P\in S_t \textit{ and } t\in T-T', \\ 0 & \textit{elsewhere},\end{cases}$$

where T' *is the set of* t *values for which* $0<\int_{S_t} g^{1/(p-1)}d\sigma<\infty$.

REMARK. We can consider $M_p(\Sigma;\pi,\kappa)$ as in §4.

References

[1] M. Brelot and G. Choquet: Espaces et lignes de Green, Ann. Inst. Fourier, 3 (1952), pp. 199-263.

[2] B. Fuglede: Extremal length and functional completion, Acta Math., 98 (1957), pp. 171-219.

[3] M. Ohtsuka: Théorèmes étoilés de Gross et leurs applications, Ann. Inst. Fourier, 5 (1955), pp. 1-28.

[4] M. Ohtsuka: Sur un théorème étoilé de Gross, Nagoya Math. J., 9 (1955), pp. 191-207.

[5] M. Ohtsuka: Dirichlet problem, extremal length and prime ends, Lecture Notes, Washington University, St. Louis, 1962-63.

[6] M. Ohtsuka: Extremal length of families of parallel segments, J. Sci. Hiroshima Univ. Ser. A-I Math., 28 (1964), pp. 39-51.

[7] S. Saks: Theory of the integral, Warszawa-Lwow, 1937.

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