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Note on F-operators in Locally Convex Spaces

Shigeaki Tôgô and Risai Shiraishi

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The theory of F-operators in Banach spaces has been developed by several authors (cf. the references in [5], [7]). According to [5], a closed, normally solvable, linear mapping with finite d-characteristic is called an F-operator. It is the purpose of this paper to generalize this notion of F-operator to locally convex spaces so that we may maintain a number of the basic results known in the case of Banach spaces. For the continuous F-operators, such an attempt has been made by H. Schaefer [9] and then by A. Deprit [4]. Our main concern here is the discussion of a general theory of F-operators: characterization of F-operators, the index theorem for a product, and so on.

§ **1.** Let E and F be locally convex Hausdorff spaces (denoted by LCS). Let u be a linear mapping with domain $\mathfrak{D}_u$ in E and rang $\mathfrak{R}_u$ in F. We denote by $\mathfrak{N}_u$ the null space of u. If u is closed, $\mathfrak{N}_u$ is a closed subspace of E. u is called *open* if $u(A)$ is an open subset of $\mathfrak{R}_u$ for each open subset A of $\mathfrak{D}_u$.

A linear mapping k of E into F is called *compact* if there is a neighbourhood U of 0 in E such that the set $k(U)$ is relatively compact.

We shall say that u is an *F-operator* when (i) $\mathfrak{N}_u$ *and* $F/\mathfrak{R}_u$ *are finite dimensional*; (ii) $\mathfrak{R}_u$ *is closed*; (iii) u *is open*. Moreover if u is continuous and $\mathfrak{D}_u=E$, we shall say that u is a continuous F-operator of E into F (According to [9], u is called a σ-homomorphism). The *index* of u is defined as $\operatorname{ind} u = \dim \mathfrak{N}_u - \operatorname{codim} \mathfrak{R}_u$.

We understand by $\tilde{\mathfrak{D}}_u$ the space $\mathfrak{D}_u$ with the weakest locally convex topology which makes the identical mapping $\mathfrak{D}_u \to \mathfrak{D}_u$ and the mapping u continuous. Then u becomes a continuous mapping of $\tilde{\mathfrak{D}}_u$ into F which we shall denote by $\tilde{u}$. As shown by F. E. Browder ([3], p. 66), $\tilde{u}$ is open if and only if u is open. Therefore u is an F-operator if and only if $\tilde{u}$ is an F-operator. With this in mind, we can show

Proposition 1 ([6], Prop. 2.1.). *Let u be a closed mapping with dense domain such that the injections $\tilde{\mathfrak{D}}_u \to E$ and $\tilde{\mathfrak{D}}_{u'} \to F'$ are compact. Then u is an F-operator.*

Proof. We have only to show that $\tilde{u}$ is an F-operator. Let v, k be the mappings of $\tilde{\mathfrak{D}}_u$ into $E \times F$ defined by $v(e)=\{e, u(e)\}$ and $k(e)=\{e, 0\}$. Then v is a monomorphism with closed range and, by assumption, k is compact.

Owing to a theorem of L. Schwartz ([10], A–16, p. 197), the mapping $v-k$: $e\rightarrow\{0, u(e)\}$ is an open mapping with closed range and with finite dimensional null space. Similarly, $\mathfrak{N}_{u'}$ is finite dimensional. Consequently, $\tilde{u}$ is an F-operator. The proof is complete.

REMARK 1. Any open linear mapping u with closed range and finite dimensional null space is closed. Consequently an F-operator is closed. Indeed, let u_1 be the restriction of u to a topological supplement E_1 of $\mathfrak{N}_u$ in $\mathfrak{D}_u$. Then u_1 is one-to-one and open. Hence the inverse mapping u_1^{-1} of $\mathfrak{R}_u$ onto E_1 is continuous. Now the graph $\mathfrak{G}_u$ of u can be written as

$$\begin{aligned}\mathfrak{G}_u &= \{(e, u(e)); e\in E_1\}+\{(e, 0); e\in\mathfrak{N}_u\}\\ &= \{(u_1^{-1}(f), f); f\in\mathfrak{R}_u\}+\{(e, 0); e\in\mathfrak{N}_u\}.\end{aligned}$$

Since u_1^{-1} is continuous and $\mathfrak{R}_u$ is closed, the subset $\{(u_1^{-1}(f), f); f\in\mathfrak{R}_u\}$ is closed in $E\times F$. It is known that if M is a closed linear subspace of an LCS G and N is a finite dimensional subspace of G, then $M+N$ is closed in G ([2], p. 28). Therefore it follows that $\mathfrak{G}_u$ is closed.

We note that if E and F are Banach spaces, owing to the closed graph theorem the notion of F-operator coincides with the one defined by I. C. Gohberg and M. G. Krein in [5] (p. 195).

For our later purpose we need the following lemma (cf. [4] and [8]). The proof goes along the same line with modifications as in the corresponding proof given in A. P. Robertson and W. Robertson ([8], p. 144).

For two mappings u_1, u_2, we shall use the notation $u_1\leqq u_2$ if u_2 is an extension of u_1.

LEMMA 1. *Let E and F be LCS's. Suppose that u is a closed linear mapping with domain in E and range in F, that v is a continuous linear mapping of F into E and that k is a compact linear mapping of E into itself such that*

$$v\circ u\leqq I_E-k,$$

where I_E denotes the identity mapping of E.

Then (i) *$\mathfrak{N}_u$ is finite dimensional*; (ii) *u is open*; (iii) *$\mathfrak{R}_u$ is closed in F.*

PROOF. Since k is compact, there exist a disked neighbourhood U of 0 in E and a compact set K of E such that $k(U)\subset K$.

(i) If $e\in U\cap\mathfrak{N}_u$, $v(u(e))=0$ and so $e=k(e)\in k(U)\subset K$. Hence $U\cap\mathfrak{N}_u\subset K$. Thus $\mathfrak{N}_u$ has a precompact neighbourhood and so is finite dimensional ([2], p. 30).

(ii) We shall consider the continuous linear mapping $\tilde{u}$ of $\tilde{\mathfrak{D}}_u$ into F. Now it is sufficient to show that $\tilde{u}$ is open. If $\tilde{u}$ is not open, there exists some disked neighbourhood W of 0 in $\tilde{\mathfrak{D}}_u$, which we may clearly suppose contained in U, such that $\tilde{u}(W)$ is not a neighbourhood of 0 in $\mathfrak{R}_u$. Let $\mathfrak{W}$ be a base of disked neighbourhoods of 0 for $\mathfrak{R}_u$. Then each $V \in \mathfrak{W}$ meets $\mathfrak{R}_u \setminus \tilde{u}(W) = \tilde{u}(\mathfrak{D}_u \setminus (W + \mathfrak{N}_u))$; if e is a common element, there is a λ with $0 < \lambda \leqq 1$ and $\lambda e \in \tilde{u}(2W) \setminus \tilde{u}(W)$. Putting $A = 2W \setminus (W + \mathfrak{N}_u)$, V also meets $\tilde{u}(A)$. Let $\mathfrak{F}$ be an ultrafilter on $\mathfrak{D}_u$ containing the sets $A \cap \tilde{u}^{-1}(V)$, $V \in \mathfrak{W}$. The set A belongs to $\mathfrak{F}$ and $\tilde{u}(\mathfrak{F}) = u(\mathfrak{F}) \xrightarrow{F} 0$. Now $k(\mathfrak{F})$ contains $k(A) \subset k(2W) \subset 2K$, hence converges in E to an element $e_0 \in 2K$. Since $I_E = k + v \circ \tilde{u}$ on $\tilde{\mathfrak{D}}_u$, $\mathfrak{F} \xrightarrow{E} e_0 + v(0) = e_0$. Since u is closed, we see that $e_0 \in \mathfrak{D}_u$, $u(e_0) = 0$ and $\mathfrak{F} \xrightarrow{\tilde{\mathfrak{D}}_u} e_0$. But $A \in \mathfrak{F}$ and so $e_0 \in \bar{A}$, the closure of A under the topology of $\tilde{\mathfrak{D}}_u$. Thus $e_0 \in \mathfrak{N}_u$ and so $e_0 \in \mathfrak{N}_u \cap \bar{A}$. But $(\mathfrak{N}_u + W) \cap A = \emptyset$, and this contradiction proves that $\tilde{u}$ is open.

(iii) If $f_0 \in \bar{\mathfrak{R}}_u$, the sets $\mathfrak{R}_u \cap (f_0 + W)$ with $W \in \mathfrak{W}$ form the base of a Cauchy filter on $\mathfrak{R}_u$, where $\mathfrak{W}$ is a base of disked neighbourhoods of 0 for F. Since $\tilde{u}(U)$ is a neighbourhood of 0 in $\mathfrak{R}_u$, there exists an element e_0 such that $(e_0 + U) \cap \tilde{u}^{-1}(f_0 + W) \neq \emptyset$ for each $W \in \mathfrak{W}$. Let $\mathfrak{G}$ be an ultrafilter on $\mathfrak{D}_u$ containing the sets $(e_0 + U) \cap \tilde{u}^{-1}(f_0 + W)$ with $W \in \mathfrak{W}$. Then $\tilde{u}(\mathfrak{G}) = u(\mathfrak{G}) \xrightarrow{F} f_0$. Now $k(\mathfrak{G})$ contains $k(e_0 + U) \subset k(e_0) + K$, a compact set in E, and so $k(\mathfrak{G})$ converges in E to an element $e_1 \in k(e_0) + K$. Since $I_E = k + v \circ \tilde{u}$ on $\tilde{\mathfrak{D}}_u$, $\mathfrak{G} \xrightarrow{E} e_1 + v(f_0)$. It follows since u is closed that $e_1 + v(f_0) \in \mathfrak{D}_u$ and $u(e_1 + v(f_0)) = f_0 \in \mathfrak{R}_u$. Therefore $\mathfrak{R}_u$ is closed. This proves the lemma.

Now we shall show a theorem concerning the characterizations of F-operator, which is a generalization of the corresponding result of F. V. Atkinson ([1], p. 4).

THEOREM 1. *Let E and F be LCS's and u be a linear mapping with domain in E and range in F. Then the following statements on u are equivalent*:

(i) *u is an F-operator*;

(ii) *u is closed and there exist a continuous linear mapping v of F into E and compact linear mappings k_1 and k_2 of E and of F into themselves respectively satisfying the relations*:

$$v \circ u \leqq I_E - k_1, \qquad u \circ v = I_F - k_2;$$

(iii) *there exist a continuous linear mapping v of F into $\tilde{\mathfrak{D}}_u$ and compact linear mappings k_1 and k_2 of $\tilde{\mathfrak{D}}_u$ and of F into themselves respectively satisfying the relations*:

$$v \circ \tilde{u} = I_{\tilde{\mathfrak{D}}_u} - k_1, \qquad \tilde{u} \circ v = I_F - k_2.$$

PROOF. (i)$\Rightarrow$(ii). Let E_1 be a topological supplement of $\mathfrak{N}_u$ in E. We put $u_1=u\,|(E_1\cap\mathfrak{D}_u)$, the restriction of u to $E_1\cap\mathfrak{D}_u$, which is open and one-to-one and so that $v_1=u_1^{-1}$ is a continuous linear mapping of $\mathfrak{R}_u$ onto $E_1\cap\mathfrak{D}_u$. By i and p we denote the injection of $E_1\cap\mathfrak{D}_u$ into E and a continuous projection of F onto $\mathfrak{R}_u$ respectively. We put $v=i\circ v_1\circ p$, a continuous linear mapping of F into E. Since $\mathfrak{N}_u$ is finite dimensional, the projection k_1 with $k_1(E_1)=0$ of E onto $\mathfrak{N}_u$ is of finite rank and so compact. An arbitrary element e in $\mathfrak{D}_u$ can be written uniquely in the form: $e=e_1+e_2$, $e_1\in E_1\cap\mathfrak{D}_u$, $e_2\in\mathfrak{N}_u$. Thus we have for any $e\in\mathfrak{D}_u$

$$v\circ u(e)=v\circ u(e_1+e_2)=v_1\circ u_1(e_1)=e_1=I_E(e)-k_1(e).$$

Consequently, $v\circ u\leqq I_E-k_1$.

Let F_1 be a topological supplement of $\mathfrak{R}_u$ in F. F_1 is finite dimensional. Let k_2 be a continuous projection of F onto F_1 such that $k_2(\mathfrak{R}_u)=0$. k_2 is of finite rank and so compact. An arbitrary element f in F can be written uniquely as the sum of $f_1\in\mathfrak{R}_u$ and $f_2\in F_1$. Thus we have for any $f\in F$

$$u\circ v(f)=u\circ v(f_1+f_2)=u_1\circ v_1(f_1)=f_1=I_F(f)-k_2(f),$$

which means that $u\circ v=I_F-k_2$.

Since an F-operator is closed, (i) implies (ii).

(i)$\Rightarrow$(iii). Now, if u is an F-operator, as remarked already, $\tilde{u}$ is also an F-operator of $\tilde{\mathfrak{D}}_u$ into F. Therefore we can infer in a similar way as in the above proof that (i) implies (iii).

(ii)$\Rightarrow$(i). By virtue of Lemma 1, it follows from $v\circ u\leqq I_E-k_1$ that $\mathfrak{N}_u$ is finite dimensional, $\mathfrak{R}_u$ is closed and u is open. Therefore we have only to show that $F/\mathfrak{R}_u$ is finite dimensional. Since $u\circ v=I_F-k_2$, it follows that $\mathfrak{R}_u\supset\mathfrak{R}_{I_F-k_2}$. But it is known that $F/\mathfrak{R}_{I_F-k_2}$ is finite dimensional ([8], p. 144). Therefore $F/\mathfrak{R}_u$ is finite dimensional. Consequently, u is an F-operator. Hence (ii) implies (i).

The implication (iii)$\Rightarrow$(i) may be proved in a similar manner as in the case (ii)$\Rightarrow$(i). The proof is omitted.

Thus the proof of the theorem is complete.

REMARK 2 ([1], Theorem 1). Theorem 1 remains true if we assume that k_1 and k_2 are of finite rank. In fact, a continuous mapping of finite rank is compact and k_1, k_2 constructed in the proof of (i)$\Rightarrow$(ii) are of finite rank. v being continuous, so it is known that v is also continuous when we impose on E and F another topology such as weak topology, or Mackey topology. Therefore if u is an F-operator, then u is also an F-operator in weak topology or in Mackey topology.

REMARK 3 ([1], Theorem 1). Let u be a closed linear mapping with domain in E and range in F. Then u is an F-operator if and only if there exist continuous linear mappings v_1 and v_2 of F into G and of H into E respectively such that $v_1 \circ u$ and $u \circ v_2$ are F-operators. In fact, the proof of "only if" part is a direct consequence of Theorem 1. Conversely, suppose $v_1 \circ u$ and $u \circ v_2$ are F-operators. By Theorem 1, there exist continuous linear mappings w_1 and w_2 of G into E and of F into H respectively such that

$$w_1 \circ v_1 \circ u \leqq I_E - k_1, \qquad u \circ v_2 \circ w_2 = I_F - k_2,$$

where k_1, k_2 are compact. Similar arguments used in the proof of (ii)$\Rightarrow$(i) show that u is an F-operator.

As an application of Theorem 1, we show

PROPOSITION 2. *Let E and F be LCS's. Let u be a closed liner mapping with dense domain in E and range in F. Then u' is an F-operator if u is an F-operator. If E, F have γ-topology and if u' as a mapping with domain in F'_c and range in E'_c is an F-operator, then u is also an F-operator. In any case,* ind $u = -$ind u'.

PROOF. From Remark 2 after Theorem 1, there exists a continuous linear mapping v of F into E such that

$$v \circ u \leqq I_E - k_1, \qquad u \circ v = I_F - k_2,$$

where k_1 and k_2 are of finite rank. Then we have for any $f' \in \mathfrak{D}_{u'}$ and any $f \in F$

$$<v' \circ u'(f'), f> = <f', u \circ v(f)> = <f', (I_F - k_2)(f)> = <(I_{F'} - k'_2)f', f>,$$

which means that $v' \circ u' \leqq I_{F'} - k'_2$. Putting $f' = v'(e')$ for any $e' \in E'$, we have for any $e \in \mathfrak{D}_u$

$$<u(e), f'> = <u(e), v'(e')> = <v \circ u(e), e'>$$
$$= <(I_E - k_1)(e), e'> = <e, (I_{E'} - k'_1)(e')>.$$

Hence we see that $u'(f') = (I_{E'} - k'_1)(e')$ and so $u' \circ v' = I_{E'} - k'_1$. Now, k'_1 and k'_2 are also of finite rank and u' is closed. Therefore by Theorem 1 it follows that u' is an F-operator. Moreover, ind $u = \dim \mathfrak{N}_u - \operatorname{codim} \mathfrak{R}_u = \dim (\mathfrak{R}_{u'})^{\perp} - \operatorname{codim} (\mathfrak{N}_{u'})^{\perp} = \operatorname{codim} \mathfrak{R}_{u'} - \dim \mathfrak{N}_{u'} = -$ind u'. As made in Remark 2, u' is also an F-operator if we impose on E', F' the topology of uniform convergence on compact disks.

To prove the second part of the proposition, let u' be an F-operator in the indicated sense, then $u'' = u$ becomes an F-operator from the preceding discussion. The proof is complete.

Corollary. *If E and F are Banach spaces, a closed linear mapping with dense domain in E and range in F is an F-operator if and only if u' is an F-operator.*

Proof. We have only to show that if u' is an F-operator, then u is also an F-operator. Then $\mathfrak{R}_{u'}$ is closed, so u is an open mapping with closed range ([3], p. 57), and $\dim \mathfrak{N}_{u'} < +\infty$ and $\operatorname{codim} \mathfrak{R}_{u'} < +\infty$ imply that $\dim \mathfrak{N}_u < +\infty$ and $\operatorname{codim} \mathfrak{R}_u < +\infty$. Consequently, u is an F-operator.

§ **2.** Now we are in a position to prove the following theorem concerning the product of F-operators ([5], Theorem 2.1). For bounded operators in a Banach space the theorem was first proved by F. V. Atkinson ([1], p. 8), and for unbounded operators by I. C. Gohberg and M. G. Krein (cf. [5], Theorem 2.1).

Theorem 2. *Let E, F and G be LCS's. If u_1 and u_2 are F-operators with domain in E and range in F and with domain in F and range in G respectively, then $u_2 \circ u_1$ is also an F-operator and*

$$\operatorname{ind} u_2 \circ u_1 \geqq \operatorname{ind} u_1 + \operatorname{ind} u_2,$$

where the equality holds if and only if $F = \mathfrak{R}_{u_1} + \mathfrak{D}_{u_2}$. The condition is satisfied if $\mathfrak{D}_{u_2}$ is dense in F.

Proof. By Theorem 1 there exist continuous linear mappings v_1, v_2, k_1, k_2 such that

$$v_1 \circ \tilde{u}_1 = I_{\tilde{\mathfrak{D}}_{u_1}} - k_1, \qquad v_2 \circ u_2 \leqq I_F - k_2,$$

where $\mathfrak{D}_{v_1} \subset F$, $\mathfrak{R}_{v_1} \subset \tilde{\mathfrak{D}}_{u_1}$, $\mathfrak{D}_{v_2} \subset G$, $\mathfrak{R}_{v_2} \subset F$ and k_1, k_2 are compact mappings of $\tilde{\mathfrak{D}}_{u_1}$ and of F into themselves respectively. Then we have

$$\begin{aligned} v_1 \circ v_2 \circ u_2 \circ \tilde{u}_1 &\leqq v_1 \circ (I_F - k_2) \circ \tilde{u}_1 = v_1 \circ \tilde{u}_1 - v_1 \circ k_2 \circ \tilde{u}_1 \\ &= I_{\tilde{\mathfrak{D}}_{u_1}} - k_1 - v_1 \circ k_2 \circ \tilde{u}_1. \end{aligned}$$

$u_2 \circ \tilde{u}_1$ is closed since $\tilde{u}_1$ is continuous and u_2 is closed. Therefore by Lemma 1 we see that $\mathfrak{N}_{u_2 \circ \tilde{u}_1}$ is finite dimensional, $\mathfrak{R}_{u_2 \circ \tilde{u}_1}$ is closed and $u_2 \circ \tilde{u}_1$ is open. On account of the definition of $\tilde{u}_1$, these properties are also enjoyed by $u_2 \circ u_1$.

On the other hand, we have

$$\mathfrak{N}_{u_2 \circ u_1} = (u_2 \circ u_1)^{-1}(0) = u_1^{-1}\big(u_2^{-1}(0) \cap \mathfrak{R}_{u_1}\big) = u_1^{-1}(\mathfrak{N}_{u_2} \cap \mathfrak{R}_{u_1}),$$

which implies that

$$\dim \mathfrak{N}_{u_2\circ u_1}=\dim \mathfrak{N}_{u_1}+\dim(\mathfrak{N}_{u_2}\cap\mathfrak{R}_{u_1}) \tag{1}$$

and

$$\operatorname{codim} \mathfrak{R}_{u_2\circ u_1}=\operatorname{codim} \mathfrak{R}_{u_2}+\dim \mathfrak{R}_{u_2}/\mathfrak{R}_{u_2\circ u_1}, \tag{2}$$

where

$$\begin{aligned}\dim \mathfrak{R}_{u_2}/\mathfrak{R}_{u_2\circ u_1} &=\dim \mathfrak{D}_{u_2}/\{\mathfrak{D}_{u_2}\cap(\mathfrak{R}_{u_1}+\mathfrak{N}_{u_2})\}\\ &=\dim(\mathfrak{D}_{u_2}+\mathfrak{R}_{u_1})/(\mathfrak{R}_{u_1}+\mathfrak{N}_{u_2})\\ &\leqq\dim F/(\mathfrak{R}_{u_1}+\mathfrak{N}_{u_2})\\ &=\dim F/\mathfrak{R}_{u_1}-\dim(\mathfrak{R}_{u_1}+\mathfrak{N}_{u_2})/\mathfrak{R}_{u_1}.\\ &=\operatorname{codim} \mathfrak{R}_{u_1}-\dim \mathfrak{N}_{u_2}+\dim(\mathfrak{R}_{u_1}\cap\mathfrak{N}_{u_2}).\end{aligned} \tag{3}$$

Consequently, from the equations (1), (2) and (3) we see that $\operatorname{codim}\mathfrak{R}_{u_2\circ u_1}$ is finite and

$$\operatorname{ind} u_2\circ u_1\geqq\operatorname{ind} u_1+\operatorname{ind} u_2.$$

In view of the relations (3), $\operatorname{ind} u_2\circ u_1=\operatorname{ind} u_1+\operatorname{ind} u_2$ holds if and only if $F=\mathfrak{D}_{u_2}+\mathfrak{R}_{u_1}$. The last statement of the theorem is almost clear. Thus the proof is complete.

Remark 4. It is easy to verify that if, in the theorem 2, u_1 and u_2 have dense domains, then $u_2\circ u_1$ has also dense domain and $(u_2\circ u_1)'=u_1'\circ u_2'$.

A linear mapping k with domain $\mathfrak{D}_k$ in E and range in F will be called *u-compact* if $\mathfrak{D}_k\supset\mathfrak{D}_u$ and there exist two neighbourhoods U and V of 0 in $\mathfrak{D}_u$ and in $\mathfrak{R}_u$ respectively such that k maps $U\cap u^{-1}(V)$ into a compact subset of F, that is, the corresponding mapping $\bar{k}$ of $\widetilde{\mathfrak{D}}_u$ into F is compact.

We next prove the following

Theorem 3. *Let E and F be LCS's and u be an F-operator with domain in E and range in F. Let k be a u-compact linear mapping. Then $u+k$ is an F-operator and*

$$\operatorname{ind}(u+k)=\operatorname{ind} u.$$

Proof. By Theorem 1, there exist a continuous linear mapping v of F into $\widetilde{\mathfrak{D}}_u$ and compact linear mappings k_1 and k_2 of $\widetilde{\mathfrak{D}}_u$ and of F into themselves respectively such that

$$v\circ\tilde{u}=I_{\tilde{\mathfrak{D}}_u}-k_1, \qquad \tilde{u}\circ v=I_F-k_2.$$

We now denote by $\bar{k}$ the restriction of k to $\tilde{\mathfrak{D}}_u$. By definition, $\bar{k}$ is compact. We have

$$v\circ(\tilde{u}+\bar{k})=v\circ\tilde{u}+v\circ\bar{k}=I_{\tilde{\mathfrak{D}}_u}-k_3,$$

$$(\tilde{u}+\bar{k})\circ v=\tilde{u}\circ v+\bar{k}\circ v=I_F-k_4,$$

where $k_3=k_1-v\circ\bar{k}$ and $k_4=k_2-\bar{k}\circ v$. Taking into account that the mappings k_3 and k_4 are compact, it follows from Theorem 1 that $\tilde{u}+\bar{k}$ is an F-operator. Thus we can easily conclude that $u+k$ is also an F-operator.

Now we note that the mapping v is an F-operator. Applying Theorem 2 to the products $\tilde{u}\circ v$ and $(\tilde{u}+\bar{k})\circ v$ we have

$$\text{ind } \tilde{u}+\text{ind } v=\text{ind}(I_F-k_2), \tag{4}$$

$$\text{ind}(\tilde{u}+\bar{k})+\text{ind } v=\text{ind}(I_F-k_4). \tag{5}$$

According to Proposition 4 in [8] (p. 151), $\text{ind}(I_F-k_2)=\text{ind}(I_F-k_4)=0$. Therefore, from the equations (4) and (5) we obtain

$$\text{ind}(\tilde{u}+\bar{k})=\text{ind } \tilde{u}.$$

Consequently,

$$\text{ind}(u+k)=\text{ind } u.$$

Thus the proof is complete.

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Department of Mathematics,
Faculty of Science,
Hiroshima University

and

Faculty of General Education,
Hiroshima University